Research Article
The Natural Filtration of Finite Dimensional Modular Lie Superalgebras of Special Type

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1. Introduction

Although many structural features of nonmodular Lie superalgebras (see [1–3]) are well understood, there seem to be very few general results on modular Lie superalgebras. The treatment of modular Lie superalgebras necessitates different techniques which are set forth in [4, 5]. In [6], four series of modular graded Lie superalgebras of Cartan type were constructed, which are analogous to the finite dimensional modular Lie algebras of Cartan type [7] or the four series of infinite dimensional Lie superalgebras of Cartan type defined by even differential forms over a field of characteristic zero [8]. Recent works on the modular Lie superalgebras of Cartan type can also be found in [9–13] and references therein.

It is well known that filtration techniques are of great importance in the structure and the classification theories of Lie (super)algebras (see [1–3, 14, 15]). For some classes of modular Lie (super)algebras, the filtrations have been well investigated, for example, the natural filtrations of finite dimensional modular Lie algebras of Cartan type [16, 17] and of finite dimensional simple modular Lie superalgebras W, S, and H of Cartan type [18, 19].

The original motivation for this paper comes from the researches of structures for the finite dimensional modular Lie superalgebras W(n, m) and H(n, m) which were first introduced in [20, 21], respectively. The starting point of our studies is to construct a class of finite dimensional modular Lie superalgebras of special type, which is denoted by S(n, m).

A brief summary of the relevant concepts and notations in the finite dimensional modular Lie superalgebras S(n, m) is presented in Section 2. In Section 3, by using the ad-nilpotent elements of S(n, m), we show that the natural filtration of S(n, m) is invariant under its automorphisms.

2. Preliminaries

Throughout this paper, F denotes an algebraic closed field of characteristic \( p > 2 \), and \( n \) is an integer greater than 3. In addition to the standard notation \( \mathbb{Z} \), we write \( \mathbb{N} \) and \( \mathbb{N}_0 \) to denote the sets of positive integers and nonnegative integers, respectively.

Let \( \Lambda(n) \) be the Grassmann algebra over \( F \) in \( n \) variables \( x_1, x_2, \ldots, x_n \). Set \( \mathbb{B}_k = \{ (i_1, i_2, \ldots, i_k) | 1 \leq i_1 < i_2 < \cdots < i_k \leq n \} \) and \( \mathbb{B}(n) = \bigcup_{k=0}^n \mathbb{B}_k \), where \( \mathbb{B}_0 = \emptyset \). For \( u = (i_1, i_2, \ldots, i_k) \in \mathbb{B}_k \), set \( [u] = k \), \( [u] = \{i_1, i_2, \ldots, i_k\} \) and \( x^u = x_{i_1} x_{i_2} \cdots x_{i_k} \). Then \( x^u \mid u \in \mathbb{B}(n) \) is an \( F \)-basis of \( \Lambda(n) \).

Let \( \Pi \) denote the prime field of \( F \); that is, \( \Pi = \{0, 1, \ldots, p-1\} \). Suppose that the set \( \{z_1, z_2, \ldots, z_m\} \) is an \( \Pi \)-linearly independent finite subset of \( F \). Let \( G = \{ \sum_{i=1}^m \lambda_i z_i | \lambda_i \in \Pi \} \). Then \( G \) is an additive subgroup of \( F \). Let \( \mathbb{F}[y_1, y_2, \ldots, y_m] \) be the truncated polynomial algebra satisfying \( y_i^p = 1 \) for all \( i = 1, 2, \ldots, m \). For every element \( \lambda = \sum_{i=1}^m \lambda_i z_i \in G \), define \( y^\lambda = y_1^{\lambda_1} y_2^{\lambda_2} \cdots y_m^{\lambda_m} \). Then \( y^\lambda y^\eta = y^\lambda y^\eta \) for all \( \lambda, \eta \in G \). Let \( \mathbb{T}(m) \) denote \( \mathbb{F}[y_1, y_2, \ldots, y_m] \). Then \( \mathbb{T}(m) = \{ \sum_{\lambda \in G} d_\lambda y^\lambda \mid d_\lambda \in \mathbb{F} \} \).
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Let \( \mathcal{U} = \Lambda(n) \otimes T(m) \). Then \( \mathcal{U} \) is an associative superalgebra with \( Z_2 \)-gradation induced by the trivial \( Z_2 \)-gradation of \( T(m) \) and the natural \( Z_2 \)-gradation of \( \Lambda(n) \); that is, \( \mathcal{U} = \mathcal{U}_0 \oplus \mathcal{U}_1 \), where \( \mathcal{U}_0 = \Lambda(n_0) \otimes T(m_0) \) and \( \mathcal{U}_1 = \Lambda(n_1) \otimes T(m_1) \).

