Research Article

Infinite-Dimensional Modular Lie Superalgebra \( \Omega \)

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1. Introduction

The theory of modular Lie superalgebras has obtained many important results during the last twenty years (e.g., see [1–4]). But the complete classification of the simple modular Lie superalgebras remains an open problem. We know that filtration structures play an important role both in the classification of modular Lie algebras and nonmodular Lie superalgebras (see [5–8]). The natural filtrations of finite-dimensional modular Lie algebras of Cartan type were proved to be invariant in [9, 10]. The similar result was obtained for the infinite-dimensional case [11]. In the case of finite-dimensional modular Lie superalgebras of Cartan type, the invariance of the natural filtration was discussed in [12, 13]. The same conclusion was obtained for some infinite-dimensional modular Lie superalgebras of Cartan type (see [14–17]).

In the present paper, we consider the infinite-dimensional modular Lie superalgebra \( \Omega(r, m, q) \), which was studied in paper [18]. Denote the natural filtration by \( (\Omega(r, m, q))_{(i)} \). We show that the filtration is invariant under automorphisms by determining ad-nilpotent elements and subalgebras generated by certain ad-nilpotent elements. We are thereby able to obtain an intrinsic characterization of Lie superalgebra \( \Omega(r, m, q) \).

The paper is organized as follows. In Section 2, we recall some necessary definitions concerning the modular Lie superalgebra \( \Omega \). In Section 3, we establish some technical lemmas which will be used to determine the invariance of the filtration. In Section 4, we prove that the natural filtration \( (\Omega(r, m, q))_{(i)} \) is invariant. Furthermore, we obtain the sufficient and necessary conditions of \( \Omega(r, m, q) \equiv \Omega(r', m', q') \); that is, all the Lie superalgebras are classified up to isomorphisms.

2. Preliminaries

Throughout the work \( \mathbb{F} \) denotes an algebraically closed field of characteristic \( p > 3 \) and \( \mathbb{F} \) is not equal to its prime field \( \Pi \). Let \( \mathbb{Z}_2 = \{0, 1\} \) be the ring of integers modulo 2. Let \( \mathbb{N} \) and \( \mathbb{N}_0 \) denote the sets of positive integers and nonnegative integers, respectively. For \( m > 0 \), let \( \mathbb{E} = \{z_1, \ldots, z_m\} \in \mathbb{F} \) be a subset of \( \mathbb{F} \) that is linearly independent over the prime field \( \Pi \), and let \( H \) be the additive subgroup generated by \( \mathbb{E} \) that does not contain 1. If \( \lambda \in H \), then we let \( \lambda = \sum_{i=1}^{m} \lambda_i z_i \) and \( y^\lambda = y_1^{\lambda_1} \cdots y_m^{\lambda_m} \), where \( 0 \leq \lambda_i < p \).

Given \( n \in \mathbb{N} \) and \( r = 2n + 2 \), we put \( M = \{1, \ldots, r-1\} \). Let \( \mu_1, \ldots, \mu_{r-1} \in \mathbb{F} \) and \( \mu_1 = 0, \mu_j + \mu_{r-j} = 1, j = 2, \ldots, n + 1 \). If \( k_i \in \mathbb{N}_0 \), then \( k_i \) can be uniquely expressed in \( p \)-adic form \( k_i = \sum_{v=0}^{\infty} e_v(k_i) p^v \), where \( 0 \leq e_v(k_i) < p \). We set \( x_i^{k_i} = \prod_{v=0}^{\infty} x_i^{e_v(k_i)} \). We define a truncated polynomial algebra

\[
A = \mathbb{F} [x_{10}, x_{11}, \ldots, x_{20}, x_{21}, \ldots, x_{r-1,0}, x_{r-1,1}, \ldots, y_1, \ldots, y_m]
\]
such that
\[ x^p_j = 0, \ \forall i \in M, \ j = 0, 1, \ldots, \\
 y^p_j = 1, \ i = 1, \ldots, m. \]  
(2)
For \( k_i, k'_i \in \mathbb{N}_0 \), it is easy to see that
\[ x^k_j x^{k'}_j = x^{k+k'}_j \neq 0 \iff \varepsilon_i(k_i) + \varepsilon_i(k'_i) < p, \]
(3)
Let \( Q = \{(k_1, \ldots, k_{r-1}) \mid k_i \in \mathbb{N}_0, i \in M \}. \) If \( k = (k_1, \ldots, k_{r-1}) \in Q \), we set \( x^k = x^1_1 x^2_2 \cdots x^{k_{r-1}}_{r-1} \).

