Research Article

Some New Nonlinear Weakly Singular Inequalities and Applications to Volterra-Type Difference Equation

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Some new nonlinear weakly singular difference inequalities are discussed, which generalize some known weakly singular inequalities and can be used in the analysis of nonlinear Volterra-type difference equations with weakly singular kernel. An application to the upper bound of solutions of a nonlinear difference equation is also presented.

1. Introduction

The discrete version of the well-known Gronwall-Bellman inequality is an important tool in the development of the theory of difference equations as well as the analysis of the numerical schemes of differential equations. A great deal of interest has been given to these inequalities, and many results on their generalizations have been found; for example, see [1–4]. Among them, one of the fundamental cases is Pachpatte's result [3] for the difference inequality:

$$u(n) \leq a(n) + \sum_{s=0}^{n-1} f(s)u(s). \quad (1)$$

In particular, due to the study of the behavior and numerical solutions for the singular integral equations, some discrete weakly singular integral inequalities also have drawn more and more attention [5–7]. Dixon and McKee [8] investigated the convergence of discretization methods for the Volterra integral and integrodifferential equations, by using the following inequality:

$$x_i \leq \psi_i + Mh^{1-\alpha} \sum_{j=0}^{i-1} \frac{x_j}{(i-j)^\alpha}, \quad i = 1, 2, \dots, N, \quad (2)$$

$$n > 0, \quad Nh = T.$$

Henry [9] presented a linear integral inequality with weakly kernel:

$$x(t) \leq a(t) + \int_0^t (t-s)^{\beta-1} b(s)x(s)ds \quad (3)$$

to investigate some qualitative properties for a parabolic equation. The corresponding discrete version was discussed by Slodička [10]. But he studied the case $\tau_k = \tau$, that is, the case of constant differences. Furthermore, the first formulation of the inequality with a nonlinearity and τ_k nonconstant was studied in [6], in which the general nonlinear discrete case as follows:

$$x_n \leq a_n + \sum_{k=0}^{n-1} (t_n - t_k)^{\beta-1} \tau_k b_k \omega(x_k) \quad (4)$$

was considered. However, his results are based on the so-called “(q) condition”: (1) ω satisfies $e^{-qt}[\omega(u)]^q \leq R(t)\omega(e^{-qt}u)^q$; (2) there exists $c > 0$ such that $a_n e^{-\tau t_n} \leq c$. Recently, a new nonlinear difference inequality:

$$x_n^\alpha \leq a_n + \sum_{k=0}^{n-1} (t_n - t_k)^{\beta-1} \tau_k b_k x_k^\lambda \quad (5)$$

was discussed by Yang et al. [11]. For other new weakly singular inequalities, lots of work can be found, for example, in [12–22] and references therein.

In this paper, we investigate the new nonlinear weakly singular inequality:

$$x_n \leq a_n + \sum_{k=0}^{n-1} (t_n - t_k)^{\beta-1} \tau_k b_k \omega(x_k), \quad (6)$$

where $0 < \beta \leq 1$, $t_0 = 0$, $\tau_k = t_{k+1} - t_k$, $\sup_{k \in \mathbb{N}} \tau_k = \tau$, and $\lim_{t \rightarrow \infty} t_k = \infty$. Compared to the existing result, our result does not need the so-called “(q) condition” proposed in [6] and can be used to obtain pointwise explicit bounds on solutions for a class of more general weakly singular inequalities of Volterra type. Finally, we also present an application to Volterra-type difference equation with weakly singular kernel.

2. Preliminaries

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}_+ = (0, \infty)$, and $\mathbb{N} = \{0, 1, 2, \dots\}$. $C(X, Y)$ denotes the collection of continuous functions from the set X to the set Y . As usual, the empty sum is taken to be 0.

