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Research Article

Certain Subclasses of Multivalent Analytic Functions

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Two new subclasses $H_{p,k}(\lambda, A, B)$ and $Q_{p,k}(\lambda, A, B)$ of multivalent analytic functions are introduced. Distortion inequalities and inclusion relation for $H_{p,k}(\lambda, A, B)$ and $Q_{p,k}(\lambda, A, B)$ are obtained. Some results of the partial sums of functions in these classes are also given.

1. Introduction

Throughout this paper, we assume that

$$N = \{1, 2, 3, \ldots\}, \quad k \in N \setminus \{1\}, \quad -1 \le B < 0, \qquad B < A \le 1, \quad 0 \le \lambda \le 1. \tag{1}$$

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=2p}^{\infty} a_n z^n \quad (p \in N), \tag{2}$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$.

For functions f and g analytic in U, we say that f is subordinate to g in U and write $f(z) \prec g(z)$ $(z \in U)$, if there exists a Schwarz function $w(z)$ in U such that

$$|w(z)| \le |z|, \quad f(z) = g(w(z)) \quad (z \in U). \tag{3}$$

Let

$$f_j(z) = z^p + \sum_{n=2p}^{\infty} a_{n,j} z^n \in A(p) \quad (j = 1, 2). \tag{4}$$

Then the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^p + \sum_{n=2p}^{\infty} a_{n,1} a_{n,2} z^n = (f_2 * f_1)(z). \tag{5}$$

The following lemma will be required in our investigation.

Lemma 1. *Let $f \in A(p)$ defined by (2) satisfy*

$$\sum_{n=2p}^{\infty} \left[n(1-B) - p\lambda(1-A)\delta_{n,p,k} \right] |a_n| \le p(A-B). \tag{6}$$

Then

$$\frac{zf'(z)}{(1-\lambda)z^p + \lambda f_{p,k}(z)} \prec p\frac{1+Az}{1+Bz} \quad (z \in U), \tag{7}$$

where

$$f_{p,k}(z) = \frac{1}{k}\sum_{j=0}^{k-1} \varepsilon_k^{-jp} f\left(\varepsilon_k^j z\right), \quad \varepsilon_k = \exp\left(\frac{2\pi i}{k}\right), \tag{8}$$

$$\delta_{n,p,k} = \begin{cases} 0 & \left(\dfrac{n-p}{k} \notin N\right), \\ 1 & \left(\dfrac{n-p}{k} \in N\right). \end{cases} \tag{9}$$

Proof. For $f \in A(p)$ defined by (2), the function $f_{p,k}(z)$ in (8) can be expressed as

$$f_{p,k}(z) = z^p + \sum_{n=2p}^{\infty} \delta_{n,p,k} a_n z^n \tag{10}$$

with

$$\delta_{n,p,k} = \frac{1}{k}\sum_{j=0}^{k-1}\varepsilon_k^{j(n-p)} = \begin{cases} 0 & \left(\dfrac{n-p}{k} \notin N\right), \\ 1 & \left(\dfrac{n-p}{k} \in N\right). \end{cases} \tag{11}$$

In view of (1) and (9), we see that

$$pA\lambda\delta_{n,p,k} - nB \geq -B\left(n - p\lambda\delta_{n,p,k}\right) \geq 0 \quad (n \geq 2p). \tag{12}$$

Let inequality (6) hold. Then from (10) and (12) we deduce that

$$\begin{aligned} &\left|\frac{zf'(z)/\left((1-\lambda)z^p + \lambda f_{p,k}(z)\right) - p}{pA - Bzf'(z)/\left((1-\lambda)z^p + \lambda f_{p,k}(z)\right)}\right| \\ &\quad = \left|\frac{\sum_{n=2p}^{\infty}\left(n - p\lambda\delta_{n,p,k}\right)a_n z^{n-p}}{p(A-B) + \sum_{n=2p}^{\infty}\left(pA\lambda\delta_{n,p,k} - nB\right)a_n z^{n-p}}\right| \\ &\quad \leq \frac{\sum_{n=2p}^{\infty}\left(n - p\lambda\delta_{n,p,k}\right)|a_n|}{p(A-B) - \sum_{n=2p}^{\infty}\left(pA\lambda\delta_{n,p,k} - nB\right)|a_n|} \\ &\quad \leq 1 \quad (|z| = 1). \end{aligned} \tag{13}$$

Hence, by the maximum modulus theorem, we arrive at (7). □

We now consider the following two subclasses of $A(p)$.