Let \( \Lambda(q) \) be the Grassmann superalgebra over \( F \) in \( q \) variables \( \xi_1, \ldots, \xi_q \), where \( q \in \mathbb{N} \) and \( q > 1 \). Denote by \( \Omega \) the tensor product \( A \otimes F \Lambda(q) \). The trivial \( \mathbb{Z}_2 \)-gradation of \( A \) and the natural \( \mathbb{Z}_2 \)-gradation of \( \Lambda(q) \) induce a \( \mathbb{Z}_2 \)-gradation of \( \Omega \) such that \( \Omega \) is an associative superalgebra:
\[
\Omega_\nu = A \otimes F \Lambda(q)_\nu, \quad \nu = 0, 1, \ldots.
\]
(4)
For \( f \in A \) and \( g \in \Lambda(q) \), we abbreviate \( f \odot g \) to \( fg \). Let
\[
B_k = \{ (i_1, i_2, \ldots, i_k) \mid r + 1 \leq i_1 < i_2 < \cdots < i_k \leq r + q \},
\]
(5)
and where \( B_0 = \emptyset \). Given \( u = (i_1, \ldots, i_k) \in B_k \), we set \( |u| = k, \{u\} = \{i_1, \ldots, i_k\} \) and \( \xi^u = \xi_{i_1} \cdots \xi_{i_k} |\{\emptyset\} = 0, \xi^\sigma = 1\).

Then \( x^k y^\lambda \xi^u \mid k \in Q, \lambda \in H, u \in B(q) \) is an \( F \)-basis of \( \Omega \).

If \( |f| \) appears in some expression in this paper, we always regard \( x \) as a \( \mathbb{Z}_2 \)-homogeneous element and \( |f| \) as the \( \mathbb{Z}_2 \)-degree of \( f \).

Let \( s = r + q, T = \{r + 1, \ldots, s\} \) and \( R = M \cup T \). Put \( M_1 = \{2, \ldots, r\} \) and \( x_i = x^i \mid \bar{i} \in M \). Define \( \bar{i} = \bar{0}, \bar{i} = \bar{1}, \) and \( \bar{i} = \bar{1} \), if \( i \in T \). Let
\[
i' = \begin{cases} i + n, & 2 \leq i \leq n + 1, \\
i - n, & n + 2 \leq i \leq r - 1, \\
i, & r + 1 \leq i \leq s, \end{cases}
\]
and
\[
[i] = \begin{cases} 1, & 2 \leq i \leq n + 1, \\
-1, & n + 2 \leq i \leq r - 1, \\
1, & r + 1 \leq i \leq s. \end{cases}
\]
(6)
Let \( D_1, D_2, \ldots, D_s \) be the linear transformations of \( \Omega \) such that
\[
D_i (x^k y^\lambda \xi^u) = \left\{ \begin{array}{ll} k^* x^{k-x_i} y^\lambda \xi^u, & i \in M, \\
x^k y^\lambda \partial^u \xi^u / \partial x_i, & i \in T, \end{array} \right. \]
where \( k^* \) is the first nonzero number of \( \varepsilon_i(k_i), \varepsilon_j(k_j), \ldots, \varepsilon_s(k_s) \). Then \( D_1, D_2, \ldots, D_s \) are superderivations of the superalgebra \( \Omega \) and \( |D_i| = \bar{i} \).

Let \( I \) be the identity mapping of \( \Omega \). Let \( f \in \Omega \) be a \( \mathbb{Z}_2 \)-homogeneous element and \( g \in \Omega \); we define a bilinear operation in \( \Omega \) such that
\[
[f, g] = D_1 (f \bar{\partial} (g) - \bar{\partial} (f) D_1 (g)) + \sum_{i \in M \cup T} [i] (-1)^{[i]/2} D_i (f \bar{\partial} (g)).
\]
(9)
Then \( \Omega \) becomes a simple Lie superalgebra. If \( 2n + 4 - q \neq 0 \) (mod \( p \)), we see that \( \lambda + 2^{-1} q - n - 2 \neq 0 \) in the sequel, we always assume that \( 2n + 4 - q \neq 0 \) (mod \( p \)). In some cases, we denote \( \Omega \) by \( \Omega(r, m, q) \) in detail and call \( \Omega(r, m, q) \) the Lie superalgebra of \( \Omega \)-type.