Lemma 1 (Discrete Jensen inequality, [11]). *Let $A_1, A_2, \dots, A_n$ be nonnegative real numbers, and let $r > 1$ be a real number. Then,*

$$(A_1 + A_2 + \dots + A_n)^r \leq n^{r-1} (A_1^r + A_2^r + \dots + A_n^r). \quad (7)$$

Lemma 2 (Discrete Hölder inequality, [11]). *Let a_i, b_i ($i = 1, 2, \dots, n$) be nonnegative real numbers, and let p, q be positive numbers such that $(1/p) + (1/q) = 1$ (or $p = 1, q = \infty$). Then,*

$$\sum_{i=1}^n a_i b_i \leq \left(\sum_{i=1}^n a_i^p \right)^{1/p} \left(\sum_{i=1}^n b_i^q \right)^{1/q}. \quad (8)$$

Furthermore, take $p = q = 2$; then, one gets the discrete Cauchy-Schwarz inequality.

Lemma 3. *Suppose that $\omega(u) \in C(\mathbb{R}_+, \mathbb{R}_+)$ is nondecreasing. Let a_n, c_n be nonnegative and nondecreasing in n . If y_n is nonnegative such that*

$$y_n \leq a_n + c_n \sum_{k=0}^{n-1} b_k \omega(y_k), \quad n \in \mathbb{N}. \quad (9)$$

Then,

$$y_n \leq \Omega^{-1} \left[\Omega(a_n) + c_n \sum_{k=0}^{n-1} b_k \right], \quad 0 \leq n \leq M, \quad (10)$$

where $\Omega(v) = \int_{v_0}^v (1/\omega(s)) ds$, $v \geq v_0$, Ω^{-1} is the inverse function of Ω , and M is defined by

$$M = \sup \left\{ i : \Omega(a_i) + c_i \sum_{k=0}^{i-1} b_k \in \text{Dom}(\Omega^{-1}) \right\}. \quad (11)$$

3. Main Results

Assume that

(A₁) a_n, b_n are nonnegative functions for $n \in \mathbb{N}$, respectively;

(A₂) $\omega(u) \in C(\mathbb{R}_+, \mathbb{R}_+)$ is nondecreasing and $\omega(0) = 0$.

Define $\tilde{a}_n = \max_{0 \leq k \leq n, k \in \mathbb{N}} a_k$ and $\tau = \max_{0 \leq k \leq n-1, k \in \mathbb{N}} \tau_k$, where τ_k is the variable time step.

Theorem 4. *Under assumptions (A₁) and (A₂), if x_n is nonnegative such that (6), then*

(1) *for $0 < \beta \leq 1/2$, letting $p = 1 + \beta$ and $q = (1 + \beta)/\beta$, one has*

$$x_n \leq \left[\Omega^{-1} \left(\Omega(2^{q-1} \tilde{a}_n^q) + 2^{q-1} \tau^{1-(q/p)\beta} \right. \right. \\ \left. \left. \times K^{q/p}(\beta) e^{q\tau t_n} \sum_{k=0}^{n-1} e^{-q\tau t_k} b_k^q \right) \right]^{1/q}, \quad (12)$$

for $0 \leq n \leq N_1$, where $\Omega(u) = \int_{u_0}^u (1/\omega^q(s^{1/q})) ds$, $u \geq u_0 \geq 0$, Ω^{-1} is the inverse function of Ω ,

$$K(\beta) = (1 + \beta)^{-\beta^2} \Gamma(\beta^2), \quad (13)$$

and N_1 is the largest integer number such that

$$\Omega(2^{q-1} \tilde{a}_n^q) + 2^{q-1} \tau^{1-(q/p)\beta} K^{q/p}(\beta) e^{q\tau t_n} \\ \times \sum_{k=0}^{n-1} e^{-q\tau t_k} b_k^q \in \text{Dom}(\Omega^{-1}); \quad (14)$$

