Research Article

Bogdanov-Takens Bifurcation of a Delayed Ratio-Dependent Holling-Tanner Predator Prey System

Xia Liu, 1 Yanwei Liu, 2 and Jinling Wang 1

1 College of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, China
2 Institute of Systems Biology, Shanghai University, Shanghai 200444, China

Correspondence should be addressed to Xia Liu; liuxiapost@163.com

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A delayed predator prey system with refuge and constant rate harvesting is discussed by applying the normal form theory of retarded functional differential equations introduced by Faria and Magalhães. The analysis results show that under some conditions the system has a Bogdanov-Takens singularity. A versal unfolding of the system at this singularity is obtained. Our main results illustrate that the delay has an important effect on the dynamical behaviors of the system.

1. Introduction

It is well known that the multiple bifurcations will occur when a predator prey system (ODE) with more interior positive equilibria, such as Bogdanov-Takens bifurcation, Hopf bifurcation, and backward bifurcation; see [1–5] for example. However, when the predator prey systems with delay and Bogdanov-Takens bifurcation are researched relative few (see [6–8] and the reference therein) using similar methods as [6–8], the authors of [9–11] consider the Bogdanov-Takens bifurcation of some delayed single inertial neuron or oscillator models.

Motivated by the works of [5, 6], we mainly consider the Bogdanov-Takens bifurcation of the following system:

\[
\begin{align*}
\dot{x} &= rx \left(1 - \frac{x}{K}\right) - \frac{axy}{Ay + x - \overline{m}} - \overline{h}, \\
\dot{y} &= sy \left(1 - \frac{by(t - \overline{\tau})}{x(t - \overline{\tau}) - \overline{m}}\right),
\end{align*}
\]

(1)

where \(x\) and \(y\) stand for prey and predator population (or densities) at time \(t\), respectively. The predator growth is of logistic type with growth rate \(r\) and carrying capacity \(K\) in the absence of predation; \(a\) and \(A\) stand for the predator capturing rate and half saturation constant, respectively; \(s\) is the intrinsic growth rate of predator; however, carrying capacity \(x/b\) (\(b\) is the conversion rate of prey into predators) is the function on the population size of prey. The parameters \(x, A, \overline{m}, \overline{h}, s, b, \) and \(\overline{\tau}\) are all positive constants. \(\overline{m}\) is a constant number of prey using refuges; \(\overline{h}\) is the rate of prey harvesting.

For simplicity, we first rescale system (1). Let \(X = x - \overline{m}, Y = y;\) system (1) can be written as (still denoting \(X, Y\) as \(x, y\))

\[
\begin{align*}
\dot{x} &= r(x + \overline{m}) \left(1 - \frac{x + \overline{m}}{K}\right) - \frac{axy}{Ay + x - \overline{m}} - \overline{h}, \\
\dot{y} &= sy \left(1 - \frac{by(t - \overline{\tau})}{x(t - \overline{\tau}) - \overline{m}}\right).
\end{align*}
\]

(2)

Next, let \(\overline{\tau} = rt, X(\overline{\tau}) = x(t)/K, \) and \(Y(\overline{\tau}) = ay(t)/rK;\) then system (2) takes the form (still denoting \(X, Y, \overline{\tau}\) as \(x, y, t\))

\[
\begin{align*}
\dot{x} &= (x + m)(1 - x - m) - \frac{xy}{ay + x} - h, \\
\dot{y} &= \delta y \left(\beta - \frac{y(t - \overline{\tau})}{x(t - \overline{\tau})}\right),
\end{align*}
\]

(3)

where \(m = \overline{m}/K, a = Ar/\alpha, \delta = \overline{h}/\alpha, \beta = \alpha/br, h = \overline{h}/r,\) and \(\overline{\tau} = rt.\)
When \( \tau = 0 \), we have known that for some parameter values system (3) exhibits Bogdanov-Takens bifurcation (see [5]). Summarizing the methods used by [6] and the formulae in [12], the sufficient conditions which depend on delay to guarantee that system (3) has a Bogdanov-Takens singularity will be given. Therefore, the delay has effect on the occurrence of Bogdanov-Takens bifurcation.

In the next section we will compute the normal form and give the versal unfolding of system (3) at the degenerate equilibrium.