Now we give a \( \mathbb{Z} \)-gradation of \( \Omega \):
\[
\Omega_j = \text{span}_F \left\{ x^k y^\lambda \xi^u \mid \sum k_j + 2 k_1 + |u| - 2 = j \right\}.
\]
(10)
where \( \Omega_{[i]} = \bigoplus \Omega_j \) for all \( i \geq -2 \). Then \( \Omega = \Omega_{[-2]} \supset \Omega_{[-1]} \supset \cdots \) are called the natural filtration of \( \Omega \).

3. Ad-nilpotent Elements

Let \( L \) be a Lie superalgebra. Recall that an element \( y \in L \) is called ad-nilpotent if there exists a \( t \in \mathbb{N} \) such that \( (ad_y)^t (L) = 0 \). If \( y \in L \) is ad-nilpotent, it is also called ad-nilpotent in brief. Let \( G \) be a subset of \( L \). Put \( \text{nil}(G) = \{x \in G \mid x \text{ is ad-nilpotent}\} \), and \( \text{nil}(G) \) is the subalgebra of \( L \) generated by \( \text{nil}(G) \).

Let \( a \in \mathbb{N}_0 \) and \( a = \sum \varepsilon_i(a) 2^i \) be the \( p \)-adic expression of \( a \), where \( 0 \leq \varepsilon_i(a) < p \). Then
\[
\text{pad}(a) = (\text{pad}_0(a), \text{pad}_1(a), \text{pad}_2(a), \ldots)
\]
(11)
is called the \( p \)-adic sequence of \( a \), where \( \text{pad}_i(a) = \varepsilon_i(a) \) for all \( v \in \mathbb{N}_0 \). For \( k = (k_1, k_2, \ldots, k_{r-1}) \in \mathbb{Q} \), we define the \( p \)-adic matrix of \( k \) to be
\[
\text{pad}(k) = \begin{pmatrix} \text{pad}(k_1) \\
\text{pad}(k_2) \\
\vdots \\
\text{pad}(k_{r-1}) \end{pmatrix}.
\]
(12)
Since \( \text{pad}(k) \) is a \((r-1) \times \infty\) matrix with only finitely many nonzero elements, we can set
\[
ht(k) = \max \left\{ j \in \mathbb{N} \mid \exists i \in M, \text{pad}_j(k_i) \neq 0 \right\}.
\]
(13)
If \( z = \sum k_{\lambda, \mu} a_{\lambda, \mu} x^k y^\lambda \xi^u \in \Omega \) is a nonzero element with \( a_{\lambda, \mu} \in F \), then we may assume
\[
ht(z) = \max \left\{ \text{ht}(k) \mid a_{\lambda, \mu} \neq 0 \right\}.
\]
(14)
For \( c, d \in \mathbb{N}_0 \), we define \( \|k\|_c,d := \sum \varepsilon_j(c) \text{pad}_j(k_i) \) and \( \|k\|_d := \|k\|_c,d \). Now for any \( t \in \mathbb{N} \) and \( x^k y^\lambda \xi^u \in \Omega \), define
\[
\delta_t (x^k y^\lambda \xi^u) = \|k\|_1 + 2\|k\|_1 + |u| + \text{pad}_0 (k_1).
\]
(15)
Lemma 1. Let \( k, k' \in Q \) and \( t \in \mathbb{N}_0 \). Then the following statements hold.

(i) \( x^k x^{k'} \neq 0 \Rightarrow \text{pad}(k) + \text{pad}(k') = \text{pad}(k + k') \).

(ii) If \( k_i^2 \neq 0 \), then \( \|k' - e_i\|_1 + 2\|k' - e_i\|_{1,2} \geq \|k\|_b + 2\|k\|_{1,2} - 1 \), where \( i \in M_1 \).

(iii) Let \( x^k y^\lambda \xi u \in \Omega_{[1]} \) and \( t \geq ht(x^k y^\lambda \xi u) \). Then \( \delta_t(x^k y^\lambda \xi u) \geq 3 \).

