Research Article

A Fourth-Order Block-Grid Method for Solving Laplace’s Equation on a Staircase Polygon with Boundary Functions in $C^{k,\lambda}$

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1. Introduction

In the last two decades, among different approaches to solve the elliptic boundary value problems with singularities, a special emphasis has been placed on the construction of combined methods, in which differential properties of the solution in different parts of the domain are used (see [1, 2], and references therein).

In [2–7], a new combined difference-analytical method, called the block-grid method (BGM), is proposed for the solution of the Laplace equation on polygons, when the boundary functions on the sides causing the singular vertices are given as algebraic polynomials of the arc length. In the BGM, the given polygon is covered by a finite number of overlapping sectors around the singular vertices (“singular” parts) and rectangles for the part of the polygon which lies at a positive distance from these vertices (“nonsingular” part). The special integral representation in each “singular” part is approximated, and they are connected by the appropriate order gluing operator with the finite difference equations used in the “nonsingular” part of the polygon.

In [8, 9], the restriction on the boundary functions to be algebraic polynomials on the sides of the polygon causing the singular vertices in the BGM was removed. It was assumed that the boundary function on each side of the polygon is given from the Hölder classes $C^{k,\lambda}$, $0 < \lambda < 1$, and on the “nonsingular” part the 5-point scheme is used when $k = 2$ [8] and the 9-point scheme is used when $k = 6$ [9]. For the 5-point scheme a simple linear interpolation with 4 points is used. For the 9-point scheme an interpolation with 31 points is used to construct a gluing operator connecting the subsystems. Moreover, to connect the quadrature nodes which are at a distance of less than $4h$ from boundary of the polygon, a special representation of the harmonic function through the integrals of Poisson type for a half plane is used (see [9]).

In this paper the BGM is developed for the Dirichlet problem when the boundary function on each side of the polygon is from $C^{k,\lambda}$, by using the 9-point scheme on the “nonsingular” part with 16-point gluing operator for all quadrature nodes, including those near the boundary. The
paper is organized as follows: in Section 2, the boundary value problem and the integral representations of the exact solution in each "singular" part are given. In Section 3, to support the aim of the paper, a Dirichlet problem on the rectangle for the known exact solution from $C^{k,\lambda}$, $k = 3, 4$, is solved using the 9-points scheme and the numerical results are illustrated. In Section 4, the system of block-grid equations and the convergence theorems are given. In Section 5 a highly accurate approximation of the coefficient of the leading singular term of the exact solution (stress intensity factor) is given. In Section 6 the method is illustrated for solving the problem in L-shaped polygon with the corner singularity. The conclusions are summarized in Section 7.

2. Dirichlet Problem on a Staircase Polygon

Let $G$ be an open simply connected polygon, $\gamma_j$, $j = 1, 2, \ldots, N$, its sides, including the ends, enumerated counterclockwise, $\gamma = \gamma_1 \cup \cdots \cup \gamma_N$ the boundary of $G$, and $\alpha, \pi$, ($\alpha_j = 1/2$, $1, 3/2, 2$), the interior angle formed by the sides $\gamma_{j-1}$ and $\gamma_j$, $j = 1, 2, 3, 4$. Denote by $A_j = \gamma_{j-1} \cap \gamma_j$ the vertex of the $j$th angle and by $r_j, \theta_j$ a polar system of coordinates with a pole in $A_j$, where the angle $\theta_j$ is taken counterclockwise from the side $\gamma_j$. We consider the boundary value problem

$$\Delta u = 0 \quad \text{on } G, \quad u = \varphi_j(s) \quad \text{on } \gamma_j, \quad 1 \leq j \leq N, \quad (1)$$

where $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$, $\varphi_j$ is a given function on $\gamma_j$ of the arc length $s$ taken along $\gamma$, and $\varphi_j \in C^{4,\lambda}(\gamma_j)$, $0 < \lambda < 1$; that is, $\varphi_j$ has the fourth-order derivative on $\gamma_j$, which satisfies a Hölder condition with exponent $\lambda$.

At some vertices $A_j$, $(s = s_j)$ for $\alpha_j = 1/2$ the conjugation conditions

$$\varphi_j^{(2\alpha)}(s_j) = (-1)^q \varphi_j^{(2\alpha)}(s_j), \quad q = 0, 1 \quad (2)$$

are fulfilled. For the remaining vertices $A_j$, the values of $\varphi_{j-1}$ and $\varphi_j$ at $A_j$ might be different. Let $E$ be the set of all $j$, $(1 \leq j \leq N)$ for which $\alpha_j \neq 1/2$ or $\alpha_j = 1/2$, but (2) is not fulfilled. In the neighborhood of $A_j$, $j \in E$, we construct two fixed block sectors $T_{j_i} = T_j(r_{j_i}) \subset G$, $i = 1, 2$, where $0 < r_{j_2} < r_{j_1} < \min\{s_{j+1} - s_j, s_j - s_{j-1}\}$, $T_j(r) = \{(r, \theta) : 0 < r_j < r, \ 0 < \theta < \alpha_j \pi\}$.