Definition 2. A function $f \in A(p)$ defined by (2) is said to be in the class $H_{p,k}(\lambda, A, B)$ if and only if it satisfies the coefficient inequality (6).

Definition 3. A function $f \in A(p)$ defined by (2) is said to be in the class $Q_{p,k}(\lambda, A, B)$ if and only if it satisfies

$$\sum_{n=2p}^{\infty} n\left[n(1-B) - p\lambda(1-A)\delta_{n,p,k}\right]|a_n| \leq p^2(A-B). \tag{14}$$

It is obvious from Definitions 2 and 3 that

$$f(z) \in Q_{p,k}(\lambda, A, B) \text{ iff } \frac{zf'(z)}{p} \in H_{p,k}(\lambda, A, B). \tag{15}$$

If we write

$$\begin{aligned} \alpha_n &= \frac{n(1-B) - p\lambda(1-A)\delta_{n,p,k}}{p(A-B)}, \\ \beta_n &= \frac{n}{p}\alpha_n \quad (n \geq 2p), \end{aligned} \tag{16}$$

then it is easy to verify that

$$\begin{aligned} \frac{\partial\beta_n}{\partial\lambda} &= \frac{n}{p}\frac{\partial\alpha_n}{\partial\lambda} \leq 0, \quad \frac{\partial\beta_n}{\partial A} = \frac{n}{p}\frac{\partial\alpha_n}{\partial A} < 0, \\ \frac{\partial\beta_n}{\partial B} &= \frac{n}{p}\frac{\partial\alpha_n}{\partial B} \geq 0. \end{aligned} \tag{17}$$

Thus we obtain the following inclusion relations:

$$\begin{aligned} Q_{p,k}(\lambda, A, B) \subset H_{p,k}(\lambda, A, B) \subseteq H_{p,k}(1, 1, -1), \\ Q_{p,k}(\lambda, A, B) \subseteq Q_{p,k}(1, 1, -1). \end{aligned} \tag{18}$$

Therefore, by Lemma 1, we see that each function in the classes $H_{p,k}(\lambda, A, B)$ and $Q_{p,k}(\lambda, A, B)$ is starlike with respect to k-symmetric points. Analytic functions which are starlike with respect to symmetric points and related functions have been extensively investigated in [1–6]. Recently, several authors have obtained many important properties and characteristics of multivalent analytic functions (see, e.g., [7–11]).

The main object of this paper is to present some distortion inequalities of functions in the classes $H_{p,k}(\lambda, A, B)$ and $Q_{p,k}(\lambda, A, B)$ which we have introduced here. In particular some results of inclusion relation and convolution of functions in these classes are also given. Further we derive several interesting results of the partial sums of functions in these classes.

2. Main Results

Theorem 4. *Let $p/k \notin N$ and suppose that either*

(a) $1 - B \geq p(1-A)$ *and* $0 \leq \lambda \leq 1$, *or*

(b) $1 - B < p(1-A)$ *and* $0 \leq \lambda \leq (1-B)/p(1-A)$.

(i) *If $f \in H_{p,k}(\lambda, A, B)$, then, for $z \in U$,*

$$|z|^p - \frac{A-B}{2(1-B)}|z|^{2p} \leq |f(z)| \leq |z|^p + \frac{A-B}{2(1-B)}|z|^{2p}. \tag{19}$$

The bounds in (19) *are best possible for the function f defined by*

$$f(z) = z^p + \frac{A-B}{2(1-B)}z^{2p}. \tag{20}$$

(ii) *If $f \in Q_{p,k}(\lambda, A, B)$, then, for $z \in U$,*

$$\begin{aligned} &p|z|^{p-1} - \frac{p(A-B)}{2(1-B)}|z|^{2p-1} \\ &\quad \leq |f'(z)| \leq p|z|^{p-1} + \frac{p(A-B)}{2(1-B)}|z|^{2p-1}. \end{aligned} \tag{21}$$

The bounds in (21) *are best possible for the function f defined by*

$$f(z) = z^p + \frac{A-B}{4(1-B)}z^{2p}. \tag{22}$$

Proof. Let $p/k \notin N$. For $n \geq 2p$ $(n \in N)$ and $(n-p)/k \notin N$, we have $\delta_{n,p,k} = \delta_{2p,p,k} = 0$, and so

$$\frac{n(1-B) - p\lambda(1-A)\delta_{n,p,k}}{p(A-B)} \geq \frac{2(1-B)}{A-B}. \tag{23}$$