Proof. (i) We see that

\[
\text{pad}(k) + \text{pad}(k') = \begin{pmatrix}
\text{pad}(k_1) + \text{pad}(k'_1) \\
\text{pad}(k_2) + \text{pad}(k'_2) \\
\vdots \\
\text{pad}(k_n) + \text{pad}(k'_n)
\end{pmatrix}
\]

Note that \( x^k x^{k'} \neq 0 \iff e_v(k_i) + e_v(k'_i) < 1 \iff \text{pad}_v(k_i) + \text{pad}_v(k'_i) < 1 \iff \text{pad}_v(k_i) + \text{pad}_v(k'_i) \neq 0 \pmod{p} \), for all \( i \in M, v \in \mathbb{N}_0 \). By the uniqueness of \( p \)-adic expression, we have \( x^k x^{k'} \neq 0 \iff \text{pad}_v(k_i) + \text{pad}_v(k'_i) = \text{pad}_v(k_i + k'_i) \), for all \( i \in M, v \in \mathbb{N}_0 \). Thus

\[
x^k x^{k'} \neq 0 \iff \text{pad}(k) + \text{pad}(k') = \begin{pmatrix}
\text{pad}(k_1) + \text{pad}(k'_1) \\
\text{pad}(k_2) + \text{pad}(k'_2) \\
\vdots \\
\text{pad}(k_n) + \text{pad}(k'_n)
\end{pmatrix} = \text{pad}(k + k')
\]

as desired.

(ii) If \( \text{pad}_v(k'_i) \neq 0 \), then \( \text{pad}_v(k'_i - 1) = \text{pad}_v(k'_i - 1, \text{pad}_v(k'_i)) \). We see that \( \|k' - e_i\|_1 = \|k\|_1 - 1 \), \( \|k' - e_i\|_{1,2} = \|k\|_{1,2} \). So (ii) holds.

If \( \text{pad}_v(k'_i) = 0 \) and \( \text{pad}_v(k'_i) \neq 0 \) for \( b \geq 1 \), then we can assume that

\[
\text{pad}(k'_i) = (0, \ldots, 0, \text{pad}_b(k'_i), \text{pad}_{b+1}(k'_i), \ldots).
\]

Hence \( \text{pad}((k'_i - 1) = (p-1, \ldots, p-1, \text{pad}_b(k'_i), \text{pad}_{b+1}(k'_i), \ldots) \).

Lemma 2. Let \( x^k y^\lambda \xi u \in \Omega, x^k y^\lambda \xi u \in \Omega_{[1]}, t \geq \max\{1, \ht(x^k y^\lambda \xi u)\}, i \in M_1 \cup T \). Then the following statements hold.

(i) If \( x^k y^\lambda \xi u D_i(x^k y^\lambda \xi u) \neq 0 \), then \( \delta_t(x^k y^\lambda \xi u D_i(x^k y^\lambda \xi u)) \geq \delta_t(x^k y^\lambda \xi u) + 1 \).

(ii) If \( x^k y^\lambda \xi u D_i(x^k y^\lambda \xi u) \neq 0 \), then \( \delta_t(x^k y^\lambda \xi u D_i(x^k y^\lambda \xi u)) \geq \delta_t(x^k y^\lambda \xi u) + 1 \).

(iii) If \( D_i(x^k y^\lambda \xi u D_j(x^k y^\lambda \xi u) \neq 0 \), then \( \delta_t(D_i(x^k y^\lambda \xi u D_j(x^k y^\lambda \xi u)) \geq \delta_t(x^k y^\lambda \xi u) + 1 \).

Proof. (i) As \( x^k y^\lambda \xi u k_1 x^{k-\xi} y^\lambda \xi u = k_1 x^k y^{k-\varepsilon} y^{n+\lambda} \xi u \neq 0 \), \( x^k x^{k-\varepsilon} y^{n+\lambda} \xi u \neq 0 \). Then we have

\[
\|k' - e_i\|_1 + 2\|k' - e_i\|_{1,2} + |u| + \text{pad}_b(k'_i + (k_i - 1))
\]

\[
\|k' - e_i\|_1 + 2\|k' - e_i\|_{1,2} + |u| + \text{pad}_b(k'_i + (k_i - 1)).
\]
By the equality above and Lemma 1, we get \( \text{pad}(k'+(k-e_1)) = \text{pad}(k') + \text{pad}(k-e_1) \). Thus

\[
\|k' + (k-e_1)\|_1 = \|k'\|_1 + \|k-e_1\|_1,
\]

\[
\|k' + (k-e_1)\|_{ij} = \|k'\|_{ij} + \|k-e_1\|_{ij}.
\]

Hence

\[
\delta_i \left( x^k y^{\lambda} \xi^u \right)
\]

\[
= \delta_i \left( k' + k, x^{(k'+e_1)+(k-e_1)} y^{\lambda+\eta} \xi^{(\xi)} \right)
\]

\[
= \|k' + (k-e_1)\|_1 + 2\|k\|_1 + |v| + \text{pad}_0 (k_1)
\]

\[
\geq \delta_i \left( x^k y^{\lambda} \xi^u \right) + \|k\|_1 + |v| + \text{pad}_0 (k_1).
\]

\[
\geq \delta_i \left( x^k y^{\lambda} \xi^u \right) + 1.
\]