For \( f \in \Lambda(n) \) and \( \alpha \in T(m) \), we abbreviate \( f \otimes \alpha \) as \( f \alpha \).

Then the elements \( x^\alpha y^\beta \) with \( u \in \mathfrak{B}(n) \) and \( \lambda \in G \) form an \( F \)-basis of \( \mathcal{U} \). It is easy to see that \( \mathcal{U} = \mathcal{U}_0 \oplus \mathcal{U}_1 \), where \( \mathcal{U}_0 = \Lambda(n_0) \otimes T(m_0) \) and \( \mathcal{U}_1 = \Lambda(n_1) \otimes T(m_1) \).

In this paper, \( A = \mathcal{U}_0 \oplus \mathcal{U}_1 \) is a superalgebra (or \( Z_2 \)-graded linear space), let \( \text{Der} A \) be the derivation superalgebra of \( A \) (see [1] or [2] for the definition) and \( h_g(A) = A_0 \cup A_1 \); that is, \( h_g(A) \) is the set of all \( Z_2 \)-homogeneous elements of \( A \). If deg \( x \) occurs in some expression, we regard \( x \) as a \( Z_2 \)-homogeneous element and deg \( x \) as the \( Z_2 \)-degree of \( x \).

We define \( f(\mathcal{U}) = f(U) + h_g(\mathcal{U}) \) for all \( f \in \mathcal{U} \) such that \( f(U) \) is a \( Z_2 \)-homogeneous element and \( h_g(\mathcal{U}) \) is \( Z_2 \)-graded superalgebra. If \( A_0 \) and \( A_1 \) be the linearmapsuchthatfor all \( f \in U \) and \( g \in h_g(\mathcal{U}) \), we set \( f(\mathcal{U}) = f(U) + h_g(\mathcal{U}) \). Since the multiplication of \( \mathcal{U} \) is \( Z_2 \)-graded superalgebra, the following equalities are easy to verify:

\[
D_i (f) = -2D_i (f) D_0, \\
D_{ij} (f) = D_{ij} (f), \\
[D_{ik}, D_{lj}] = -D_{ij} (D_k (f)), \\
[D_{ij}, g] = \sum_{s=1}^{2} (-1)^{\text{deg}D_s} D_s (f_i g_j).
\] (5)

By the definition of linear map \( D_{r_1 r_2} \), the following equalities are easy to verify:

\[
D_i (f) = -2D_i (f) D_0, \\
D_{ij} (f) = D_{ij} (f), \\
[D_{ik}, D_{lj}] = -D_{ij} (D_k (f)), \\
[D_{ij}, g] = \sum_{s=1}^{2} (-1)^{\text{deg}D_s} D_s (f_i g_j).
\] (4)

where \( f, g \in h_g(\mathcal{U}) \); \( i, j \in \mathbb{Y} \); and \( f_i, g_j \) and as in (3).

The equality (5) shows that \( S(n, m) \) is a subalgebra of \( W(n, m) \). Hereafter, \( S(n, m) \) and \( S(n, m) \) will be simply denoted by \( S \) and \( S \), respectively.

Put \( A = \{D_{ij}(x^\alpha y^\beta) \mid i, j \in \mathbb{Y}, \lambda \in G \} \) and \( B = \{D_{ij}(x^\alpha y^\beta) \mid i, j, k \in \mathbb{Y}, \eta \in G \} \).

**Proposition 1.** The Lie superalgebra \( S \) is generated by \( A \cup B \).

**Proof.** Suppose that \( A \cup B \) generate the subalgebra \( Q \) of \( S \). Since \( A \) and \( B \) are subsets of \( S \), it follows that \( Q \subseteq S \).