(2) *for $1/2 < \beta \leq 1$, letting $p = 2$ and $q = 2$, one has*

$$x_n \leq \left[\Omega^{-1} \left(\Omega(2 \tilde{a}_n^2) + B(\beta) \tau^{2-2\beta} e^{2\tau t_n} \sum_{k=0}^{n-1} e^{-2\tau t_k} b_k^2 \right) \right]^{1/2}, \quad (15)$$

for $0 \leq n \leq N_2$, where $\Omega(u) = \int_{u_0}^u (1/\omega^2(s^{1/2})) ds$, $u \geq u_0 \geq 0$,

$$B(\beta) = 4^{1-\beta} \Gamma(2\beta - 1), \quad \beta > \frac{1}{2}, \quad (16)$$

and N_2 is the largest integer number such that

$$\Omega(2 \tilde{a}_n^2) + B(\beta) \tau^{2-2\beta} e^{2\tau t_n} \sum_{k=0}^{n-1} e^{-2\tau t_k} b_k^2 \in \text{Dom}(\Omega^{-1}). \quad (17)$$

Proof. By definition of $\tilde{a}_n$ and assumption (A₁), $\tilde{a}_n$ is nonnegative and nondecreasing and $\tilde{a}_n \geq a_n$. It follows from (6) that

$$x_n \leq \tilde{a}_n + \sum_{k=0}^{n-1} (t_n - t_k)^{\beta-1} \tau_k b_k \omega(x_k). \quad (18)$$

- (1) If $0 < \beta \leq 1/2$, using Lemma 2 with the indices $p = 1 + \beta$, $q = (1 + \beta)/\beta$ for (18), we get

$$\begin{aligned} x_n &\leq \tilde{a}_n + \sum_{k=0}^{n-1} (t_n - t_k)^{\beta-1} \tau_k^{1/p} \tau_k^{1/q} e^{\tau t_k} e^{-\tau t_k} b_k \omega(x_k) \\ &\leq \tilde{a}_n + \tau^{1/q} \sum_{k=0}^{n-1} (t_n - t_k)^{\beta-1} \tau_k^{1/p} e^{\tau t_k} e^{-\tau t_k} b_k \omega(x_k) \\ &\leq \tilde{a}_n + \tau^{1/q} \left[\sum_{k=0}^{n-1} (t_n - t_k)^{p(\beta-1)} \tau_k e^{p\tau t_k} \right]^{1/p} \\ &\quad \times \left[\sum_{k=0}^{n-1} e^{-q\tau t_k} b_k^q \omega^q(x_k) \right]^{1/q}. \end{aligned} \quad (19)$$

By Lemma 1, the inequality above yields

$$\begin{aligned} x_n^q &\leq 2^{q-1} \tilde{a}_n^q + 2^{q-1} \tau \left[\sum_{k=0}^{n-1} (t_n - t_k)^{p(\beta-1)} \tau_k e^{p\tau t_k} \right]^{q/p} \\ &\quad \times \left[\sum_{k=0}^{n-1} e^{-q\tau t_k} b_k^q \omega^q(x_k) \right]. \end{aligned} \quad (20)$$

Consider that

$$\begin{aligned} &\sum_{k=0}^{n-1} (t_n - t_k)^{p(\beta-1)} \tau_k e^{p\tau t_k} \\ &\leq \int_0^{t_n} (t_n - s)^{p(\beta-1)} e^{p\tau s} ds \\ &= e^{p\tau t_n} \int_0^{t_n} \eta^{p(\beta-1)} e^{-p\tau \eta} d\eta, \\ &= \frac{e^{p\tau t_n}}{(p\tau)^{1+p(\beta-1)}} \int_0^{p\tau t_n} \sigma^{p(\beta-1)} e^{-\sigma} d\sigma \leq K(\beta) \tau^{-\beta} e^{p\tau t_n}, \end{aligned} \quad (21)$$

where $K(\beta) = (1 + \beta)^{-\beta^2} \Gamma(\beta^2)$ and $\Gamma(z) = \int_0^\infty u^{z-1} e^{-u} du$, ($\text{Re } z > 0$) is the well-known Γ -function. Thus, we have

$$\begin{aligned} x_n^q &\leq 2^{q-1} \tilde{a}_n^q + 2^{q-1} \tau^{1-(q/p)\beta^2} \\ &\quad \times K^{q/p}(\beta) e^{q\tau t_n} \sum_{k=0}^{n-1} e^{-q\tau t_k} b_k^q \omega^q(x_k). \end{aligned} \quad (22)$$