For \( i \in T \), it is easily seen that \( \varepsilon_i(k' + e_1) + \varepsilon_i(k-e_1) = \varepsilon_i(k') + \varepsilon_i(k) < p \). Also by Lemma 1, we obtain

\[
\delta_i \left( D_1 \left( x^k y^{\lambda} \xi^u \right) \right)
\]

\[
= \delta_i \left( x^k y^{\lambda} \xi^u \right) + 3 - 2
\]

\[
= \delta_i \left( x^k y^{\lambda} \xi^u \right) + 1.
\]

Hence Lemma 2 holds.

\( \Box \)

**Lemma 3.** Let \( x^k y^{\lambda} \xi^u \in \Omega, x^k y^{\lambda} \xi^u \in \Omega[I] \) and \( t \geq \max(1, \text{ht}(x^k y^{\lambda} \xi^u)) \). Let \( x^k y^{\lambda} \xi^u \) be a nonzero summand of \( \{x^k y^{\lambda} \xi^u, x^k y^{\lambda} \xi^u\} \). Then \( \delta_i(x^k y^{\lambda} \xi^u) \geq \delta_i(x^k y^{\lambda} \xi^u) + 1 \).

**Proof.** By a direct computation, we obtain that

\[
\left[ x^k y^{\lambda} \xi^u, x^k y^{\lambda} \xi^u \right]
\]

\[
= D_1 \left( x^k y^{\lambda} \xi^u \right) \partial \left( x^k y^{\lambda} \xi^u \right) - D_1 \left( x^k y^{\lambda} \xi^u \right) \partial \left( x^k y^{\lambda} \xi^u \right)
\]

\[
+ \sum_{i \in I} \{i\} D_i \left( x^k y^{\lambda} \xi^u \right) D_i \left( x^k y^{\lambda} \xi^u \right),
\]

which satisfies the conditions of Lemma 2.

\( \Box \)

**Lemma 4.** \( \Omega[I] \subseteq \text{nil}(\Omega) \).

**Proof.** Given \( t \in \mathbb{N} \), put

\[
l_t = (r-1)(t+1) + 2(r-1) t (p-1) + (p+1) + n + 1.
\]

Clearly, we have \( \delta_i(x^k y^{\lambda} \xi^u) < l_t \) for all standard basis element \( x^k y^{\lambda} \xi^u \) of \( \Omega \).

Let \( t \in \mathbb{N} \) such that \( t \geq \text{ht}(z) \). For any \( z = \sum_{k \in \mathbb{Z}} x^k y^{\lambda} \xi^u \in \Gamma[I] \) with \( 0 \neq \xi_{k_1} \in \mathbb{F} \), we have

\[
\text{adx} \left( x^k y^{\lambda} \xi^u \right) = \left[ z, x^k y^{\lambda} \xi^u \right] = D_1 \left( z \right) \partial \left( x^k y^{\lambda} \xi^u \right)
\]

\[
- \partial \left( z \right) D_1 \left( x^k y^{\lambda} \xi^u \right)
\]

\[
+ \sum_{i \in I} \{i\} D_i \left( z \right) D_i \left( x^k y^{\lambda} \xi^u \right).
\]

Note that \( t \geq \max(1, \text{ht}(x^k y^{\lambda} \xi^u)) \). By using Lemma 3 repeatedly we see that \( \text{adx} \left( x^k y^{\lambda} \xi^u \right) = 0 \). Hence \( \Omega[I] \subseteq \text{nil}(\Omega) \).

\( \Box \)

**Lemma 5.** (i) If \( f = \sum_{i=1}^n f_i \in \text{nil}(\Omega) \), where \( f_i \in \Omega \), then \( f_i \in \text{nil}(\Omega) \).

(ii) If \( f = \sum_{i=1}^n f_i \in \text{nil}(\Omega_2) \), then \( f_2 = 0 \).

(iii) If \( f = \sum_{i=1}^n f_i \in \text{nil}(\Omega_2) \), then \( f_2 = 0 \).

(iv) \( \text{Nil}(\Omega_1) \cap \Omega_2 = \text{Nil}(\Omega_2) \cap \Omega_1 \cap \Omega_2 \).

(v) \( \text{Nil}(\Omega_2) = \text{Nil}(\Omega_1) \cap \Omega_2 \).