Next we will consider the reverse inclusion.

It is easy to see that \( D_{ij}(x^\alpha y^\beta) = -y^\beta D_i(x^\alpha y^\beta) \) for all distinct elements \( i, k \) of \( \mathbb{Y} \) and \( \lambda \in G \). Therefore, \( zd(D_{ij}(x^\alpha y^\beta)) = -1 \) and \( S_{n-3} \subseteq Q \).

A direct calculation shows that

\[
\begin{align*}
[D_{ij}(x^\alpha y^\beta), D_{kl}(x_\gamma y_\eta)] &= [-D_j(x^\alpha y^\beta) D_k - D_k(x^\alpha y^\beta) D_j, -y^\eta D_\lambda] \\
&= (-1)^{\text{deg} D_i} (D_j(x^\alpha y^\beta) D_k + D_k(x^\alpha y^\beta) D_j) \\
&= (-1)^{\text{deg} D_i} D_i (x^\alpha y^\beta)] \in S,
\end{align*}
\]

for all distinct elements \( i, j, k, l \) of \( \mathbb{Y} \) and \( \lambda, \eta \in G \). It follows from \( zd(D_{ij}(x^\alpha y^\beta)) = n - 3 \) that \( S_{n-3} \subseteq Q \).

For distinct elements \( i, j, k, l, \) \( \alpha \) of \( \mathbb{Y} \) and \( \lambda, \eta, \zeta \in G \), we have

\[
[D_{ij}(x^\alpha y^\beta), D_{kl}(x_\gamma y_\eta)] = (-1)^{\text{deg} D_i} D_i (x^\alpha y^\beta)] \in S,
\]

and \( zd(D_{ij}(x^\alpha y^\beta)) = n - 4 \). Thus \( S_{n-4} \subseteq Q \).

By the same methods above, we may obtain \( D_{ij}(x^\alpha y^\beta) \in S \) for \( u \in \mathfrak{B}(n) \); that is, \( S_0 \subseteq Q \) for \( i < j \leq n - 5 \).

According to \( D_{ij}(x_\gamma y_\eta) D_l \in S_{l-1} \) and \( x^\gamma y^\eta D_l \in S_0 \), we have

\[
x^\gamma y^\eta D_l = [x_\gamma x_\gamma y^\eta D_l, y^\gamma D_l] \in Q.
\] (8)

Hence \( S_0 \subseteq Q \).

In conclusion, \( S \subseteq Q \). Therefore, the desired result follows immediately.
3. The Natural Filtration of \(S(n,m)\)

Adopting the notion of [22], the element \(x\) of Lie superalgebra \(S\) is called ad-nilpotent if \(adx\) is a nilpotent linear transformation. The set of all ad-nilpotent elements of \(S\) is denoted by \(\text{nil}(S)\). Let \(S_{(j)} = a_{\geq j}S_{j}\). Then

\[
S = S_{(-1)} \supseteq S_{(0)} \supseteq S_{(1)} \supseteq \cdots \supseteq S_{(n-2)} \supseteq S_{(n-1)} = 0
\]  

(9)

is a descending filtration of \(S\), which is called the natural filtration of \(S\). We also call \(S_{(k)} | k \in \mathbb{Z}\) a filtration of \(S\) for short, where \(S_{(k)} = S\) if \(k \leq -1\) and \(S_{(k)} = 0\) if \(k > n - 2\). Since \(S\) is \(\mathbb{Z}\)-graded and finite dimensional, we may easily obtain \(S_{-1} \subseteq \text{nil}(S)\) and \(S_{(1)} \subseteq \text{nil}(S)\).

Let \(M_{n}(\mathbb{F})\) denote the set of all \(n \times n\) matrices over \(\mathbb{F}\). Notice that \(\dim \Gamma(m) = p^{m}\). Without loss of generality, we may suppose that \(\{y_{1}, \ldots, y_{p^{m}}\}\) is a standard \(\mathbb{F}\)-basis of \(\Gamma(m)\).