For $n \geq 2p$ $(n \in N)$ and $(n-p)/k \in N$, we have $\delta_{n,p,k} = 1$ and

$$\frac{n(1-B) - p\lambda(1-A)\delta_{n,p,k}}{p(A-B)} \geq \frac{(2p+1)(1-B) - p\lambda(1-A)}{p(A-B)}. \tag{24}$$

If either (a) or (b) is satisfied, then

$$\frac{(2p+1)(1-B) - p\lambda(1-A)}{p(A-B)} \geq \frac{2(1-B)}{A-B} > 1. \tag{25}$$

(i) If

$$f(z) = z^p + \sum_{n=2p}^{\infty} a_n z^n \in H_{p,k}(\lambda, A, B), \tag{26}$$

then it follows from (23) to (25) that

$$\frac{2(1-B)}{A-B} \sum_{n=2p}^{\infty} |a_n| \leq 1. \tag{27}$$

Hence we have

$$\begin{aligned} |f(z)| &\leq |z|^p + |z|^{2p} \sum_{n=2p}^{\infty} |a_n| \\ &\leq |z|^p + \frac{A-B}{2(1-B)} |z|^{2p}, \end{aligned} \tag{28}$$

$$\begin{aligned} |f(z)| &\geq |z|^p - |z|^{2p} \sum_{n=2p}^{\infty} |a_n| \\ &\geq |z|^p - \frac{A-B}{2(1-B)} |z|^{2p} > 0 \end{aligned} \tag{29}$$

for $z \in U$.

(ii) If

$$f(z) = z^p + \sum_{n=2p}^{\infty} a_n z^n \in Q_{p,k}(\lambda, A, B), \tag{30}$$

then (23) to (25) yield

$$\frac{2(1-B)}{p(A-B)} \sum_{n=2p}^{\infty} n|a_n| \leq 1. \tag{31}$$

This leads to (21). The proof of the theorem is complete. □

Theorem 5. *Let*

$$\frac{p}{k} \notin N, \quad 1-B < p(1-A), \qquad \frac{1-B}{p(1-A)} < \lambda \leq 1. \tag{32}$$

(i) *If* $f(z) = z^p + a_{2p} z^{2p} + \cdots \in H_{p,k}(\lambda, A, B)$, *then, for* $z \in U$,

$$\begin{aligned} |f(z)| &\leq |z|^p + |a_{2p}| |z|^{2p} \\ &\quad + \frac{p(A-B) - 2p(1-B)|a_{2p}|}{(2p+1)(1-B) - p\lambda(1-A)} |z|^{2p+1}, \end{aligned} \tag{33}$$

$$\begin{aligned} |f(z)| &\geq |z|^p - |a_{2p}| |z|^{2p} \\ &\quad - \frac{p(A-B) - 2p(1-B)|a_{2p}|}{(2p+1)(1-B) - p\lambda(1-A)} |z|^{2p+1}. \end{aligned} \tag{34}$$

Equalities in (33) *are attained, for example, by*

$$f(z) = z^p + \frac{p(A-B)}{(2p+1)(1-B) - p\lambda(1-A)} z^{2p+1}. \tag{35}$$

(ii) *If* $f(z) = z^p + a_{2p} z^{2p} + \cdots \in Q_{p,k}(\lambda, A, B)$, *then, for* $z \in U$,

$$\begin{aligned} |f'(z)| &\leq p|z|^{p-1} + 2p|a_{2p}| |z|^{2p-1} \\ &\quad + \frac{p^2\left[(A-B) - 4(1-B)|a_{2p}|\right]}{(2p+1)(1-B) - p\lambda(1-A)} |z|^{2p}, \\ |f'(z)| &\geq p|z|^{p-1} - 2p|a_{2p}| |z|^{2p-1} \\ &\quad - \frac{p^2\left[(A-B) - 4(1-B)|a_{2p}|\right]}{(2p+1)(1-B) - p\lambda(1-A)} |z|^{2p}. \end{aligned} \tag{36}$$

Equalities in (36) *are attained, for example, by*

$$f(z) = z^p + \frac{p^2(A-B)}{(2p+1)\left[(2p+1)(1-B) - p\lambda(1-A)\right]} \times z^{2p+1}. \tag{37}$$