**Proof.** (i) See Lemma 5 in [14].

(ii) By (i), we see that \( f_2 \) is ad-nilpotent. If \( f_2 \neq 0 \), then \( [y^i, x^m] = -D_1(x^{(m+1)}y^{\lambda-1}) = (\lambda-1)k_1^*x^{(m-1)}y^{\lambda-1} \). By a direct computation, we obtain \( (\lambda-1)^m(1-\lambda)^m \neq 0 \), contradicting the nilpotency of \( f_2 \). Hence \( f_2 = 0 \), as desired.

\( \Box \)
(iii) Clearly, \( f_{-1} \) is ad-nilpotent by virtue of (i). If \( f_{-1} \neq 0 \), then we can suppose that \( f_{-1} = \sum_{i} y_i x_i \neq 0 \), where \( y_i \in F \). Thus there exists some \( y_i \neq 0 \). By computation, we have

\[
\left[ \Sigma y_i x_i y^i, x^{m\varepsilon} \right] = \sum_{j} \left( \delta_{jj} (m_j + m_j) \right) D_j^{m_j} D_j^{m_j} \left( x^{m\varepsilon} \right) = [j] y_j k^j x^{(m-1)j} x^j y^j.
\]

Similarly, we get \( \left( \text{ad} f_{-1} \right)^m \left( x^{m\varepsilon} \right) = [j]^{m_j} \left( k^j \right)^m y^m \neq 0 \), a contradiction. Consequently, \( f_{-1} = 0 \).

(iv) Suppose that \( f = f_0 + f\{1\} \) is an arbitrary element of \( \text{Nil} \left( \Omega_0 \cap \Omega_T \right) \), where \( f_0 \in \Omega_0 \cap \Omega_T \) and \( f\{1\} \in \Omega_1 \cap \Omega_T \). By (i) of this lemma, we see that \( f_0 \in \text{Nil} \left( \Omega_0 \cap \Omega_T \right) \subseteq \text{Nil} \left( \Omega_0 \cap \Omega_T \right) \). Hence \( \text{Nil} \left( \Omega_0 \cap \Omega_T \right) \subseteq \text{Nil} \left( \Omega_0 \cap \Omega_T \right) \). Clearly, \( \text{Nil} \left( \Omega_0 \cap \Omega_T \right) \subseteq \text{Nil} \left( \Omega_0 \cap \Omega_T \right) \). Thus \( \text{Nil} \left( \Omega_0 \cap \Omega_T \right) \subseteq \text{Nil} \left( \Omega_0 \cap \Omega_T \right) \). This shows that \( \text{Nil} \left( \Omega_0 \cap \Omega_T \right) \subseteq \text{Nil} \left( \Omega_0 \cap \Omega_T \right) \), as desired.

(v) It is obvious that \( \text{Nil} \left( \Omega_0 \cap \Omega_T \right) \subseteq \text{Nil} \left( \Omega_0 \cap \Omega_T \right) \). Conversely, we assume that \( f \neq 0 \).

Then by (v) of this lemma, \( f_{-2} = 0 \). Hence \( f = f_{-1} + f\{1\} \in \text{Nil} \left( \Omega_0 \right) \). It follows from (iii) that \( f_{-1} = 0 \). Noting that \( \text{Nil} \left( \Omega_1 \right) \subseteq \text{Nil} \left( \Omega_0 \right) \), we have \( f = f\{1\} \subseteq \text{Nil} \left( \Omega_0 \cap \Omega_T \right) \). Therefore, \( \text{Nil} \left( \Omega_0 \right) \subseteq \text{Nil} \left( \Omega_0 \cap \Omega_T \right) \), and the assertion holds.

**Lemma 6.** Let \( i, j \in M_i \). Suppose that \( x^k y^\lambda \xi^u \) is an arbitrary standard element of \( \Omega \). Then the following statements hold.

(i) \( x^k y^\lambda \xi^u \in \text{Nil} \left( \Omega_0 \cap \Omega_T \right) \).

(ii) If \( [i] = [j] \) and \( i \neq j \), then \( x_i x_j y^\lambda \in \text{Nil} \left( \Omega_0 \cap \Omega_T \right) \).

(iii) If \( [i] \neq [j] \) and \( i \neq j \), then \( x_i x_j y^\lambda \in \text{Nil} \left( \Omega_0 \cap \Omega_T \right) \).

(iv) \( x_i x_j x^k y^\lambda \in \text{Nil} \left( \Omega_0 \cap \Omega_T \right) \).