If \(z = \sum_{i=1}^{n} \sum_{t=1}^{m} a_{ij}x_{i}y_{t}D_{j} \in S_{0}\), where \(a_{ij} \in \mathbb{F}\); then let

\[
\rho(z) = \left(\begin{array}{ccc}
A_{1} & \cdots & A_{p^{m}} \\
\vdots & \ddots & \vdots \\
A_{p^{m}} & \cdots & A_{n}
\end{array}\right) \in M_{n}(\mathbb{F}).
\]

(10)

\[\text{Lemma 2. Suppose that } z = \sum_{i=1}^{n} \sum_{t=1}^{m} a_{ij}x_{i}y_{t}D_{j} \in S_{0}. \text{ If } z \text{ is ad-nilpotent, then } \rho(z) \text{ is a nilpotent matrix.}\]

\[\text{Proof. Let } \Gamma \text{ be the representation of } S_{0} \text{ with values in } S_{-1}. \text{ Then } \Gamma(z) = adz \text{ and the matrix of } \Gamma(z) \text{ over the basis } \{y_{1}D_{1}, \ldots, y_{p^{m}}D_{1}, \ldots, y_{p^{m}}D_{n}\} \text{ of } S_{-1} \text{ is } A = \left(-A_{1} \cdots -A_{p^{m}}\right) \in M_{n}(\mathbb{F}). \text{ Since } z \text{ is ad-nilpotent, the representation } \Gamma(z) \text{ is a nilpotent linear transformation. It implies that } A \text{ is nilpotent. Therefore, } \rho(z) = -A \text{ is a nilpotent matrix.}\]

\[\text{Lemma 3. Let } z = \sum_{i=k}^{n-1} z_{i}, \text{ where } z_{i} \in S_{i} \text{ and } k \leq n - 1. \text{ If } z \in \text{nil}(S) \text{ and } k \geq 0, \text{ then } z_{k+1} \in \text{nil}(S).\]

\[\text{Proof. Suppose that } z = z_{k} + z', \text{ where } z_{k} \in S_{k} \text{ and } z' \in \oplus_{i=k+1}^{n-1} S_{i} \subseteq S_{(k+1)}. \text{ Since } z \in \text{nil}(S), \text{ we may assume that } (adz)' = 0. \text{ Let } x \text{ be a } \mathbb{Z}\text{-homogeneous element of } S \text{ with } \mathbb{Z}\text{-degree } i. \text{ On the other hand,}\]

\[
(adz)'(x) = \left(ad\left(z_{k} + z'\right)\right)'(x) = (adz_{k})'(x) + h,
\]

(11)

which implies \((adz_{k})'(x) + h = 0\). It is easy to see that \((adz_{k})'(x) \in S_{(k+i)}\) and \(h \in S_{(k+i+1)} = \oplus_{j=k+i+1} S_{j}\). Thus \((adz_{k})'(x) = 0\). Since \(x\) is an arbitrary \(\mathbb{Z}\)-homogeneous element of \(S\), we have \((adz)'(S) = 0\). Then \((adz)' = 0\); that is, \(z_{k} \in \text{nil}(S)\).
we have \([z, D_{kj}(x_i x_j)] \notin \text{nil}(S)\). Then \(z \notin \Delta\). This contradicts \(z \in \Delta\). This proves our assertion. \(\square\)