Let (see [10])

$$\varphi_{j0}(t) = \varphi_j(s_j + t) - \varphi_j(s_j), \quad (3)$$

$$\varphi_{j1}(t) = \varphi_{j-1}(s_j - t) - \varphi_j(s_j),$$

$$Q_j(r, \theta_j) = \varphi_j(s_j) + \frac{\varphi_{j-1}(s_j) - \varphi_j(s_j)}{\alpha_j \pi} \theta_j, \quad (4)$$

where

$$\tilde{x}_j = r_{j}^{1/\alpha_j} \cos \left( \frac{\theta_j}{\alpha_j} \right), \quad \tilde{y}_j = r_{j}^{1/\alpha_j} \sin \left( \frac{\theta_j}{\alpha_j} \right), \quad (5)$$

$$\sigma_{jk} = \left| s_{j+k} - s_{j-k} \right|^{1/\alpha_{jk}}.$$
and at the vertices $A_j = y_{j-1} \cap y_j$ the conjugation conditions
\[
\varphi_j^{(2q)} = (-1)^q \varphi_j^{(2q)}, \quad q = 0, 1
\]
are satisfied.

Remark 3. From Theorem 3.1 in [11] it follows that the class of functions $C^{4,\lambda}(\Pi)$ is wider than $C^{4,\lambda}(\Pi)$.

Let $h > 0$, with $a/h \geq 2$, $b/h \geq 2$ integers. We assign to $\Pi^h$ a square net on $\Pi$, with step $h$, obtained with the lines $y = 0, h, 2h, \ldots$. Let $y_j^h$ be a set of nodes on the interior of $y_j$ and let
\[
y_j^h = y_j \cap y_{j+1}, \quad y_j^h = \bigcup_{q=1}^4 \left(y_j^h \cup y_j^h\right),
\]
(12)

We consider the system of finite difference equations
\[
\begin{align*}
\Delta u &= 0 \quad \text{on } \Pi^h, \\
u_h &= \varphi_j \quad \text{on } y_j^h, \quad j = 1, 2, 3, 4,
\end{align*}
\]
(13)

where
\[
Bu(x, y) = \frac{1}{5} \left( (u(x+h, y) + u(x, y+h) + u(x-h, y) + u(x, y-h))
\right.
\]
\[
+ \left( (u(x+h, y+h) + u(x-h, y+h)
\right.
\]
\[
+ u(x-h, y-h) + u(x+h, y-h))
\]
\[
\times 20^{-1}.
\]
(14)

On the basis of the maximum principle the unique solvability of the system of finite difference equations (13) follows (see [12, Chapter 4]).

Everywhere below we will denote constants which are independent of $h$ and of the co-factors on their right by $C_0, C_1, \ldots$, generally using the same notation for different constants for simplicity.

Theorem 4. Let $u$ be the solution of problem (9). If $u \in C^{4,\lambda}(\Pi)$, then
\[
\max_{\Pi^h} |u_h - u| \leq ch^4,
\]
(15)

where $u_h$ is the solution of the system (13).

Proof. For the proof of this theorem see [13].

Let $\Pi' = \{(x, y) : -0.25 < x < 0.25, \; 0 < y < 1\}$ and let $y'$ be the boundary of $\Pi'$. We consider the Dirichlet problem
\[
\Delta u = 0 \quad \text{on } \Pi',
\]
\[
u = \nu' \quad \text{on } y',
\]
(16)

where $\nu = r^{k+\lambda} \cos(k + \lambda)\theta, r = \sqrt{x^2 + y^2}, \; 0 < \lambda < 1$, is the exact solution of this problem, which is from $C^{4,\lambda}(\Pi')$.

We solve the problem (16) by approximating 9-point scheme when $k = 3, 4$ for the different values of $\lambda$.

In Figure 1, the order of numerical convergence
\[
\gamma^h_{9p} = \max_{\Pi^h} \frac{|u_{2p} - u|}{\max_{\Pi^h} |u_{2p-1} - u|}
\]
(17)
of the 9-point solution $u_h$, for different $h = 2^{-p} \Psi$ and $p = 4, 5, 6, 7$, is demonstrated. These results show that the order of numerical convergence, when the exact solution $u \in C^{4,\lambda}(\Pi)$, depends on $k$ and $\lambda$ and is $O(h^4)$ when $k = 4$, which supports estimation (15). Moreover, this dependence also requires the use of fourth-order gluing operator for all quadrature nodes in the construction of the system of block-grid equations, when the given boundary functions are from the Hölder classes $C^{4,\lambda}$.