Let $v_n = x_n^q$, $A_n = 2^{q-1} \tilde{a}_n^q$, and $C_n = 2^{q-1} \tau^{1-(q/p)\beta^2} K^{q/p}(\beta) e^{q\tau t_n}$. Obviously, A_n , C_n are nondecreasing for $n \in \mathbb{N}$ and $\omega^q(v_k^{1/q})$ satisfies the assumption (A_2) . Equation (22) can be rewritten as

$$v_n \leq A_n + C_n \sum_{k=0}^{n-1} e^{-q\tau t_k} b_k^q \omega^q(v_k^{1/q}), \quad (23)$$

which is similar to inequality (9). Using Lemma 3, from (23), we have

$$v_n \leq \Omega^{-1} \left[\left(\Omega(A_n) + C_n \sum_{k=0}^{n-1} e^{-q\tau t_k} b_k^q \right) \right], \quad (24)$$

for $0 \leq n \leq N_1$, where N_1 is the largest integer number such that

$$\Omega(A_n) + C_n \sum_{k=0}^{n-1} e^{-q\tau t_k} b_k^q \in \text{Dom}(\Omega^{-1}). \quad (25)$$

Therefore, by $x_n = v_n^{1/q}$, (12) holds for $0 \leq n \leq N_1$.

- (2) If $1/2 < \beta \leq 1$, applying Cauchy-Schwarz inequality for (18), that is, $p = q = 2$, we get

$$\begin{aligned} x_n &\leq \tilde{a}_n + \sum_{k=0}^{n-1} (t_n - t_k)^{\beta-1} \tau_k^{1/2} \tau_k^{1/2} e^{\tau t_k} e^{-\tau t_k} b_k \omega(x_k) \\ &\leq \tilde{a}_n + \tau^{1/2} \sum_{k=0}^{n-1} (t_n - t_k)^{\beta-1} \tau_k^{1/2} e^{\tau t_k} e^{-\tau t_k} b_k \omega(x_k) \\ &\leq \tilde{a}_n + \tau^{1/2} \left[\sum_{k=0}^{n-1} (t_n - t_k)^{2(\beta-1)} \tau_k e^{2\tau t_k} \right]^{1/2} \\ &\quad \times \left[\sum_{k=0}^{n-1} e^{-2\tau t_k} b_k^2 \omega^2(x_k) \right]^{1/2}. \end{aligned} \quad (26)$$

By Lemma 1, the inequality above yields

$$\begin{aligned} x_n^2 &\leq 2\tilde{a}_n^2 + 2\tau \left[\sum_{k=0}^{n-1} (t_n - t_k)^{2(\beta-1)} \tau_k e^{2\tau t_k} \right] \\ &\quad \times \left[\sum_{k=0}^{n-1} e^{-2\tau t_k} b_k^2 \omega^2(x_k) \right]. \end{aligned} \quad (27)$$

Because

$$\begin{aligned} &\sum_{k=0}^{n-1} (t_n - t_k)^{2(\beta-1)} \tau_k e^{2\tau t_k} \\ &\leq \int_0^{t_n} (t_n - s)^{2(\beta-1)} e^{2\tau s} ds \\ &= \frac{e^{2\tau t_n}}{(2\tau)^{2\beta-1}} \int_0^{2\tau t_n} \sigma^{2(\beta-1)} e^{-\sigma} d\sigma \\ &\leq \frac{1}{2} B(\beta) \tau^{1-2\beta} e^{2\tau t_n}, \end{aligned} \quad (28)$$

where $B(\beta) = 4^{1-\beta} \Gamma(2\beta - 1)$, $\beta > 1/2$, it follows from (27) that

$$x_n^2 \leq 2\tilde{a}_n^2 + B(\beta) \tau^{2-2\beta} e^{2\tau t_n} \left[\sum_{k=0}^{n-1} e^{-2\tau t_k} b_k^2 \omega^2(x_k) \right]. \quad (29)$$