**Lemma 5.** Let \(z = \sum_{i=1}^{n-1} z_i\), where \(z_i \in S_i\). If \(z \notin \Delta\), then \(z_0 = 0\).

**Proof.** Assume that \(z_0 \neq 0\). Let \(z_0 = \sum_{i=1}^{k} \sum_{j=1}^{n} a_{ijq} x_j y_z D_j\), \(a_{ijq} \in F\), and
\[
\begin{align*}
l &= \min \{i | a_{ijq} \neq 0, i, j \in Y\}, \\
t &= \min \{j | a_{ijq} \neq 0, i, j \in Y\}.
\end{align*}
\]
(15) (i) Suppose that \(l \leq t\). Let
\[
k = \max \{j | a_{ijq} \neq 0, j \in Y\}.
\]
(16) Then \(a_{kq} \neq 0\). It is easy to see that \(t \leq k\). Since \(l \leq t\), we have \(l \leq k\). Therefore,
\[
z_0 = \sum_{j=q-1}^{k} \sum_{i=1}^{n} a_{ijq} x_j y_z D_j + \sum_{i=1}^{l} \sum_{j=q+1}^{n} a_{ijq} x_j y_z D_j.
\]
(17) Assume that \(l = k\). It follows from \(t \leq k\) that \(t \leq l\). Then we have \(t = l\) which implies that
\[
z_0 = \sum_{j=q+1}^{n} a_{ijq} x_j y_z D_j + \sum_{i=1}^{n} \sum_{j=q+1}^{n} a_{ijq} x_j y_z D_j.
\]
(18) Therefore,
\[
\rho(z_0) = a_0 E_g + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} E_{ij}
\]
\[
+ a_{(l+n)(l+n)} E_{(l+n)(l+n)} + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} E_{ij}
\]
\[
+ \cdots + a_{(l+n(p-1))(l+n(p-1))n} E_{(l+n(p-1))(l+n(p-1))n}
\]
\[
+ \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ijq} E_{ij}
\]
(19) where \(A_k = a_{ik} E_{(l+n)(l+n)(l+n)}\) is an \((l+n) \times (l+n)\) matrix and \(q \in \{1, \ldots, p\}\). Since \(a_0 \neq 0\), we have \(A_k\) not being a nilpotent matrix. Then \(p(z_0)\) is not a nilpotent matrix and \(z_0 \notin \text{nil}(S)\). Lemma 3 shows that \(z \notin \text{nil}(S)\). It is a contradiction of \(z \notin \Delta\); that is, \(l > k\).

Suppose that \(h \in Y\) and \(h \neq l, k\). Let \(d = [z_0, x_h D_j]\). By equality (2), we obtain
\[
d = \sum_{q=1}^{p} \left( a_{pq} x_j y_z D_j + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ijq} x_j y_z D_j - \sum_{j=1}^{n} a_{ijq} x_j y_z D_j \right).
\]
(20) Since \(l \neq k\), \(\rho(d)\) also has the matrix form as \(p(z_0)\), it follows from \(a_{kl} \neq 0\) that \(A_k\) is not a nilpotent matrix. Then \(\rho(d)\) is not nilpotent. So \(z \notin \text{nil}(S)\) and \([z, x_h D_j] \notin \text{nil}(S)\). It is a contradiction of \(z \notin \Delta\).

(ii) Suppose that \(t < l\). Let \(k = \max \{i | a_{ik} \neq 0\}\) and \(d' = [z, x_h D_k]\). Imitating (i), we may prove that \(\rho(d')\) is also not nilpotent. Then the desired result follows. \(\square\)

**Lemma 6.** (i) If \(z \in S_\infty \cap \text{nil}(S)\) and \(h \in S_{(1)}\), then \(z + h \in \text{nil}(S)\).

(ii) Suppose that \(i, j\) are distinct elements of \(Y\); then \(x_i y_j D_i \in \text{nil}(S)\) for all \(\lambda \in G\).

(iii) Suppose that \(i, j, k\) are distinct elements of \(Y\); then \(ax_i y_j D_k + bx_i y^n D_k \in \text{nil}(S)\), where \(a, b \in F\) and \(\lambda, \eta\) are arbitrary elements of \(G\).

**Proof.** (i) A direct verification shows that \(\{adz\} \cup \{adS_{(1)}\}\) is a weakly closed subset of nilpotent elements of \(pl(S)\), where \(pl(S)\) is the general linear Lie superalgebra of \(S\). It was shown in [23, Theorem 1 of Chapter II] that each element of \(span\{\{adz\} \cup \{adS_{(1)}\}\}\) is a nilpotent linear transformation of \(S\). Then \(adz + adh\) is nilpotent. So \(z + h\) is \(ad\)-nilpotent.