**Proof.** (i) By a direct computation, we get

\[
\left( \text{ad}_x y^j \right) \left( x^k y^\lambda \xi^u \right) = [x^k y^\lambda, x^j y^\lambda \xi^u] = \left[ (1 - 2\mu_i - \lambda) x^k y^\lambda D_1 + 2 [i] x_i y^\lambda D_j \right] \left( x^k y^\lambda \xi^u \right) = (31)
\]

and \( x^k y^\lambda D_1 \circ x^j y^\lambda D_j = x^k x_j y^\lambda D_1 + x^k x^j y^\lambda D_1 \). It follows from the binomial theorem that

\[
\left( \text{ad}_x y^j \right)^{\rho^u} \left( x^k y^\lambda \xi^u \right) = \left( (1 - 2\mu_i - \lambda) x^k y^\lambda D_1 \right)^{\rho^u} + 2 [i] x_i y^\lambda D_j \left( x^k y^\lambda \xi^u \right) = 0.
\]

(ii) Also by a direct calculation, we have

\[
\left( \text{ad}_x y^j \right) \left( x^k y^\lambda \xi^u \right) = [x^k y^\lambda, x^j y^\lambda \xi^u] = \left( (1 - 2\mu_i - \lambda) x_i y^\lambda D_j \right) \left( x^k y^\lambda \xi^u \right)
\]

\[
+ 2 [i] x_i y^\lambda D_j \left( x^k y^\lambda \xi^u \right) = 0.
\]

Put

\[
A = y\lambda \xi^j y^\lambda D_1, \quad B = y\lambda \xi^j y^\lambda D_j, \quad C = y\lambda \xi^j y^\lambda D_1, \quad D = \chi \xi y^\lambda D_1, \quad E = \chi \xi y^\lambda D_k.
\]

**Lemma 7.** Suppose that \( i, j, k \in T \) are different from each other, and \( \gamma, \chi \in F \). Then

(i) \( f = \gamma \xi y^\lambda D_1 + \chi \xi^j y^\lambda D_j \) for \( y^\lambda + \chi^2 = 0 \).

(ii) \( \xi y^\lambda \xi^u \in \text{Nil} \left( \Omega_0 \cap \Omega_T \right) \).

**Proof.** (i) Set \( x^k y^\lambda \xi^u \) be an arbitrary standard element of \( \Omega \). Then

\[
\left( \text{ad} f \right) \left( x^k y^\lambda \xi^u \right) = \left[ f, x^k y^\lambda \xi^u \right]
\]

\[
= [f, x^k y^\lambda \xi^u] = [\gamma \xi y^\lambda, x^k y^\lambda \xi^u] + \chi \xi y^\lambda D_j \left( x^k y^\lambda \xi^u \right) + \gamma \xi y^\lambda D_1 \left( x^k y^\lambda \xi^u \right) + \chi \xi y^\lambda D_k \left( x^k y^\lambda \xi^u \right).
\]

Put

\[
A = y\lambda \xi^j y^\lambda D_1, \quad B = y\lambda \xi^j y^\lambda D_j, \quad C = y\lambda \xi^j y^\lambda D_1, \quad D = \chi \xi y^\lambda D_1, \quad E = \chi \xi y^\lambda D_k.
\]
Obviously,
\[ A^2 = B^2 = C^2 = D^2 = E^2 = F^2 = 0, \]
\[ AB = BA = AC = CA = AD = DA = AF = FA = BD = DB = BE = EB = CD = DC = CF = FC = DF = FD = 0. \]  
(37)

Hence
\[ (ad f)^2 = AE + EA + BC + CB + BF + FB + CE + EC + DE + ED + EF + FE. \]
(38)

Noting that \( y^2 + x^2 = 0 \), we obtain
\[ (ad f)^3 = BCB + CBC + EFE + FEF. \]
(39)

Similarly, \( (ad f)^4 = 0 \), and then \( f = y \xi \xi_j y^\lambda + x \xi \xi_k y^\lambda \in \text{nil}(\Omega) \).

(ii) Let \( k \in T \setminus \{i, j\} \). It follows from (i) that
\[ f_1 = y \xi \xi_j y^\lambda + x \xi \xi_k y^\lambda \in \text{nil}(\Omega_0 \cap \Omega_7), \]
\[ f_2 = x \xi \xi_j y^\lambda + y \xi \xi_k y^\lambda \in \text{nil}(\Omega_0 \cap \Omega_7), \]
where \( y^2 = -1 \).