Let $v_n = x_n^2$, $A_n = 2\tilde{a}_n^2$, and $C_n = B(\beta)\tau^{2-2\beta}e^{2\tau t_n}$. Similarly, A_n, C_n also are nondecreasing for $n \in \mathbb{N}$ and $\omega^2(v_k^{1/2})$ also satisfies the assumption (A_2) . Equation (29) can be rewritten as

$$v_n \leq A_n + C_n \left(\sum_{k=0}^{n-1} e^{-2\tau t_k} b_k^2 \omega^2(v_k^{1/2}) \right), \quad (30)$$

which also is similar to inequality (9). Using Lemma 3, from (30), we have

$$v_n \leq \left[\Omega^{-1} \left(\Omega(A_n) + C_n \sum_{k=0}^{n-1} e^{-2\tau t_k} b_k^2 \right) \right], \quad (31)$$

for $0 \leq n \leq N_2$, and N_2 is the largest integer number such that

$$\Omega(A_n) + C_n \sum_{k=0}^{n-1} e^{-2\tau t_k} b_k^2 \in \text{Dom}(\Omega^{-1}). \quad (32)$$

Clearly, by $x_n = v_n^{1/2}$, (15) also holds for $0 \leq n \leq N_2$. $\square$

Remark 5. Here, we note that the most significant work in the study of weakly singular inequalities is Medved's method, originally presented in the paper [6] and also applied in the paper [18]. But his result holds under the assumption " $\omega(u)$ satisfies the condition (q)," that is, " $e^{-qt}[\omega(u)]^q \leq R(t)\omega(e^{-qt}u^q)$, where $R(t)$ is a continuous, nonnegative function." In our result, the condition (q) is eliminated.

Corollary 6. Under assumptions (A_1) and (A_2) , let $\nu > 0$, $\mu > 0(\nu > \mu)$. If x_n is nonnegative such that

$$x_n^\nu \leq a_n + \sum_{k=0}^{n-1} (t_n - t_k)^{\beta-1} \tau_k b_k x_k^\mu, \quad (33)$$

then

- (1) if $0 < \beta \leq 1/2$, let $p = 1 + \beta$ and $q = (1 + \beta)/\beta$, and one gets

$$x_n \leq \left[\left(2^{q-1} \tilde{a}_n^q \right)^{(\nu-\mu)/\nu} + \frac{\nu-\mu}{\nu} 2^{q-1} \tau^{1-(q/p)\beta^2} \times K^{q/p}(\beta) e^{q\tau t_n} \sum_{k=0}^{n-1} e^{-q\tau t_k} b_k^q \right]^{1/((\nu-\mu)q)} \quad (34)$$

for $n \geq 0$, where $K(\beta)$ is defined as in Theorem 4;

- (2) if $1/2 < \beta \leq 1$, let $p = q = 2$, and one gets

$$x_n \leq \left[\left(2\tilde{a}_n^2 \right)^{(\nu-\mu)/\nu} + \frac{\nu-\mu}{\nu} B(\beta) \tau^{2-2\beta} \times e^{2\tau t_n} \sum_{k=0}^{n-1} e^{-2\tau t_k} b_k^2 \right]^{1/2(\nu-\mu)}, \quad (35)$$

for $n \geq 0$, where $B(\beta)$ is defined as in Theorem 4

Proof. Let $z_n = x_n^\nu$, then $x_n = z_n^{1/\nu}$ and $x_n^\mu = z_n^{\mu/\nu}$. From (33), we have

$$z_n \leq a_n + \sum_{k=0}^{n-1} (t_n - t_k)^{\beta-1} \tau_k b_k z_k^{\mu/\nu}. \quad (36)$$