4. System of Block-Grid Equations

In addition to the sectors $T_j^1$ and $T_j^2$ (see Section 2) in the neighborhood of each vertex $A_j, j \in E$ of the polygon $G$, we construct two more sectors $T_j^3$ and $T_j^4$, where $0 < r_{j2} < r_{j3} < r_{j2}$, $r_{j3} = (r_{j2} + r_{j4})/2$ and $T_k^1 \cap T_j^1 = \emptyset, k \neq l, k, l \in E$, and let $G_T = G \setminus (\bigcup_{j \in E} T_j^3)$.

We cover the given solution domain (a staircase polygon) by the finite number of sectors $T_j^1, j \in E$, and rectangles $\Pi_k \subset G_T$, $k = 1, 2, \ldots, M$, as is shown in Figure 2, for the case of L-shaped polygon, where $j = 1, M = 4$ (see also [2]). It is assumed that for the sides $a_{ik}/a_{2k}$, $a_{ik}$ of $\Pi_k$ the quotient $a_{ik}/a_{2k}$ is rational and $G = (\bigcup_{k=1}^M \Pi_k) \cup (\bigcup_{j \in E} T_j^3)$. Let $\ell_k$ be the boundary of the rectangle $\Pi_k$, let $V_j$ be
the curvilinear part of the boundary of the sector $T_j^2$, and let $t_{kj} = \eta_k \cap T_j^2$. We choose a natural number $n$ and define the quantities $n(j) = \max\{4, n(m_j)\}$, $\beta_j = \alpha_j \pi / n(j)$, and $\theta_j^m = (m - 1/2) \beta_j$, $j \in E$, $1 \leq m \leq n(j)$. On the arc $V_j$, we take the points $(r_j^m, \theta_j^m)$, $1 \leq m \leq n(j)$, and denote the set of these points by $V_j^n$. We introduce the parameter $h \in (0, x_0/4]$, where $x_0$ is a gluing depth of the rectangles $\Pi_k$, $k = 1, 2, \ldots, M$, and define a square grid on $\Pi_k$, $k = 1, 2, \ldots, M$, with maximal possible step $h_k \leq \min\{h, \min\{a_{1kj}, a_{2kj}\}/6\}$ such that the boundary $\eta_k$ lies entirely on the grid lines. Let $\Pi_k^h$ be the set of grid nodes on $\Pi_k$, let $\eta_k^h$ be the set of nodes on $\eta_k$, and let $\Pi_k^h = \Pi_k^h \cup \eta_k^h$. We denote the set of nodes on the closure of $\eta_k \cap G_j$ by $\eta_k^h$, the set of nodes on $t_{kj}$ by $t_{kj}^h$, and the remaining set of nodes on $\eta_k$ by $\eta_k^{1h}$.

Let $\omega^{kn} = \left(\pi_{k=1}^{M_k} \eta_k^h\right) \cup \left(\bigcup_{j \in E} V_j^n\right)$, 

$$\Pi_k^h = \omega^{kn} \cup \left(\bigcup_{m=1}^{M_k} \Pi_k\right).$$

Let $\varphi = \{\varphi_j\}_{j=1}^{N}$, where $\varphi_j \in C^{4,\lambda}(y_j)$, $0 < \lambda < 1$, is the given function in (1). We use the matching operator $S^\delta$ at the points of the set $\omega^{kn}$ constructed in [14]. The value of $S^\delta(u_h, \varphi)$ at the point $P \in \omega^{kn}$ is expressed linearly in terms of the values of $u_h$ at the points $P_k$ of the grid constructed on $\Pi_k(P)$, $(P \in \Pi_k(P))$ some part of whose boundary located in $G$ is the maximum distance away from $P$, and in terms of the boundary values of $\varphi^m$, $m = 0, 1, 2, 3$ at a fixed number of points. Moreover $S^\delta(u_h, 0)$ has the representation

$$S^\delta(u_h, 0) = \sum_{0 \leq j \leq 15} \xi_j u_{h,j},$$

where $u_{h,k} = u_h(P_k)$,

$$\xi_j \geq 0, \quad \sum_{0 \leq j \leq 15} \xi_j = 1,$$  \hspace{1cm} (20)

$$u - S^\delta(u, \varphi) = O(h^4).$$  \hspace{1cm} (21)