(ii) To prove \((adx_i y_j D_k)^p = 0\), we may assume without loss of generality that \(i < j\). Set \(\eta\) to be an arbitrary element of \(G\). If \(k \neq i\), then
\[
(adx_i y_j D_k)^p = (x_i y_j D_k)^p = x_i y_j D_k
\]
(21)
\[
= 0.
\]
\[
(adx_i y^k D_k)^3 = (x_i y^k D_k)
\]
(22)
\[
= [x_i y^k D_j, [x_i y^k D_j, [x_i y^k D_j, x_i y^k D_j]]]
\]
\[
= [x_i y^k D_j, [x_i y^k D_j, [x_i y^k D_j, x_i y^k D_j]]] = 0.
\]
\[
(adx_i y^k D_j)^3 = (x_i y^k D_j)^3
\]
(23)
\[
= [x_i y^k D_j, [x_i y^k D_j, [x_i y^k D_j, x_i y^k D_j]]]
\]
\[
= [x_i y^k D_j, [x_i y^k D_j, [x_i y^k D_j, x_i y^k D_j]]] = 0.
\]

For \(p > 2\) we obtain \((adx_i y^k D_j)^p(x_i y^k D_k) = 0\). Therefore \((adx_i y^k D_j)^p(S) = 0\). This yields \((adx_i y^k D_j)^p = 0\). Thus \(x_i y^k D_j \in \text{nil}(S)\).

(iii) According to (ii) and \([x_i y^k D_k, x_i y^k D_k] = 0\), \(\{adx_i y^k D_k, adx_i y^k D_k\}\) is a weakly closed subset of nilpotent elements of \(pl(S)\). So \(ax_i y^k D_k + bx_i y^n D_k \in \text{nil}(S)\), where \(a, b \in F\). \(\square\)
Lemma 7. If $i, j, k$ are distinct elements of $Y$, then $x_i x_j y^k D_k \in \Delta$ for all $\lambda \in G$.

Proof. Suppose that $l \in Y \setminus \{i, j, k\}$. Then $x_i x_j y^k D_k \in S_{(1)} \subseteq \text{nil}(S)$. Let $z = \sum_{i=1}^{n-2} z_i$, where $z_i \in S_i$. Assume that $[x_i x_j y^k D_k, z] = f_0 + f_1$, where $f_0 = [x_i x_j y^k D_k, z_{-1}] \in S_0$ and $f_1 \in S_{(1)}$. Let $z_{-1} = \sum_{i=1}^{n-1} \sum_{\eta \in G} a_{\eta} y^\eta D_i$. Then

$$f_0 = \left( x_i x_j y^k D_k \right) \sum_{i=1}^{n} \sum_{\eta \in G} a_{\eta} y^\eta D_i = \sum_{\eta \in G} \left( a_{\eta} x_i x_j y^{\eta+k} D_k - a_{\eta} x_j x_i y^{\eta+k} D_k \right).$$

By (iii) of Lemma 6, we have $f_0 \in S_0 \cap \text{nil}(S)$. By (i) of Lemma 6, it follows that $f_0 + f_1 \in \text{nil}(S)$. We finally obtain $x_i x_j y^k D_k \in \Delta$ for all $\lambda \in G$.

Let $Q = \{ z \in \text{nil}(S) | \text{ad}(\Delta) \subseteq \Delta \}$.

Lemma 8. $Q = S_{(1)}$.

Proof. By the definition of $\Delta$, we have $S_{(2)} \subseteq \Delta$. Lemmas 4 and 5 show that $\Delta \subseteq S_{(1)}$. Then $[S_{(1)}, \Delta] \subseteq [S_{(1)}, S_{(1)}] \subseteq S_{(2)} \subseteq \Delta$. Thus $S_{(1)} \subseteq Q$.

Next we will prove $Q \subseteq S_{(1)}$. Let $z \in Q$ and $z = \sum_{i=1}^{n-2} z_i$, where $z_i \in S_i$. Assume that $z_{-1} = \sum_{i=1}^{n-1} \sum_{\eta \in G} a_{\eta} y^\eta D_i \neq 0, a_{\eta} \in F$. Without loss of generality, we may suppose that $a_0 \neq 0$. Let $d = x_i x_j y^k D_k$, where $i, j, k$ are distinct elements of $Y$ and $\eta$ is an arbitrary element of $G$. By Lemma 7, we have $d \in \Delta$. Let $[z, d] = h_0 + h_1$, where $h_0 = \sum_{i=1}^{n-1} d_i \in S_0$ and $h_1 \in S_{(1)}$. Since $a_0 \neq 0$, we have $h_0 = \sum_{\eta \in G} a_{\eta} x_i x_j y^{\eta+k} D_k - a_{\eta} x_j x_i y^{\eta+k} D_k \neq 0$. Lemma 5 implies that $h_0 + h_1 \notin \Delta$. It is a contradiction of $z \in Q$. Hence $z_{-1} = 0$.