2. Bogdanov-Takens Bifurcation

System (3) can also be written as

\[
\begin{align*}
\dot{x} &= \tau \left( (x + m)(1 - x - m) - \frac{xy}{ay + x} - h \right), \\
\dot{y} &= \tau \delta y \left( \beta - \frac{y(t - 1)}{x(t - 1)} \right).
\end{align*}
\]

Let

\[
0 < m < \frac{1}{2} \left( 1 - \frac{\beta}{a\beta + 1} \right),
\]

\[
h = \tilde{h} = \frac{1}{4} \left( \frac{\beta}{a\beta + 1} - 1 \right)^2 + \frac{m\beta}{a\beta + 1}.
\]

Then \( P_x = (x_*, y_*) \) is an interior positive equilibrium of systems (3) and (4), where \( x_* = -\frac{1}{2}(\beta/(a\beta + 1) + 2m - 1), \ y_* = \beta x_* \).

In order to discuss the properties of system (4) in the neighborhood of the equilibrium \( P_x = (x_*, y_*) \), let \( \bar{x} = x - x_* \), \( \bar{y} = y - y_* \); then \( P_x \) is translated to \((0,0)\), and system (4) becomes (still denoting \( \bar{x}, \bar{y} \) as \( x, y \))

\[
\begin{align*}
\dot{\bar{x}} &= \tau \delta \beta \bar{x} - \frac{\tau}{(a\beta + 1)^2} \bar{x}^2 + g_1 \bar{x} y + g_2 \bar{x} y + g_3 \bar{x} y^2 + h.o.t, \\
\dot{\bar{y}} &= \tau \delta \beta^2 \bar{x} (t - 1) - \tau \delta \beta \bar{y} (t - 1) + g_4 \bar{x}^2 (t - 1) + g_5 \bar{x} y (t - 1) + g_6 \bar{y} (t - 1) y + g_7 \bar{y} y (t - 1) y + h.o.t,
\end{align*}
\]

where h.o.t denotes the higher order terms and

\[
\begin{align*}
g_1 &= \left( \frac{2a\delta \beta \tau}{(\beta + 2m - 1)(a\beta + 1)} + \tau \right), \\
g_2 &= \frac{4a\delta \beta \tau}{(\beta + 2m - 1)(a\beta + 1)}, \\
g_3 &= \frac{-2a\delta \tau}{(\beta + 2m - 1)(a\beta + 1)}, \\
g_4 &= \frac{2\tau \delta \beta^2 (a\beta + 1)}{(\beta + 2m - 1)(a\beta + 1)}. \\
g_5 &= \frac{-2\tau \delta \beta (a\beta + 1)}{(\beta + 2m - 1)(a\beta + 1)}, \\
g_6 &= \frac{-2\tau \delta \beta (a\beta + 1)}{(\beta + 2m - 1)(a\beta + 1)}, \\
g_7 &= \frac{2\tau \delta \beta (a\beta + 1)}{(\beta + 2m - 1)(a\beta + 1)}.
\end{align*}
\]

The characteristic equation of the linearized part of system (6) is

\[
F(\lambda) = \lambda^2 + \left[ \frac{\tau \delta \beta e^{-\lambda}}{(a\beta + 1)^2} \right] \lambda.
\]

Clearly, if

\[
\delta = \frac{1}{(a\beta + 1)^2},
\]

\[
\tau \neq \frac{1}{\delta \beta},
\]

then \( \lambda = 0 \) is double zero eigenvalue; if \( \delta = 1/(a\beta + 1)^2 \) and \( \tau = 1/\delta \beta \), that is, \( \tau = (a\beta + 1)^2/\beta \), then \( \lambda = 0 \) is triple zero eigenvalue. We will mainly discuss the first case in this paper.

According to the normal form theory developed by Faria and Magalhães [13], first, rewrite system (4) as \( \tilde{X}(t) = L(X(t)) \), where \( X(t) = (x_1(t), x_2(t)) \), \( L(\phi) = L(\phi_{t}(0)) \), and \( \phi = (\phi_1, \phi_2) \). Take \( A_0 \) as the infinitesimal generator of system. Let \( \Lambda = \{0\} \), and denote by \( P \) the invariant space of \( A_0 \) associated with the eigenvalue \( \lambda = 0 \); using the formal adjoint theory of RFDE in [13], the phase space \( C_1 \) can be decomposed by \( \Lambda \) as \( C_1 = P \oplus Q \). Define \( \Phi, \Psi \) as the bases for \( P \) and \( P^* \), the space associated with the eigenvalue \( \lambda = 0 \) of the adjoint equation, respectively, and to be normalized such that \( (\Psi, \Phi) = I, \Phi = \Phi^*J, \) and \( \Psi = -J\Psi \), where \( \Phi \) and \( \Psi \) are \( 2 \times 2 \) matrices.