Let $\omega_n^{kn} \subset \omega^{kn}$ be the set of such points $P \in \omega^{kn}$, for which all points $P_k$ in expression (19) are in $\cup_{k=1}^{M} \Pi_{r_j}\Pi_{\theta_j}$. If some points $P_k$ in (19) emerge through the side $y_m$, then the set of such points $P$ is denoted by $\omega_n^{kn}$. According to the construction of $S^\delta$ in [14], the expression $S^\delta(u_h, \varphi)$ at each point $P \in \omega^{kn} = \omega_n^{kn} \cup \omega_D^{kn}$ can be expressed as follows:

$$S^\delta(u_h, \varphi) = \begin{cases} S^\delta(u_h, \varphi) & P \in \omega_n^{kn}, \\
S^\delta(u_h - \sum_{k=0}^{3} a_k \Re e^k, P) & P \in \omega_D^{kn}, \end{cases}$$

where

$$a_k = \frac{1}{k!} \frac{d}{ds} \left|_{s=s_p} \varphi_0(m(s)) \right|, \quad k = 0, 1, 2, 3$$  \hspace{1cm} (23)

and $s_p$ corresponds to such a point $Q \in y_m$ for which the line $PQ$ is perpendicular to $y_m$.

Let

$$Q_j = Q_j(r_j, \theta_j), \quad Q_j^0 = Q_j(r_j, \theta_j^0).$$

The quantities in (24) are given by (4) and (5), which contain integrals that have not been computed exactly in the general case. Assume that approximate values $Q_j^0$ and $Q_j^2$ of the quantities in (24) are known with accuracy $\varepsilon > 0$; that is,

$$|Q_j^0 - Q_j| \leq c_1 \varepsilon, \quad |Q_j^{2e} - Q_j^0| \leq c_2 \varepsilon,$$  \hspace{1cm} (25)

where $j \in E$, $1 \leq q \leq n(j)$, and $c_1$, $c_2$ are constants independent of $\varepsilon$.

Consider the system of linear algebraic equations

$$u_h^e = Bu_h^e \quad \text{on} \quad \Pi_k^h,$$

$$u_h^e = \varphi_q \quad \text{on} \quad \eta_k^h \cap y_m,$$

$$u_h^e(r_j, \theta_j) = Q_j^0 + \beta_j \sum_{q=1}^{n(j)} (u_h^e(r_j, \theta_j^q) - Q_j^0) \times R_j(r_j, \theta_j^q) \quad \text{on} \quad (r_j, \theta_j) \in t_{kj}^h,$$

where $1 \leq k \leq M, 1 \leq m \leq N$, and $j \in E$. 

![Figure 2: Description of BGM for the L-shaped domain.](image-url)
Table 1: The order of convergence in the “nonsingular” part when \( h = 2^{-\epsilon} \) and \( \epsilon = 5 \times 10^{-13} \).

<table>
<thead>
<tr>
<th>( (2^{-\epsilon}, n) )</th>
<th>( | \xi^n_h |_{C^{0,1}} )</th>
<th>( \mathcal{R}^{C_{0,1}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (2^{-4}, 60) )</td>
<td>( 1.609 \times 10^{-8} )</td>
<td>15.577</td>
</tr>
<tr>
<td>( (2^{-3}, 170) )</td>
<td>( 1.033 \times 10^{-9} )</td>
<td>16.690</td>
</tr>
<tr>
<td>( (2^{-3}, 130) )</td>
<td>( 1.191 \times 10^{-9} )</td>
<td>16.259</td>
</tr>
<tr>
<td>( (2^{-4}, 150) )</td>
<td>( 7.136 \times 10^{-11} )</td>
<td>16.259</td>
</tr>
<tr>
<td>( (2^{-4}, 140) )</td>
<td>( 1.136 \times 10^{-9} )</td>
<td>16.259</td>
</tr>
<tr>
<td>( (2^{-4}, 170) )</td>
<td>( 6.991 \times 10^{-11} )</td>
<td>16.259</td>
</tr>
<tr>
<td>( (2^{-4}, 100) )</td>
<td>( 2.169 \times 10^{-10} )</td>
<td>17.096</td>
</tr>
<tr>
<td>( (2^{-4}, 130) )</td>
<td>( 1.269 \times 10^{-11} )</td>
<td>17.096</td>
</tr>
</tbody>
</table>

Table 2: The order of convergence in the “singular” part when \( h = 2^{-\epsilon} \) and \( \epsilon = 5 \times 10^{-13} \).