Proof. Note that $(1-B)/p(1-A) < \lambda \leq 1$ implies that

$$\frac{2(1-B)}{A-B} > \frac{(2p+1)(1-B) - p\lambda(1-A)}{p(A-B)}. \tag{38}$$

(i) For $f(z) = z^p + a_{2p} z^{2p} + \cdots \in H_{p,k}(\lambda, A, B)$, it follows from (23), (24), and (38) that

$$\frac{2(1-B)}{A-B} |a_{2p}| + \frac{(2p+1)(1-B) - p\lambda(1-A)}{p(A-B)} \sum_{n=2p+1}^{\infty} |a_n| \leq 1. \tag{39}$$

From this we can get (33).

(ii) For $f(z) = z^p + a_{2p}z^{2p} + \cdots \in Q_{p,k}(\lambda, A, B)$, from (23), (24), and (38) we deduce that

$$\frac{4(1-B)}{A-B}\left|a_{2p}\right| + \frac{(2p+1)(1-B) - p\lambda(1-A)}{p^2(A-B)}\sum_{n=2p+1}^{\infty} n\left|a_n\right| \leq 1. \tag{40}$$

Hence we have (36). The proof of the theorem is complete. □

Theorem 6. *Let $p/k \in N$.*

(i) *If $f \in H_{p,k}(\lambda, A, B)$, then, for $z \in U$,*

$$\begin{aligned} &|z|^p - \frac{A-B}{2(1-B)-\lambda(1-A)}|z|^{2p} \\ &\quad \leq \left|f(z)\right| \leq |z|^p + \frac{A-B}{2(1-B)-\lambda(1-A)}|z|^{2p}. \end{aligned} \tag{41}$$

The bounds in (41) *are sharp for the function f defined by*

$$f(z) = z^p + \frac{A-B}{2(1-B)-\lambda(1-A)}z^{2p}. \tag{42}$$

(ii) *If $f \in Q_{p,k}(\lambda, A, B)$, then, for $z \in U$,*

$$\begin{aligned} &p|z|^{p-1} - \frac{p(A-B)}{2(1-B)-\lambda(1-A)}|z|^{2p-1} \\ &\quad \leq \left|f'(z)\right| \leq p|z|^{p-1} + \frac{p(A-B)}{2(1-B)-\lambda(1-A)}|z|^{2p-1}. \end{aligned} \tag{43}$$

The bounds in (43) *are sharp for the function f defined by*

$$f(z) = z^p + \frac{A-B}{4(1-B)-2\lambda(1-A)}z^{2p}. \tag{44}$$

Proof. Let $p/k \in N$. For $n \geq 2p$ $(n \in N)$ and $(n-p)/k \in N$, we have $n = 2p + k(l-1)$ $(l \in N)$, $\delta_{n,p,k} = \delta_{2p,p,k} = 1$, and

$$\frac{n(1-B) - p\lambda(1-A)\delta_{n,p,k}}{p(A-B)} \geq \frac{2(1-B)-\lambda(1-A)}{A-B}. \tag{45}$$

For $n \geq 2p$ $(n \in N)$ and $(n-p)/k \notin N$, we have $\delta_{n,p,k} = \delta_{2p+1,p,k} = 0$, and so

$$\begin{aligned} \frac{n(1-B) - p\lambda(1-A)\delta_{n,p,k}}{p(A-B)} &\geq \frac{(2p+1)(1-B)}{p(A-B)} \\ &\geq \frac{2(1-B)-\lambda(1-A)}{A-B}. \end{aligned} \tag{46}$$

(i) If

$$f(z) = z^p + \sum_{n=2p}^{\infty} a_n z^n \in H_{p,k}(\lambda, A, B), \tag{47}$$

then it follows from (45) and (46) that

$$\frac{2(1-B)-\lambda(1-A)}{A-B}\sum_{n=2p}^{\infty}\left|a_n\right| \leq 1. \tag{48}$$

Hence we have

$$\begin{aligned} \left|f(z)\right| &\leq |z|^p + |z|^{2p}\sum_{n=2p}^{\infty}\left|a_n\right| \\ &\leq |z|^p + \frac{A-B}{2(1-B)-\lambda(1-A)}|z|^{2p}, \\ \left|f(z)\right| &\geq |z|^p - |z|^{2p}\sum_{n=2p}^{\infty}\left|a_n\right| \\ &\geq |z|^p - \frac{A-B}{2(1-B)-\lambda(1-A)}|z|^{2p} \end{aligned} \tag{49}$$

for $z \in U$.