Assume that $0 \neq z_0 = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} a_{ij} x_i y^k D_j - a_{ij} D_j \in F$, and suppose that $l$ and $t$ are as the definitions in (15). We may suppose that $l < t$ (the proof is similar to the case $t < l$) and let $k$ be as the definition in (16). In a similar way to the first part of the proof in Lemma 5, we have $l < k$. Suppose that $h \in Y \setminus \{l, k, t\}$ and $d_1 = x_l x_k D_l$. Lemma 7 shows that $d_1 \in \Delta$. Let $\{z, d_1\} = g_1 + g_2$, where $g_1 = [z_0, d_1] \in S_1$ and $g_2 \in S_{(2)}$.

Using equality (2), we have

$$g_1 = \sum_{q=1}^{n} \left( a_{kq} x_l y^q D_l \right) - \sum_{i=1}^{n} a_{ij} x_i y^k D_k - \sum_{j=1}^{k} a_{ij} x_l y^k D_j - a_{ij} D_j = \sum_{q=1}^{n} \left( a_{kq} x_l y^q D_l \right).$$

If $h < t$, then $a_{ihq} = 0$ in the above equality, where $i \in Y \setminus \{1, \ldots, l - 1\}$. Thus

$$[D_h, g_1] = -\sum_{q=1}^{n} \left( a_{kq} x_l y^q D_l + \sum_{i=1}^{n} a_{ihq} x_i y^q D_l \right).$$

By equality (12), the matrix $\rho([D_h, g_1])$ has the matrix form as in Lemma 5. Since $a_{bhq} \neq 0$, $A_1$ is not a nilpotent matrix. It implies that $\rho([D_h, g_1])$ is not nilpotent. Hence $[D_h, g_1] \notin \text{nil}(S)$. Lemma 3 shows that $[D_h, g_1 + g_2] \notin \text{nil}(S)$; that is, $[D_h, g_1 + g_2] \notin \Delta$. It contradicts $z \in Q$. Thus $z_0 = 0$. Therefore, $z \in S_{(1)}$, and $Q \subseteq S_{(1)}$.

According to the fact that $\Delta$ and $Q$ are invariant subspaces under the automorphisms of $S$ and Lemma 8, $S_{(1)}$ is also invariant under the automorphisms of $S$. Since

$$S_{(0)} = \{ x \in S | [x, S_{(1)}] \subseteq S_{(1)} \},$$

$$S_{(0)} = \{ x \in S_{(1)} | [x, S] \subseteq S_{(1)} \},$$

we may easily obtain the following theorem.

Theorem 9. The natural filtration of $S$ is invariant under the automorphisms of $S$.

Let $S_i = S_{(i)}/S_{(i+1)}$ for $-1 \leq i \leq n - 2$. Then $S_i$ is a $Z$-graded space. Suppose that $\mathcal{G} := \oplus_{i=-2}^{n} S_i$; then $\mathcal{G}$ is also a $Z$-graded space. Let $x + S_{(i+1)} \in S_i$ and $y + S_{(i+1)} \in S_j$. Define

$$[x + S_{(i+1)}, y + S_{(j+1)}] := [x, y] + S_{(i+j+1)}.$$
By virtue of Lemma 8, we have $Q = S_{(1)}$ and $Q' = S'_{(1)}$. Thus $\sigma(S_{(i)}) = S'_{(i)}$. By equalities (26), the desired result $\sigma(S_{(i)}) = S'_{(i)}$ for all $i \geq -1$ is obtained.

**Lemma 12.** Suppose that $S \equiv S'$ and $\sigma$ is an isomorphism from $S$ to $S'$; then $\sigma$ induces an isomorphism $\tilde{\sigma}$ from $\mathcal{S}$ to $\mathcal{S}'$ such that $\tilde{\sigma}(S_{(i)}) = S'_{(i)}$ for all $i \geq -1$.