<table>
<thead>
<tr>
<th>( (2^{-\epsilon}, n) )</th>
<th>( | \xi^n_h |_{C^{3,1}} )</th>
<th>( \mathcal{R}^{C_{3,1}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (2^{-4}, 100) )</td>
<td>( 1.931 \times 10^{-8} )</td>
<td>16.078</td>
</tr>
<tr>
<td>( (2^{-3}, 150) )</td>
<td>( 1.182 \times 10^{-9} )</td>
<td>16.789</td>
</tr>
<tr>
<td>( (2^{-3}, 130) )</td>
<td>( 1.294 \times 10^{-9} )</td>
<td>16.789</td>
</tr>
<tr>
<td>( (2^{-4}, 150) )</td>
<td>( 7.078 \times 10^{-11} )</td>
<td>17.967</td>
</tr>
<tr>
<td>( (2^{-4}, 140) )</td>
<td>( 1.312 \times 10^{-9} )</td>
<td>17.967</td>
</tr>
<tr>
<td>( (2^{-4}, 170) )</td>
<td>( 7.304 \times 10^{-11} )</td>
<td>17.967</td>
</tr>
<tr>
<td>( (2^{-4}, 100) )</td>
<td>( 2.389 \times 10^{-10} )</td>
<td>18.164</td>
</tr>
<tr>
<td>( (2^{-4}, 130) )</td>
<td>( 1.315 \times 10^{-11} )</td>
<td>18.164</td>
</tr>
</tbody>
</table>

Table 3: The minimum errors of the solution over the pairs \( (h^{-1}, n) \) in maximum norm when \( \epsilon = 5 \times 10^{-13} \).

<table>
<thead>
<tr>
<th>( (h^{-1}, n) )</th>
<th>( | \xi^n_h |_{C^{0,1}} )</th>
<th>( | \xi^n_h |_{C^{3,1}} )</th>
<th>Iteration</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (16, 70) )</td>
<td>( 1.139 \times 10^{-8} )</td>
<td>( 1.572 \times 10^{-8} )</td>
<td>22</td>
</tr>
<tr>
<td>( (32, 170) )</td>
<td>( 1.033 \times 10^{-9} )</td>
<td>( 1.184 \times 10^{-9} )</td>
<td>23</td>
</tr>
<tr>
<td>( (64, 170) )</td>
<td>( 6.990 \times 10^{-11} )</td>
<td>( 7.304 \times 10^{-11} )</td>
<td>24</td>
</tr>
<tr>
<td>( (128, 200) )</td>
<td>( 8.628 \times 10^{-12} )</td>
<td>( 8.833 \times 10^{-12} )</td>
<td>25</td>
</tr>
</tbody>
</table>

Let \( T_j^+ = T_j(r_j^+) \) be the sector, where \( r_j^* = (r_{j2} + r_{j3})/2, j \in E \), and let \( u_j^*(r_{j2}, \theta_j^*) \), \( 1 \leq q \leq m(j), j \in E \), be the solution values of the system (26) on \( V_j^h \) (at the quadrature nodes). The function

\[
U_h^e(r_j, \theta_j) = Q_j(r_j, \theta_j) + \sum_{q=1}^{n(j)} R_j(r_j, \theta_j, \theta_q^j)(u_j^*(r_{j2}, \theta_q^j) - Q_j^e),
\]

defined on \( T_j^+ \), is called an approximate solution of the problem (1) on the closed block \( \overline{T_j}^+ \), \( j \in E \).

**Definition 5.** The system (26) and (27) is called the system of block-grid equations.

**Theorem 6.** There is a natural number \( n_0 \), such that for all \( n \geq n_0 \) and for any \( \epsilon > 0 \) the system (26) has a unique solution.

**Proof.** From the estimation (2.29) in [15] follows the existence of the positive constants \( n_0 \) and \( \sigma \), such that for all \( n \geq n_0 \)

\[
\max_{(r_j, \theta_j) \in T_j^+} \beta_j \sum_{q=1}^{n(j)} R_j(r_j, \theta_j, \theta_q^j) \leq \sigma < 1.
\]

The proof is obtained on the basis of principle of maximum by taking into account (14), (19), (20), and (28).

**Theorem 7.** There exists a natural number \( n_0 \), such that for all \( n \geq n_0 \)

\[
\max_{\mathcal{C}_e} \{ |u_n^e - u| \leq c (h^4 + \epsilon),
\]

where \( \kappa > 0 \) is a fixed number, and for any \( \epsilon > 0 \) the following inequalities are valid:

\[
\begin{align*}
&\text{for integer } 1/\alpha, \text{ when } p \geq 1/\alpha, \\
&\left| \frac{\partial^p}{\partial x^p \partial y^q} (U_h^e(r_j, \theta_j) - u(r_j, \theta_j)) \right| \leq c_p (h^4 + \epsilon) \quad \text{on } T_j^3, \\
&\text{for any } 1/\alpha, \text{ if } 0 \leq p < 1/\alpha, \\
&\left| \frac{\partial^p}{\partial x^p \partial y^q} (U_h^e(r_j, \theta_j) - u(r_j, \theta_j)) \right| \leq c_p (h^4 + \epsilon) \quad \text{on } T_j^3 \setminus A_j,
\end{align*}
\]

for noninteger \( 1/\alpha, \) when \( p > 1/\alpha \). Everywhere \( 0 \leq q \leq p, u \) is the exact solution of the problem (1) and \( U_h^e(r_j, \theta_j) \) is defined by formula (27).