(ii) If

$$f(z) = z^p + \sum_{n=2p}^{\infty} a_n z^n \in Q_{p,k}(\lambda, A, B), \tag{50}$$

then (45) and (46) yield

$$\frac{2(1-B)-\lambda(1-A)}{p(A-B)}\sum_{n=2p}^{\infty} n\left|a_n\right| \leq 1. \tag{51}$$

This leads to (43). Thus we complete the proof. □

Next, we generalize the inclusion relation $Q_{p,k}(\lambda, A, B) \subset H_{p,k}(\lambda, A, B)$ which is mentioned in (18).

Theorem 7. *If $-1 \leq D < 0$, then*

$$Q_{p,k}(\lambda, A, B) \subset H_{p,k}(\lambda, C(D), D), \tag{52}$$

where

$$C(D) = D + \frac{(1-D)(A-B)}{2(1-B)}. \tag{53}$$

Proof. Since $B < A \leq 1$ and $-1 \leq D < 0$, we see that

$$D < C(D) < 1. \tag{54}$$

Let $f \in Q_{p,k}(\lambda, A, B)$. In order to prove that $f \in H_{p,k}(\lambda, C(D), D)$, we need only to find the smallest C $(D < C \leq 1)$ such that

$$\begin{aligned} &\frac{n(1-D) - p\lambda(1-C)\delta_{n,p,k}}{p(C-D)} \\ &\quad \leq \frac{n\left[n(1-B) - p\lambda(1-A)\delta_{n,p,k}\right]}{p^2(A-B)} \end{aligned} \tag{55}$$

for all $n \geq 2p$, that is, that

$$\frac{(1-D)\left(n-p\lambda\delta_{n,p,k}\right)}{p(C-D)}+\lambda\delta_{n,p,k} \leq \frac{n}{p}\left\{\frac{(1-B)\left(n-p\lambda\delta_{n,p,k}\right)}{p(A-B)}+\lambda\delta_{n,p,k}\right\}. \tag{56}$$

For $n \geq 2p$ and $(n-p)/k \in N$, (56) is equivalent to

$$C \geq D+\frac{1-D}{(n-p)\lambda/(n-p\lambda)+n(1-B)/p(A-B)} = \varphi(\lambda,n). \tag{57}$$

Noting (1), a simple calculation shows that $(\partial\varphi(\lambda,x))/\partial x<0$ for all real $x \geq 2p$ and $0 \leq \lambda \leq 1$, and so the function $\varphi(\lambda,n)$ is decreasing in n $(n \geq 2p)$. Therefore

$$\varphi(\lambda,n) \leq \begin{cases} \varphi(\lambda,2p) & \left(\frac{p}{k} \in N\right), \\ \varphi\left(\lambda,k\left(\left[\frac{p}{k}\right]+1\right)+p\right) & \left(\frac{p}{k} \notin N\right). \end{cases} \tag{58}$$

For $n \geq 2p$ and $(n-p)/k \notin N$, (56) becomes

$$C \geq D+\frac{p(1-D)(A-B)}{n(1-B)} = \varphi(0,n),$$
$$\varphi(0,n) \leq \begin{cases} \varphi(0,2p+1) & \left(\frac{p}{k} \in N\right), \\ \varphi(0,2p) & \left(\frac{p}{k} \notin N\right). \end{cases} \tag{59}$$

Consequently, by taking

$$C(D) = \varphi(0,2p) = D+\frac{(1-D)(A-B)}{2(1-B)}, \tag{60}$$

it follows from (55) to (60) that $f \in H_{p,k}(\lambda, C(D), D)$. The proof is complete. □

Remark 8. If we take $D = B$ in Theorem 7, then from (1) we have $C(D) = (A+B)/2 < A$. This shows that

$$Q_{p,k}(\lambda,A,B) \subset H_{p,k}\left(\lambda,\frac{A+B}{2},B\right) \subset H_{p,k}(\lambda,A,B). \tag{61}$$