Clearly, $\omega(z_k) = z_k^{\mu/\nu}$ satisfies the assumption (A_2) . According to the definition of Ω in Theorem 4, for $0 < \beta \leq 1/2$, letting $u_0 = 0$, we have

$$\Omega(u) = \int_{u_0}^u \frac{1}{\omega^q(s^{1/q})} ds = \int_0^u \frac{ds}{s^{\mu/\nu}} = \frac{\nu}{\nu-\mu} u^{(\nu-\mu)/\nu}, \quad (37)$$

$$\Omega^{-1}(u) = \left(\frac{\nu-\mu}{\nu} u \right)^{\nu/(\nu-\mu)}, \quad \text{Dom}(\Omega^{-1}) = [0, \infty). \quad (38)$$

It can be seen easily from (38) that $N_1 = \infty$. Substituting (37) and (38) into (12), we get

$$z_n \leq \left[\left(2^{q-1} \tilde{a}_n^q \right)^{(\nu-\mu)/\nu} + \frac{\nu-\mu}{\nu} 2^{q-1} \tau^{1-(q/p)\beta^2} \times K^{q/p}(\beta) e^{q\tau t_n} \sum_{k=0}^{n-1} e^{-q\tau t_k} b_k^q \right]^{\nu/((\nu-\mu)q)}. \quad (39)$$

In view of $x_n = z_n^{1/\nu}$, we can obtain (34). For the case that $1/2 < \beta \leq 1$, in fact, Ω and Ω^{-1} are the same as (37) and (38), respectively. So, it follows from (37), (38), and (15) that

$$x_n \leq \left[\left(2\tilde{a}_n^2 \right)^{(\nu-\mu)/\nu} + \frac{\nu-\mu}{\nu} B(\beta) \tau^{2-2\beta} \times e^{2\tau t_n} \sum_{k=0}^{n-1} e^{-2\tau t_k} b_k^2 \right]^{1/2(\nu-\mu)}, \quad (40)$$

for $n > 0$. $\square$

Remark 7. In [11], Yang et al. investigated inequality (33), under the assumption that a_n is nondecreasing. Clearly, our result does not need such condition, and we get a more concise formula.

Remark 8. Letting $\nu = 2$ and $\mu = 1$, we can get the interesting Henry version of the Ou-Iang-Pachpatte-type difference inequality [3]. Thus, our result is a more general discrete analogue for such inequality.

Corollary 9. Under assumptions (A_1) and (A_2) , if x_n is nonnegative such that

$$x_n \leq a_n + \sum_{k=0}^{n-1} (t_n - t_k)^{\beta-1} \tau_k b_k x_k, \quad (41)$$

then

- (1) if $0 < \beta \leq 1/2$, let $p = 1 + \beta$ and $q = (1 + \beta)/\beta$, and one gets

$$x_n \leq 2^{(q-1)/q} \tilde{a}_n \exp \left(2^{(q-1)/q} \tau^{1-(q/p)\beta^2} K^{q/p}(\beta) \times e^{q\tau t_n} \sum_{k=0}^{n-1} e^{-q\tau t_k} b_k^q \right), \quad (42)$$

for $n \geq 0$, where $K(\beta)$ is defined as in Theorem 4;

- (2) if $1/2 < \beta \leq 1$, let $p = q = 2$, and one gets

$$x_n \leq \sqrt{2} \tilde{a}_n \exp \left(\frac{1}{2} B(\beta) \tau^{2-2\beta} e^{2\tau t_n} \sum_{k=0}^{n-1} e^{-2\tau t_k} b_k^2 \right), \quad (43)$$

for $n \geq 0$, where $B(\beta)$ is defined as in Theorem 4.