Next we will find the \( \Phi(\theta) \) and \( \Psi(s) \) based on the techniques developed by [14].

Lemma 1 (see Xu and Huang [14]). The bases of \( P \) and its dual space \( P^* \) have the following representations:

\[
P = \text{span } \Phi, \Phi(\theta) = (\phi_1(\theta), \phi_2(\theta)), \quad -1 \leq \theta \leq 0,
\]

\[
P^* = \text{span } \Psi, \Psi(s) = \text{col } (\psi_1(s), \psi_2(s)), \quad 0 \leq s \leq 1,
\]

where \( \phi_1(\theta) = \phi_1^0 \in R^n \setminus \{0\}, \phi_2(\theta) = \phi_2^0 + \phi_0^0 \theta, \) and \( \phi_0^0 \in R^n \) and \( \psi_1(s) = \psi_1^0 \in R^{n*} \setminus \{0\}, \) and \( \psi_1(s) = \psi_1^0 - s\psi_2^0, \psi_2^0 \in R^{n*}, \) which satisfy

\[
\begin{align*}
(1) \quad (A + B)\psi_1^0 &= 0, \\
(2) \quad (A + B)\psi_2^0 &= (B + I)\psi_1^0, \\
(3) \quad \psi_1^0(A + B) &= 0, \\
(4) \quad \psi_1^0(A + B) &= \psi_2^0(B + I),
\end{align*}
\]
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It is easy to obtain

\[
\Phi(\theta) = \left( \begin{array}{c} 1 \frac{1}{\tau \delta \beta} + \theta \\ \beta \left( z_1 + x_2 \theta \right) \end{array} \right), -1 \leq \theta \leq 0,
\]

(13)

\[
\Phi(\theta) z = \left( z_1 + \frac{1}{\tau \delta \beta} + \theta \right) z_2,
\]

(14)

Hence (6) becomes

\[
\begin{align*}
\dot{z}_1 &= z_2 - \tau m_1 z_1^2 - \frac{2m_1}{\delta \beta} z_1 z_2 + l_1 z_2^2 + \text{h.o.t}, \\
\dot{z}_2 &= m \beta \delta \beta z_1 + \frac{2m}{\delta} z_1 z_2 + l_2 z_2^2 + \text{h.o.t},
\end{align*}
\]

(15)

\[
\begin{align*}
\phi_1(0) &= z_1 + \frac{1}{\tau \delta \beta} z_2, \\
\phi_1(-1) &= z_1 + \frac{1}{\tau \delta \beta} - 1 z_2, \\
\phi_2(0) &= \beta z_1, \\
\phi_2(-1) &= \beta (z_1 - z_2).
\end{align*}
\]

In the following, we will do versal unfolding of system (4) at \( P_* \):

\[
\dot{x} = \tau (x + m)(1 - x - m) - \frac{xy}{ay + x} - h,
\]

(16)

\[
\dot{y} = \tau (\delta + \mu_1) y \left( \beta + \mu - \frac{y}{x(t-1)} \right).
\]

(17)

When \( \mu_1 = \mu_2 = 0 \), system (16) has a Bogdanov-Takens singularity \( P_* \) and a two-dimensional center manifold exists.

The Taylor expansion of system (16) at \( P_* \) takes the form

\[
\begin{align*}
\dot{x} &= \tau \delta \beta x - \tau \delta y + g_1 x^2 + g_2 xy + g_3 y^2 + \text{h.o.t}, \\
\dot{y} &= w_0 \mu_2 + \tau (\delta + \mu_1) \beta^2 x (t-1) - \tau (\delta + \mu_1) \beta y (t-1) + w_2 x (t-1) y (t-1) + w_3 x (t-1) y (t-1) + \text{h.o.t},
\end{align*}
\]