Theorem 9. *Let $f \in H_{p,k}(\lambda,A,B)$. Then*

$$(f*h_\sigma)(z) \neq 0 \quad (z \in U;\ \sigma \in C,\ |\sigma|=1), \tag{62}$$

where

$$h_\sigma(z) = z^p-\frac{2(1+B\sigma)}{\sigma(A-B)}\cdot\frac{z^{2p}}{1-z}-\frac{1+B\sigma}{p\sigma(A-B)}\cdot\frac{z^{2p+1}}{(1-z)^2} + \frac{\lambda(1+A\sigma)}{\sigma(A-B)}g_{p,k}(z),$$
$$g_{p,k}(z) = \begin{cases} \dfrac{z^{2p}}{1-z^k} & \left(\dfrac{p}{k} \in N\right), \\ \dfrac{z^{k([p/k]+1)+p}}{1-z^k} & \left(\dfrac{p}{k} \notin N\right). \end{cases} \tag{63}$$

Proof. For $f \in H_{p,k}(\lambda,A,B)$, from Lemma 1 we have (7), which is equivalent to

$$\frac{zf'(z)}{(1-\lambda)z^p+\lambda f_{p,k}(z)} \neq p\frac{1+A\sigma}{1+B\sigma} \quad (z \in U;\ \sigma \in C,\ |\sigma|=1,\ 1+B\sigma \neq 0) \tag{64}$$

or to

$$p(1+A\sigma)\left[(1-\lambda)z^p+\lambda f_{p,k}(z)\right]-(1+B\sigma)zf'(z) \neq 0 \quad (z \in U;\ \sigma \in C,\ |\sigma|=1). \tag{65}$$

Obviously

$$z^p = f(z)*z^p,$$
$$\frac{zf'(z)}{p} = f(z)*\left(z^p+\frac{1}{p}\sum_{n=2p}^{\infty}nz^n\right) = f(z)*\left(z^p+\frac{2z^{2p}}{1-z}+\frac{z^{2p+1}}{p(1-z)^2}\right). \tag{66}$$

If we put

$$f_{p,k}(z) = f(z)*\left(z^p+g_{p,k}(z)\right), \tag{67}$$

then, for $p/k \in N$,

$$g_{p,k}(z) = \sum_{n=2p}^{\infty}\delta_{n,p,k}z^n = \sum_{l=0}^{\infty}z^{2p+lk} = \frac{z^{2p}}{1-z^k}, \tag{68}$$

and, for $p/k \notin N$,

$$g_{p,k}(z) = \sum_{l=1}^{\infty}z^{k([p/k]+l)+p} = \frac{z^{k([p/k]+1)+p}}{1-z^k}. \tag{69}$$

Now, making use of (65) to (69), we arrive at

$$f(z)*\left\{p(1+A\sigma)\left[(1-\lambda)z^p+\lambda\left(z^p+g_{p,k}(z)\right)\right] - (1+B\sigma)\left(pz^p+\frac{2pz^{2p}}{1-z}+\frac{z^{2p+1}}{(1-z)^2}\right)\right\} \neq 0 \tag{70}$$

for $z \in U$, $\sigma \in C$, and $|\sigma|=1$. This gives the desired result (62). The proof of the theorem is complete. □

Corollary 10. *Let $f \in Q_{p,k}(\lambda, A, B)$. Then*

$$f(z) * zh'_\sigma(z) \neq 0 \quad (z \in U;\ \sigma \in C,\ |\sigma| = 1), \tag{71}$$

where $h_\sigma(z)$ is the same as in Theorem 9.

Proof. Since $f \in Q_{p,k}(\lambda, A, B)$ if and only if

$$\frac{zf'(z)}{p} \in H_{p,k}(\lambda, A, B), \tag{72}$$

it follows from Theorem 9 that

$$\begin{aligned} f(z) * \frac{zh'_\sigma(z)}{p} &= \frac{zf'(z)}{p} * h_\sigma(z) \neq 0 \quad (z \in U;\ \sigma \in C,\ |\sigma| = 1). \end{aligned} \tag{73}$$

Thus we complete the proof. □

Finally, we derive certain results of the partial sums of functions in the classes $H_{p,k}(\lambda, A, B)$ and $Q_{p,k}(\lambda, A, B)$.

Let $f \in A(p)$ be given by (2) and define the partial sums $s_1(z)$ and $s_m(z)$ by

$$\begin{aligned} s_1(z) &= z^p, \\ s_m(z) &= z^p + \sum_{n=2p}^{2p+m-2} a_n z^n \quad (m \in N \setminus \{1\}). \end{aligned} \tag{74}$$

For simplicity we use the notation α_n $(n \geq 2p)$ defined by (16).