**Proof.** Let

\[
\xi_h^e = u_h^e - u,
\]

where \( u_h^e \) is a solution of system (26) and \( u \) is the trace on \( \overline{G_{T_j}^{h,n}} \) of the solution of (1). On the basis of (1), (26), and (34) the error \( \xi_h^e \) satisfies the system of difference equations

\[
\begin{align*}
\xi_h^e &= B_h^e + r_h^e \quad \text{on } \Pi_h^h, \\
\xi_h^e &= 0 \quad \text{on } \eta_{k+1}^h, \\
\xi_h^e(r_j, \theta_j) &= \beta_j \sum_{q=1}^{n(j)} \xi_h^e(r_{j2}, \theta_q^j) R_j(r_j, \theta_j, \theta_q^j) + r_h^2, \quad (r_j, \theta_j) \in t_{h}^j, \\
\xi_h^e &= S_h^e + \xi_h^e \quad \text{on } \omega_{k+1}^h,
\end{align*}
\]
Furthermore, the function $\zeta^e(r_j^*, \theta_j)$ continuous on $T^*_j$ is a solution of the following Dirichlet problem:

$$\Delta \zeta^e = 0 \quad \text{on } T^*_j^*, \quad \zeta^e = 0 \quad \text{on } \gamma_m \cap T^*_j^*, \quad m = j - 1, j, \quad (41)$$

Since $T^*_j \subset T^*_j^*$, on the basis of (39) and (40), from Lemma 6.12 in [16], inequalities (31)–(33) of Theorem 7 follow. \(\square\)

## 5. Stress Intensity Factor

Let, in the condition $\varphi_j \in C^{4\lambda}(\gamma_j)$, the exponent $\lambda$ be such that

$$\{\alpha_j \cdot (4 + \lambda)\} \neq 0, \quad \{2\alpha_j \cdot (4 + \lambda)\} \neq 0, \quad (42)$$

where $\{\cdot\}$ is the symbol of fractional part. These conditions for the given $\alpha_j$ can be fulfilled by decreasing $\lambda$.

On the basis of Section 2 of [11], a solution of the problem (1) can be represented in $\hat{T}^*_j$, $j \in E$, as follows:

$$u(x_j, y_j) = \bar{u}(x_j, y_j) + \sum_{k=0}^{n_{\alpha_j}} \mu_k^{(j)} \ln \left| \frac{z^k \ln z}{\alpha_j} \right| + \sum_{k=1}^{n_{\alpha_j}} \frac{\partial_{\alpha_j} \theta_j}{\alpha_j}, \quad (43)$$

where $n_{\alpha_j} = \left[\alpha_j(4+\lambda)\right]$, $[\cdot]$ is the integer part, $z = x_j + iy_j$, $\mu_k^{(j)}$ and $\theta_j^{(k)}$ are some numbers, and $\bar{u}(x_j, y_j) \in C^{4\lambda}(\gamma_j^*)$ is the harmonic on $T^*_j$. By taking $\theta_j = \alpha_j \pi/2$, from the formula (43), it follows that the coefficient $r_1^{(j)}$ which is called the stress intensity factor can be represented as

$$r_1^{(j)} = \lim_{r_j \to 0} \frac{1}{r_j^{1/\alpha_j}} \left| \frac{u(x_j, y_j) - \bar{u}(x_j, y_j)}{z^{k_{\alpha_j}}} \right| \quad (44)$$
Table 6: The stress intensity factor $\tau_{1,n}$ for $n = 70, 170, 200$ when $\varepsilon = 5 \times 10^{-13}$.