(18)

where

\[
\begin{align*}
w_0 &= -\frac{\tau \delta \beta \beta (2m - 1)(\alpha \beta + 1)}{2(\alpha \beta + 1)}, \\
w_2 &= \frac{2\tau (\delta + \mu_1) \beta^2 (\alpha \beta + 1)}{\beta + (2m - 1)(\alpha \beta + 1)}, \\
w_3 &= -\frac{2\tau (\delta + \mu_1) \beta (\alpha \beta + 1)}{\beta + (2m - 1)(\alpha \beta + 1)}, \\
w_5 &= \frac{2\tau (\delta + \mu_1) (\alpha \beta + 1)}{\beta + (2m - 1)(\alpha \beta + 1)}.
\end{align*}
\]
We decompose the enlarged phase space $\mathcal{BC}$ of system as $\mathcal{BC} = P \oplus \text{Ker}\,\pi$. Then $y$ in system can be decomposed as $y = \Phi z + u$ with $z \in \mathbb{R}^2$ and $u \in Q$. Hence, system is decomposed as

$$\dot{z} = B_1 + B_2 z + \Psi(0)G(\Phi z + u),$$

where

$$B_0 = \begin{pmatrix} 0 \\ w_0 m_2 \mu_2 \end{pmatrix}, \quad B_1 = \Psi(0)B_0 = \begin{pmatrix} w_0 m_2 \mu_1 \\ w_0 n \mu_2 \end{pmatrix}, \quad B_2 = \Psi(0) \begin{pmatrix} \tau \delta \beta \phi_1(0) - \tau \delta \phi_2(0) \\ (\delta + \mu_1) \beta^2 \phi_1(-1) - \tau (\delta + \mu_1) \beta \phi_2(-1) + \tau \delta \mu_2 \phi_2(0) \end{pmatrix}, \quad G(\phi) = \begin{pmatrix} g_1 \phi_1^2(0) + g_2 \phi(0) \phi_2(0) + g_3 \phi_2^2(0) + \text{h.o.t} \\ w_4 \phi_1^2(-1) + w_5 \phi_1(-1) \phi_2(-1) + w_6 \phi_1(-1) \phi_2(0) + w_7 \phi_2(-1) \phi_2(0) + \text{h.o.t} \end{pmatrix}.$$  

(20)

To compute the normal form of system at $P_\ast$, consider

$$\dot{z}_1 = B_1 + B_2 z + \Psi(0)G(\Phi z); \quad \text{together with } (13) \text{ we obtain}$$

$$\dot{z}_1 = w_0 m_2 \mu_2 + z_2 + m_2 \tau \delta \beta \mu_2 z_1 + \frac{m_2 \beta \mu_1}{\delta} z_2$$

$$- \tau m_1 z_1^2 \frac{2 m_1}{\delta \beta} z_2 + l_1 z_2^2 + \text{h.o.t.}$$

$$\dot{z}_2 = w_0 n \mu_2 + n \tau \delta \beta \mu_2 z_1 + \frac{n \beta \mu_1}{\delta} z_2$$

$$+ n \tau \beta z_1^2 + \frac{2 n}{\delta} z_1 z_2 + l_2 z_2^2 + \text{h.o.t.}$$

(21)

Following the normal form formula in Kuznetsov [12], system (21) can be reduced to

$$\dot{z}_1 = z_2 + \text{h.o.t.},$$

$$\dot{z}_2 = \gamma_1 z_1 + d_1 z_1^2 + d_2 z_1 z_2 + \text{h.o.t.}$$

(22)

where

$$\gamma_1 = -\frac{\tau \delta \beta [(2m-1)(a \beta + 1)]}{2(a \beta + 1)} n \mu_2$$

$$= \frac{\tau^2 \beta [(2m-1)(a \beta + 1)]}{2(a \beta + 1)^3 [(a \beta + 1)^2 - \tau \delta \beta]} \mu_2,$$

Hence system (4) exist the following bifurcation curves in a small neighborhood of the origin in the $(\mu_1, \mu_2)$ plane.

**Theorem 3.** Let (5), (9), and $\tau \neq (2 \pm \sqrt{2})/\delta \beta$ hold. Then system (4) admits the following bifurcations:

(i) a saddle-node bifurcation curve $\text{SN} = \{(\mu_1, \mu_2) : \gamma_1 = (1/4d_1)\gamma_2^2\};$

(ii) a Hopf bifurcation curve $H = \{(\mu_1, \mu_2) : \mu_2 = 0, \gamma_2 < 0\};$

(iii) a homoclinic bifurcation curve $\text{HL} = \{(\mu_1, \mu_2) : \gamma_1 = -(6/25d_1)\gamma_2^2, \gamma_2 < 0\}.$

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.
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