Proof. In (41), $\omega(u) = u$ also satisfies the assumption (A_2) . Thus, we have

$$\Omega(u) = \int_{u_0}^u \frac{ds}{s} = \ln \frac{u}{u_0}, \quad \Omega^{-1}(u) = u_0 \exp(u), \quad (44)$$

$$\text{Dom}(\Omega^{-1}) = [0, \infty).$$

Similarly to the computation in Corollary 6, the estimates (42) and (43) hold, respectively. $\square$

4. Application

In this section, we apply our results to discuss the upper bound of solution of a Volterra type difference equation with weakly singular kernel.

Consider the following the inequality:

$$x_n \leq 1 + \sum_{k=0}^{n-1} (t_n - t_k)^{-1/2} \tau_k \sqrt{x_k}. \quad (45)$$

Obviously, (45) is the special case of inequality (6), then we get

$$a_n = 1, \quad \beta = \frac{1}{2}, \quad \omega = \sqrt{u}. \quad (46)$$

Thus, we can take $p = 1 + \beta = 3/2$ and $q = (1 + \beta)/\beta = 3$; then, $q/p = 2$. Moreover,

$$\tilde{a}_n = 1,$$

$$K(\beta) = (1 + \beta)^{-\beta^2} \Gamma(\beta^2) = \left(\frac{3}{2}\right)^{-1/4} \Gamma\left(\frac{1}{4}\right), \quad (47)$$

$$\Omega(u) = \int_0^u \frac{ds}{\sqrt{s}} = 2\sqrt{u}, \quad \Omega^{-1}(u) = \frac{u^2}{4}.$$

According to Theorem 4, we obtain

$$x_n \leq \left[\Omega^{-1} \left(\Omega \left(2^{q-1} \tilde{a}_n^q \right) + 2^{q-1} \tau^{1-(q/p)\beta^2} \right) \times K^{q/p}(\beta) e^{q\tau t_n} \sum_{k=0}^{n-1} e^{-q\tau t_k} b_k^q \right]^{1/q}$$

$$= \left[\Omega^{-1} \left(\Omega(4) + 4\tau^{1/2} \left(\frac{3}{2}\right)^{-1/2} \right) \times \Gamma^2\left(\frac{1}{4}\right) e^{3\tau t_n} \sum_{k=0}^{n-1} e^{-3\tau t_k} b_k^3 \right]^{1/3}$$

$$= \left[\Omega^{-1} \left(4 + \frac{4}{3} \sqrt{6} \tau^{1/2} \Gamma^2\left(\frac{1}{4}\right) e^{3\tau t_n} \sum_{k=0}^{n-1} e^{-3\tau t_k} b_k^3 \right) \right]^{1/3}$$

$$= 4^{-1/3} \left(4 + \frac{4}{3} \sqrt{6} \tau^{1/2} \Gamma^2\left(\frac{1}{4}\right) e^{3\tau t_n} \sum_{k=0}^{n-1} e^{-3\tau t_k} b_k^3 \right)^{2/3} \quad (48)$$

for $n > 0$, which indicates that we get the upper bound of x_n .

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

- [1] R. P. Agarwal, S. Deng, and W. Zhang, "Generalization of a retarded Gronwall-like inequality and its applications," *Applied Mathematics and Computation*, vol. 165, no. 3, pp. 599–612, 2005.
- [2] S. Deng, "Nonlinear discrete inequalities with two variables and their applications," *Applied Mathematics and Computation*, vol. 217, no. 5, pp. 2217–2225, 2010.
- [3] B. G. Pachpatte, "On some fundamental integral inequalities and their discrete analogues," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 2, no. 2, article 15, 2001.
- [4] Y. Wu, X. Li, and S. Deng, "Nonlinear delay discrete inequalities and their applications to Volterra type difference equations," *Advances in Difference Equations*, vol. 2010, Article ID 795145, 14 pages, 2010.
- [5] M. Medveď, "A new approach to an analysis of Henry type integral inequalities and their Bihari type versions," *Journal of Mathematical Analysis and Applications*, vol. 214, no. 2, pp. 349–366, 1997.