Theorem 11. *Let $f \in H_{p,k}(\lambda, A, B)$ and let either*

(a) $1 - B \geq p(1 - A)$ *and* $0 \leq \lambda \leq 1$, *or*

(b) $1 - B < p(1 - A)$ *and* $0 \leq \lambda \leq (1 - B)/p(1 - A)$.

Then, for $m \in N$, we have

$$\operatorname{Re}\left(\frac{f(z)}{s_m(z)}\right) > 1 - \frac{1}{\alpha_{2p+m-1}} \quad (z \in U), \tag{75}$$

$$\operatorname{Re}\left(\frac{s_m(z)}{f(z)}\right) > \frac{\alpha_{2p+m-1}}{1 + \alpha_{2p+m-1}} \quad (z \in U). \tag{76}$$

The bounds in (75) *and* (76) *are best possible for each m.*

Proof. If either (a) or (b) is satisfied, then, for $n \geq 2p$,

$$\begin{aligned} \alpha_n &= \frac{n(1 - B) - p\lambda(1 - A)\delta_{n,p,k}}{p(A - B)} \geq \frac{1 - B}{A - B} \geq 1, \\ \alpha_{n+1} &= \frac{(n + 1)(1 - B) - p\lambda(1 - A)\delta_{n+1,p,k}}{p(A - B)} \\ &= \alpha_n + \frac{1 - B - p\lambda(1 - A)\left(\delta_{n+1,p,k} - \delta_{n,p,k}\right)}{p(A - B)} \\ &\geq \alpha_n + \frac{1 - B - p\lambda(1 - A)}{p(A - B)} \geq \alpha_n. \end{aligned} \tag{77}$$

Let $f \in H_{p,k}(\lambda, A, B)$. Then it follows from (77) that

$$\begin{aligned} &\sum_{n=2p}^{2p+m-2} |a_n| + \alpha_{2p+m-1} \cdot \sum_{n=2p+m-1}^{\infty} |a_n| \\ &\quad \leq \sum_{n=2p}^{\infty} \alpha_n |a_n| \leq 1 \quad (m \in N \setminus \{1\}). \end{aligned} \tag{78}$$

If we put

$$p_1(z) = 1 + \alpha_{2p+m-1}\left(\frac{f(z)}{s_m(z)} - 1\right) \tag{79}$$

for $z \in U$ and $m \in N \setminus \{1\}$, then $p_1(0) = 1$ and we deduce from (78) that

$$\begin{aligned} &\left|\frac{p_1(z) - 1}{p_1(z) + 1}\right| \\ &\quad = \left|\frac{\alpha_{2p+m-1} \sum_{n=2p+m-1}^{\infty} a_n z^{n-p}}{2\left(1 + \sum_{n=2p}^{2p+m-2} a_n z^{n-p}\right) + \alpha_{2p+m-1} \sum_{n=2p+m-1}^{\infty} a_n z^{n-p}}\right| \\ &\quad \leq \frac{\alpha_{2p+m-1} \sum_{n=2p+m-1}^{\infty} |a_n|}{2 - 2\sum_{n=2p}^{2p+m-2} |a_n| - \alpha_{2p+m-1} \sum_{n=2p+m-1}^{\infty} |a_n|} \\ &\quad \leq 1 \quad (z \in U;\ m \in N \setminus \{1\}). \end{aligned} \tag{80}$$

This implies that $\operatorname{Re}\{p_1(z)\} > 0$ for $z \in U$, and so (75) holds true for $m \in N \setminus \{1\}$.

Similarly, by setting

$$p_2(z) = \left(1 + \alpha_{2p+m-1}\right)\frac{s_m(z)}{f(z)} - \alpha_{2p+m-1}, \tag{81}$$

it follows from (78) that

$$\begin{aligned} &\left|\frac{p_2(z) - 1}{p_2(z) + 1}\right| \\ &\quad = \left|\left(-\left(1 + \alpha_{2p+m-1}\right) \sum_{n=2p+m-1}^{\infty} a_n z^{n-p}\right) \right. \\ &\qquad \times \left(2\left(1 + \sum_{n=2p}^{2p+m-2} a_n z^{n-p}\right)\right. \\ &\qquad \left.\left. + \left(1 - \alpha_{2p+m-1}\right) \sum_{n=2p+m-1}^{\infty} a_n z^{n-p}\right)^{-1}\right| \\ &\quad \leq \frac{\left(1 + \alpha_{2p+m-1}\right) \sum_{n=2p+m-1}^{\infty} |a_n|}{2 - 2\sum_{n=2p}^{2p+m-2} |a_n| - \left(\alpha_{2p+m-1} - 1\right) \sum_{n=2p+m-1}^{\infty} |a_n|} \\ &\quad \leq 1 \quad (z \in U;\ m \in N \setminus \{1\}). \end{aligned} \tag{82}$$

Hence we have (76) for $m \in N \setminus \{1\}$.