<table>
<thead>
<tr>
<th>$h^{-1}$</th>
<th>$\tau_{1,70}$</th>
<th>$\tau_{1,170}$</th>
<th>$\tau_{1,200}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>1.000000014856688</td>
<td>1.00000001780415</td>
<td>1.000000017197438</td>
</tr>
<tr>
<td>32</td>
<td>1.000000005800844</td>
<td>1.000000001230267</td>
<td>1.000000001236709</td>
</tr>
<tr>
<td>64</td>
<td>1.000000004138169</td>
<td>1.00000000073107</td>
<td>1.00000000079938</td>
</tr>
<tr>
<td>128</td>
<td>1.000000004053153</td>
<td>1.00000000003531</td>
<td>1.00000000003267</td>
</tr>
</tbody>
</table>

From formula (44) it follows that $\tau_{1}^{(j)}$ can be approximated by

$$
\tau_{1,n}^{(j)} = \lim_{r_{j} \to 0} \frac{1}{r_{j}} \left( \frac{\sigma_{\rho} \rho_{j0} \left( r_{j}^{\alpha} \right) dt}{t^{2}} + \frac{1}{\pi} \int_{0}^{\alpha_{j}} \frac{\sigma_{\rho j} \left( r_{j}^{\alpha} \right) dt}{t^{2}} \right) - \left( \phi_{j} \left( s_{j} \right) + \left( \phi_{j-1} \left( s_{j} \right) - \phi_{j} \left( s_{j} \right) \right) \frac{\theta_{j}}{\alpha_{j}} \right). \tag{45}
$$

Using formula (3), (4), and (27) from (45) for the stress intensity factor (see [17]), we obtain the next formula:

$$
\tau_{1,n}^{(j)} = \frac{1}{\pi} \int_{0}^{\sigma_{\rho} \rho_{j0} \left( r_{j}^{\alpha} \right) dt}{t^{2}} + \frac{1}{\pi} \int_{0}^{\sigma_{\rho j} \left( r_{j}^{\alpha} \right) dt}{t^{2}} \right) + \frac{2}{n(j)} \sum_{q=1}^{n(j)} I_{r_{j} \theta_{j}^{q}} \left( r_{j}^{\alpha} \theta_{j}^{q} \right) \sin \frac{1}{\alpha_{j}} \theta_{j}^{q}. \tag{46}
$$

This formula is obtained for the second-order BGM in [8].

6. Numerical Results

Let $G$ be L-shaped and defined as follows:

$$
G = \{ (x, y) : -1 < x < 1, -1 < y < 1 \} \setminus \Omega, \tag{47}
$$

where $\Omega = \{ (x, y) : 0 \leq x \leq 1, -1 \leq y \leq 0 \}$ and $\gamma$ is the boundary of $G$.

We consider the following problem:

$$
\Delta u = 0 \quad \text{in} \ G, \quad u = \nu(r, \theta) \quad \text{on} \ \gamma, \tag{48}
$$

where

$$
\nu(r, \theta) = r^{2/3} \sin \left( \frac{2}{3} \theta \right) + 0.0051 r^{16/3} \cos \left( \frac{16}{3} \theta \right) \tag{49}
$$

is the exact solution of this problem.

We choose a "singular" part of $G$ as

$$
G^S = \{(x, y) : -0.5 < x < 0.5, -0.5 < y < 0.5 \} \setminus \Omega_1, \tag{50}
$$

where $\Omega_1 = \{ (x, y) : 0 \leq x \leq 0.5, -0.5 \leq y \leq 0 \}$. Then $G^{NS} = G \setminus G^S$ is a "nonsingular" part of $G$.

The given domain $G$ is covered by four overlapping rectangles $\Pi_k, k = 1, \ldots, 4$, and by the block sector $T_1$. 
Figure 5: Maximum error depending on the number of quadrature nodes $n$.

Figure 6: The approximate solution $U^\varepsilon_h$ and the exact solution $u$ in the "singular" part for $\varepsilon = 5 \times 10^{-13}$. 
For the boundary of \( G^S \) on \( G \) is the polygonal line \( t_1 = abcde \). The radius \( r_{12} \) of sector \( T_{12}^r \) is taken as 0.93. According to (49), the function \( Q(r, \theta) \) in (4) is

\[
Q(r, \theta) = \frac{0.0051}{\pi} \int_0^1 \frac{\tilde{y}^2 dt}{(t-\tilde{y})^2 + \tilde{y}^2} + \frac{0.0051}{\pi} \int_0^1 \frac{\tilde{y}^2 dt}{(t-\tilde{y})^2 + \tilde{y}^2}
\]

(51)

where \( \tilde{x} = r^{2/3} \cos(2\theta/3) \) and \( \tilde{y} = r^{2/3} \sin(2\theta/3) \). Since we have only one singular point, we omit subindices in (51). We calculate the values \( Q^e(r_{12}, \theta^l) \) and \( Q^e(r, \theta) \) on the grids \( t_{12}^l \), with an accuracy of \( \epsilon \) using the quadrature formulae proposed in [10].