- [6] M. Medveď, “Nonlinear singular difference inequalities suitable for discretizations of parabolic equations,” *Demonstratio Mathematica*, vol. 33, no. 3, pp. 517–525, 2000.
- [7] K. Zheng, H. Wang, and C. Guo, “On nonlinear discrete weakly singular inequalities and applications to Volterra-type difference equations,” *Advances in Difference Equations*, vol. 2013, article 239, 2013.
- [8] J. Dixon and S. McKee, “Weakly singular discrete Gronwall inequalities,” *Zeitschrift für Angewandte Mathematik und Mechanik*, vol. 66, no. 11, pp. 535–544, 1986.
- [9] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, vol. 840 of *Lecture Notes in Mathematics*, Springer, New York, NY, USA, 1981.
- [10] M. Slodička, “Semigroup formulation of Rothe’s method: application to parabolic problems,” *Commentationes Mathematicae Universitatis Carolinae*, vol. 33, no. 2, pp. 245–260, 1992.
- [11] E.-H. Yang, Q.-H. Ma, and M.-C. Tan, “Discrete analogues of a new class of nonlinear Volterra singular integral inequalities,” *Journal of Jinan University*, vol. 28, no. 1, pp. 1–6, 2007.
- [12] W.-S. Cheung, Q.-H. Ma, and S. Tseng, “Some new nonlinear weakly singular integral inequalities of Wendroff type with applications,” *Journal of Inequalities and Applications*, vol. 2008, Article ID 909156, 12 pages, 2008.
- [13] S. Deng and C. Prather, “Generalization of an impulsive nonlinear singular Gronwall-Bihari inequality with delay,” *Journal of Inequalities in Pure and Applied Mathematics*, vol. 9, no. 2, article 34, 2008.
- [14] F. Lakhal, “A new nonlinear integral inequality of Wendroff type with continuous and weakly singular kernel and its application,” *Journal of Mathematical Inequalities*, vol. 6, no. 3, pp. 367–379, 2012.
- [15] Q. H. Ma and E. H. Yang, “Estimates on solutions of some weakly singular Volterra integral inequalities,” *Acta Mathematicae Applicatae Sinica*, vol. 25, no. 3, pp. 505–515, 2002.
- [16] Q. H. Ma and J. Pečarić, “Some new explicit bounds for weakly singular integral inequalities with applications to fractional differential and integral equations,” *Journal of Mathematical Analysis and Applications*, vol. 341, no. 2, pp. 894–905, 2008.
- [17] S. Mazouzi and N.-E. Tatar, “New bounds for solutions of a singular integro-differential inequality,” *Mathematical Inequalities & Applications*, vol. 13, no. 2, pp. 427–435, 2010.
- [18] M. Medveď, “On singular versions of Bihari and Wendroff-Pachpatte type integral inequalities and their application,” *Tatra Mountains Mathematical Publications*, vol. 38, pp. 163–174, 2007.
- [19] S. Szufła, “On the Volterra integral equation with weakly singular kernel,” *Mathematica Bohemica*, vol. 131, no. 3, pp. 225–231, 2006.
- [20] N.-E. Tatar, “On an integral inequality with a kernel singular in time and space,” *Journal of Inequalities in Pure and Applied Mathematics*, vol. 4, no. 4, article 82, 2003.
- [21] H. Wang and K. Zheng, “Some nonlinear weakly singular integral inequalities with two variables and applications,” *Journal of Inequalities and Applications*, vol. 2010, Article ID 345701, 12 pages, 2010.
- [22] K. Zheng, “Bounds on some new weakly singular Wendroff-type integral inequalities and applications,” *Journal of Inequalities and Applications*, vol. 2013, article 159, 2013.