Letter to the Editor

Comment on “Perturbation Analysis of the Nonlinear Matrix Equation $X - \sum_{i=1}^m A_i^* X^{p_i} A_i = Q$ ”

Maher Berzig¹ and Erdal Karapınar²

¹ *Tunis College of Sciences and Techniques, Tunis University, 5 Avenue Taha Hussein, P.O. Box 56, Bab Manara, Tunis, Tunisia*

² *Department of Mathematics, Atılım University, Incek, 06836 Ankara, Turkey*

Correspondence should be addressed to Erdal Karapınar; erdalkarapinar@yahoo.com

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We show that the perturbation estimate for the matrix equation $X - \sum_{i=1}^m A_i^* X^{p_i} A_i = Q$ due to J. Li, is wrong. Our discussion is supported by a counterexample.

1. Introduction and Preliminaries

The following definitions and the notations are the same as in [1]. We denote by $\mathcal{C}^{n \times n}$ the set of $n \times n$ complex matrices, by $\|\cdot\|$ the spectral norm, and by $\lambda_{\min}(M)$ the minimal eigenvalues of M .

Consider the matrix equation

$$X - \sum_{i=1}^m A_i^* X^{p_i} A_i = Q, \quad (1)$$

where $A_i \in \mathcal{C}^{n \times n}$ for $1 \leq i \leq m$. The existence and uniqueness of its positive definite solution X is proved in [2]. Next, consider the perturbed equation

$$\tilde{X} - \sum_{i=1}^m \tilde{A}_i^* \tilde{X}^{p_i} \tilde{A}_i = \tilde{Q}, \quad (2)$$

where $0 < p_i < 1$ and $\tilde{A}_i$ and $\tilde{Q}$ are small perturbations of A_i and Q , respectively. We assume that X and $\tilde{X}$ are solutions of (1) and (2), respectively. Let

$$\Delta X = \tilde{X} - X, \quad \Delta Q = \tilde{Q} - Q, \quad \Delta A_i = \tilde{A}_i - A_i. \quad (3)$$

In [3, 4], some comments on perturbation estimates for particular cases of (1) and (2) have been furnished. In this note, we focus on the following recent result obtained by J. Li.

Theorem 1 (see [1, Theorem 5]). *Let*

$$\begin{aligned} \beta &= \lambda_{\min}(Q) + \sum_{i=1}^m \lambda_{\min}(A_i^* A_i) \lambda_{\min}^{p_i}(Q), \\ b &= \beta + \|\Delta Q\| - \sum_{i=1}^m p_i \beta^{p_i} \|A_i\|^2, \end{aligned} \quad (4)$$

$$s = \sum_{i=1}^m \beta^{p_i} \|\Delta A_i\| (2 \|A_i\| + \|\Delta A_i\|).$$

If

$$0 < b < 2(\beta - s) \quad (5)$$

$$b^2 - 4(\beta - s)(s + \|\Delta Q\|) \geq 0,$$

then

$$\frac{\|\tilde{X} - X\|}{\|X\|} \leq \rho \sum_{i=1}^m \|\Delta A_i\| + \omega \|\Delta Q\|, \quad (6)$$

where

$$\rho = \frac{2s}{\sum_{i=1}^m \|\Delta A_i\| \left(b + \sqrt{b^2 - 4(\beta - s)(s + \|\Delta Q\|)} \right)}, \quad (7)$$

$$\omega = \frac{2}{b + \sqrt{b^2 - 4(\beta - s)(s + \|\Delta Q\|)}}.$$

2. Counterexample

The following counterexample shows that the perturbation estimates in Theorem 1 are not true in general. Consider

$$q = \frac{3}{4}, \quad m = 1, \quad A = \frac{1}{2}, \quad (8)$$

$$\widetilde{A} = A + \frac{1}{10}, \quad X = 1, \quad \widetilde{X} = X + \frac{1}{100}.$$

Now, we compute Q and $\widetilde{Q}$ by using

$$Q = X - A^* X^q A, \quad \widetilde{Q} = \widetilde{X} - \widetilde{A}^* \widetilde{X}^q \widetilde{A}, \quad (9)$$

so we get

$$Q = 0.75, \quad \widetilde{Q} \approx 0.64730. \quad (10)$$

Finally, using (8)–(10), we obtain that the hypothesis of Theorem 1 is satisfied, that is,

$$0 < b \approx 0.66815 < 2(\beta - s) \approx 1.69102, \quad (11)$$

$$b^2 - 4(\beta - s)(s + \|\Delta Q\|) \approx 0.43535 \geq 0,$$

whereas

$$\frac{\|\widetilde{X} - X\|}{\|X\|} \approx 0.01000 \not\leq \rho \sum_{i=1}^m \|\Delta A_i\| + \omega \|\Delta Q\| \approx 0.00491. \quad (12)$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contribution

All the authors contributed equally to this work and significantly in writing this paper. All the authors read and approved the final paper.

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