On the basis of (46) and (51), for the stress intensity factor, we have

\[
\tau_{1n}^e = \frac{0.0102}{7\pi} + \frac{2}{n(0.93)^{2/3}} \sum_{q=1}^n \left( u_h^e(0.93, \theta_j^l) - Q_{ij}^e \right) \sin \frac{2}{3} \theta_j^l.
\]

(52)

Taking the zero approximation \( u_h^{e(0)} = 0 \), the results of realization of the Schwarz iteration (see [2]) for the solution of the problem (48) are given in Tables 1, 2, 3, and 4. Tables
1 and 2 represent the order of convergence. Table 6 shows a highly accurate approximation of the stress intensity factor by the proposed fourth order BGM

\begin{equation}
R_e^{G_2} = \max_{G_2^{(s)}} \left| U_{2,\nu}^e - u \right| \max_{G_2^{(s)}} \left| U_{2,\nu}^{e(\pm 1)} - u \right|
\end{equation}

in the “nonsingular” and the order of convergence

\begin{equation}
R_e^{G_1} = \max_{G_1^{(s)}} \left| U_{2,\nu}^e - u \right| \max_{G_1^{(s)}} \left| U_{2,\nu}^{e(\pm 1)} - u \right|
\end{equation}

in the “singular” parts of G, respectively, for \( \epsilon = 5 \times 10^{-13} \), where \( \varphi \) is a positive integer. In Table 3, the minimal values over the pairs \((h^{-1}, n)\) of the errors in maximum norm, of the approximate solution when \( \epsilon = 5 \times 10^{-13} \), are presented. The similar values of errors for the first-order derivatives are presented in Table 4, when \( \partial Q/\partial x \) and \( \partial Q/\partial y \) are approximated by fourth-order central difference formula on \( G^2 \) for \( r \geq 0.2 \). For \( r < 0.2 \), the order of errors decreases down to \( 10^{-6} \), which are presented in Table 5. This happens because the integrands in (51) are not sufficiently smooth for fourth-order differentiation formula. The order of accuracy of the derivatives for \( r < 0.2 \) can be increased if we use similar quadrature rules, which we used for the integrals in (51) for the derivatives of integrands also.

Figures 3 and 4 show the dependence on \( \epsilon \) for different mesh steps \( h \). Figure 5 demonstrates the convergence of the BGM with respect to the number of quadrature nodes for different mesh steps \( h \). The approximate solution and the exact solution in the ”singular” part are given in Figure 6, to illustrate the accuracy of the BGM. The error of the block-grid solution, when the function \( Q(r, \theta) \) in (51) is calculated with an accuracy of \( \epsilon = 5 \times 10^{-13} \), is presented in Figure 7. Figures 8 and 9 show the singular behaviour of the first-order partial derivatives in the “singular” part. The ratios \( R_e^{G_2} \) and \( R_e^{G_2^{(s)}} \), when \( \varphi = 5 \) with respect to different \( n \) values for \( h^{-1} = 64 \) and for a fixed value of \( n \) of \( h^{-1} = 32 \), are illustrated in Figures 10 and 11, respectively. These ratios show that the order of the convergence in both the “singular” and the “nonsingular” parts is asymptotically equal to 16 when \( n \) is kept fixed for \( h^{-1} = 32 \), and it is selected as large as possible \((n > 100)\) for \( h^{-1} = 64 \).

7. Conclusions

In the block-grid method (BGM) for solving Laplace's equation, the restriction on the boundary functions to be algebraic polynomials on the sides of the polygon causing the singular vertices is removed. This condition is replaced by the functions from the Hölder classes \( C^{4,\lambda} \), \( 0 < \lambda < 1 \). In the integral representations around singular vertices (on the “singular” part), which are combined with the 9-point finite difference equations on the “nonsingular” part of the polygon, the boundary conditions are taken into account with the help of integrals of Poisson type for a half-plane. To connect the subsystems, a homogeneous fourth-order gluing operator is used. It is proved that the final uniform error is of order \( O(h^4 + \epsilon) \), where \( \epsilon \) is the error of the approximation of the mentioned integrals and \( h \) is the mesh step. For the \( p \)-order derivatives \((p = 0, 1, \ldots)\) of the difference between the approximate and the exact solutions, in each “singular” part \( O(h^{1/4}r_j^{(1/4) - p}) \) order is obtained. The method is illustrated in solving the problem in L-shaped polygon with the corner singularity. Dependence of the approximate solution and its errors on \( \epsilon, h \) and the number of quadrature nodes \( n \) are demonstrated. Furthermore, by the constructed approximate solution on the “singular” part of the polygon, a highly accurate formula for the stress intensity factor is given.

From the error estimation formula (33) of Theorem 7 it follows that the error of the approximate solution on the block sectors decreases as \( r_j^{(1/4)}(h^4 + \epsilon) \), which gives an additional accuracy of the BGM near the singular points.

The method and results of this paper are valid for multiply connected polygons.

References

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