For $m = 1$, replacing (78) by

$$\alpha_{2p} \sum_{n=2p}^{\infty} |a_n| \leq \sum_{n=2p}^{\infty} \alpha_n |a_n| \leq 1 \tag{83}$$

and proceeding as the above, we see that (75) and (76) are also true.

Furthermore, taking the function f defined by

$$f(z) = z^p + \frac{z^{2p+m-1}}{\alpha_{2p+m-1}} \in H_{p,k}(\lambda, A, B), \tag{84}$$

we have $s_m(z) = z^p$,

$$\begin{gathered} \operatorname{Re}\left(\frac{f(z)}{s_m(z)}\right) \longrightarrow 1 - \frac{1}{\alpha_{2p+m-1}} \quad \text{as } z \longrightarrow \exp\left(\frac{\pi i}{p+m-1}\right), \\ \operatorname{Re}\left(\frac{s_m(z)}{f(z)}\right) \longrightarrow \frac{\alpha_{2p+m-1}}{1+\alpha_{2p+m-1}} \quad \text{as } z \longrightarrow 1. \end{gathered} \tag{85}$$

Thus the proof of Theorem 11 is completed. □

Remark 12. Replacing $H_{p,k}(\lambda, A, B)$ by $Q_{p,k}(\lambda, A, B)$, it follows from Theorem 11 that inequalities (75) and (76) are true. In Theorem 13 we improve the bounds in (75) and (76) for $f \in Q_{p,k}(\lambda, A, B)$.

Theorem 13. *Let $f \in Q_{p,k}(\lambda, A, B)$ and let either (a) or (b) in Theorem 11 be satisfied. Then, for $m \in N$, one has*

$$\begin{gathered} \operatorname{Re}\left(\frac{f(z)}{s_m(z)}\right) > 1 - \frac{p}{(2p+m-1)\alpha_{2p+m-1}} \quad (z \in U), \\ \operatorname{Re}\left(\frac{s_m(z)}{f(z)}\right) > \frac{(2p+m-1)\alpha_{2p+m-1}}{p+(2p+m-1)\alpha_{2p+m-1}} \quad (z \in U). \end{gathered} \tag{86}$$

The bounds in (86) *are sharp for the function f defined by*

$$f(z) = z^p + \frac{pz^{2p+m-1}}{(2p+m-1)\alpha_{2p+m-1}} \in Q_{p,k}(\lambda, A, B). \tag{87}$$

Proof. In view of the assumptions of Theorem 13, it follows from (77) that

$$\begin{gathered} \sum_{n=2p}^{2p+m-2} |a_n| + \frac{(2p+m-1)\alpha_{2p+m-1}}{p} \sum_{n=2p+m-1}^{\infty} |a_n| \\ \leq \sum_{n=2p}^{\infty} \frac{n}{p} \alpha_n |a_n| \leq 1 \quad (m \in N \setminus \{1\}), \\ \alpha_{2p} \sum_{n=2p}^{\infty} |a_n| \leq \sum_{n=2p}^{\infty} \frac{n}{p} \alpha_n |a_n| \leq 1. \end{gathered} \tag{88}$$

If we put

$$\begin{gathered} p_1(z) = 1 + \frac{(2p+m-1)\alpha_{2p+m-1}}{p}\left(\frac{f(z)}{s_m(z)} - 1\right), \\ p_2(z) = \left(1 + \frac{(2p+m-1)\alpha_{2p+m-1}}{p}\right)\frac{s_m(z)}{f(z)} \\ - \frac{(2p+m-1)\alpha_{2p+m-1}}{p}, \end{gathered} \tag{89}$$

then (88) leads to $\operatorname{Re}(p_j(z)) > 0$ $(z \in U;\ m \in N;\ j = 1, 2)$. Hence we have (86). Sharpness can be verified easily. □

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