**Proof.** Define a linear map $\tilde{\sigma} : \mathcal{S} \to \mathcal{S}'$ such that
\[
\tilde{\sigma}(x + S_{(i+1)}) = \sigma(x) + S'_{(i+1)},
\]
where $x + S_{(i+1)} \in \mathcal{S}$. Using Proposition 11, the definition of $\tilde{\sigma}$ is reasonable and
\[
\tilde{\sigma}\left([x + S_{(i+1)}, y + S_{(j+1)}]\right) = \sigma([x, y]) + S'_{(i+j+1)}
\]
\[
= \left[\sigma(x) + S'_{(i+1)}, \sigma(y) + S'_{(j+1)}\right] = \left[\tilde{\sigma}(x + S'_{(i+1)}), \tilde{\sigma}(y + S'_{(j+1)})\right].
\]
Thus $\tilde{\sigma}$ is a homomorphism from $\mathcal{S}$ to $\mathcal{S}'$. Clearly, $\tilde{\sigma}(\mathcal{S}_{(i)}) = \mathcal{S}'_{(i)}$ for all $i \geq -1$. It follows that $\tilde{\sigma}$ is an epimorphism.

Suppose that $y \in \ker \tilde{\sigma}$; then $y \in \mathcal{S}$. So we may suppose that $y = \sum_{i=1}^{n-1} y_i$ and $y_i \in \mathcal{S}$. Since $\mathcal{S}_{(i)} = S_{(i)}/S_{(i+1)}$, let $y_i = z_i + S_{(i+1)}$, where $z_i \in S_{(i)}$. Hence $\tilde{\sigma}(y_i) = \sigma(z_i) + S'_{(i+1)}$. It follows from $\tilde{\sigma}(y) = 0$ that $\sum_{i=1}^{n-1} \tilde{\sigma}(y_i) = 0$. Thus $\tilde{\sigma}(y_i) = 0$; that is, $\sigma(z_i) + S'_{(i+1)} = 0$. It follows that $\sigma(z_i) \in S'_{(i+1)}$. By Proposition 11, we have $z_i \in \sigma^{-1}(S'_{(i+1)}) = S_{(i+1)}$. Then $y_i = z_i + S_{(i+1)} = 0$ for $-1 \leq i \leq n - 2$. Therefore, $y = 0$ and $\ker \tilde{\sigma} = 0$. Consequently, $\tilde{\sigma}$ is an isomorphism induced by $\sigma$ such that $\tilde{\sigma}(\mathcal{S}) = \mathcal{S}'$ for all $i \geq -1$.

**Theorem 13.** $S \equiv S'$ if and only if $m = m'$ and $n = n'$.

**Proof.** Because the sufficiency is obvious, it suffices to prove the necessity. Suppose that $\phi : S \to \mathcal{S}$ is the isomorphism given in the proof of Lemma 10. Similarly, there also exists the $\phi' : S' \to \mathcal{S}'$. According to the equality (28) and Lemma 12, we have
\[
\phi(S_{(i)}) = \mathcal{S}_{(i)}, \quad \phi'(S'_{(i)}) = \mathcal{S}'_{(i)}, \quad \tilde{\sigma}(\mathcal{S}_{(i)}) = \mathcal{S}'_{(i)}
\]
for $-1 \leq i \leq n - 2$. Let $\psi = (\phi')^{-1} \tilde{\sigma} \phi$. Then
\[
\psi(S_{(i)}) = (\phi')^{-1} \tilde{\sigma} \phi(S_{(i)}) = (\phi')^{-1} \sigma(S_{(i)}) = (\phi')^{-1} \mathcal{S}_{(i)} = S'_{(i)}.
\]
In particular, $\psi(S_{(-1)}) = S'_{-1}$. It follows from $\dim S_{-1} = \dim S'_{-1}$ that $np^m = n'p^{m'}$. By virtue of the definition of $S_{(i)}$, we have
\[
S_0 = \text{span}_{\mathbb{C}} \left\{ D_{x_i}(x_j x_k) : x_i \in S \right\},
\]
where $D_{x_i}(x_j x_k) = x_i x_j x_k$. Similarly, $S'_{0} = (n^2 - 1)p^{m'}$. According to $\dim S_0 = \dim S'_{0}$ and $np^{m} = n'p^{m'}$, we have $n = n'$. In conclusion, the proof is completed.

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**References**