Hence \( \xi \xi_j y^\lambda = -(1/2)(y f_1 - f_2) \in \text{nil}(\Omega_0 \cap \Omega_7) \).

\[ \square \]

Lemma 8. \( x \xi \xi_j y^\lambda \in \text{nil}(\Omega_0 \cap \Omega_7) \) for \( i \in M_1 \) and \( j \in T \).

Proof. By a direct computation, we obtain
\[ \left( \text{ad}_{x \xi_j} y^\lambda \right) \left( x^\lambda y^\lambda \xi^\mu \right) = \left[ x \xi \xi_j y^\lambda, x^\lambda y^\lambda \xi^\mu \right] \]
\[ = \left( -\frac{1}{2} - \mu_i - \lambda \right) x \xi \xi_j y^\lambda y^{D_1} + [i] \xi \xi_j y^\lambda D_{i'} - x \xi_j y^\lambda D_{i'} \]
\[ \times (x^\lambda y^\lambda \xi^\mu). \]
(41)

Put \( A = -(1/2 - \mu_i - \lambda)x \xi \xi_j y^\lambda y^{D_1}, B = [i] \xi \xi_j y^\lambda D_{i'} \) and \( C = -x \xi_j y^\lambda D_{i'} \). Observing \( A^2 = B^2 = C^2 = 0 \) and \( AB = BA = 0 \), we see that
\[ \left( \text{ad}_{x \xi_j} y^\lambda \right)^2 = AC + BC + CA + CB, \]
\[ \left( \text{ad}_{x \xi_j} y^\lambda \right)^n = 0, \quad n \in \mathbb{N}. \]
(42)

Thus \( x \xi \xi_j y^\lambda \in \text{nil}(\Omega_0 \cap \Omega_7) \), as required.

\[ \square \]

Lemma 9. The following statements hold

(i) \( \text{nil}(\Omega_0) = \text{span}_F\{x_j y^\lambda, x_i \xi \xi_j y^\lambda, \xi \xi_j y^\lambda | i, j \in M_1 \cup T\} \).

(ii) For \( q \geq 3 \), \( \text{nil}(\Omega_0 \cap \Omega_7) = \text{span}_F\{x_j y^\lambda, \xi \xi_j y^\lambda | i, j \in M_1, i \in T[i], j \in [j]\} \).

(iii) For \( q = 2 \), \( \text{nil}(\Omega_0 \cap \Omega_7) = \text{span}_F\{x_j y^\lambda | i, j \in M_1\} \).

Proof. (i) Suppose that \( f = y_1 x_1 y^\lambda + \sum_{i,j \in M_1 \cup T} (x_j y^\lambda + x_i \xi \xi_j y^\lambda) \) is an arbitrary element of \( \text{nil}(\Omega_0) \), where \( y_1 \in F \). If \( y_1 \neq 0 \), then \( [y_i x_j y^\lambda, y^\lambda] = y_1 (1 - \lambda) y_{2i} \). A direct calculation shows that \( (ad f)^\lambda(y^\lambda) = y_1^\lambda (1 - \lambda) (1 - 2 \lambda) \cdots (1 - m \lambda) y^\lambda \neq 0 \). Thus \( f \) is not ad-nilpotent, contradicting the nilpotency of \( f \). Hence \( y_1 = 0 \) and \( \text{nil}(\Omega_0) \subseteq \text{span}_F\{x_j y^\lambda, x_i \xi \xi_j y^\lambda, \xi \xi_j y^\lambda | i, j \in M_1 \cup T\} \).

Obviously, \( \text{span}_F\{x_j y^\lambda, x_i \xi \xi_j y^\lambda, \xi \xi_j y^\lambda | i, j \in M_1 \cup T\} \) is a subalgebra of \( \Omega \), which yields \( \text{Nil}(\Omega_0) \subseteq \text{span}_F\{x_j y^\lambda, x_i \xi \xi_j y^\lambda, \xi \xi_j y^\lambda | i, j \in M_1 \cup T\} \).

Conversely, we have \( \text{span}_F\{x_j y^\lambda, x_i \xi \xi_j y^\lambda, \xi \xi_j y^\lambda | i, j \in M_1 \cup T\} \) by virtue of Lemmas 6, 7, and 8. It follows that \( \text{Nil}(\Omega_0) = \text{span}_F\{x_j y^\lambda, x_i \xi \xi_j y^\lambda, \xi \xi_j y^\lambda | i, j \in M_1 \cup T\} \).

(ii) By (i) of the lemma, we have
\[ \text{Nil}(\Omega_0 \cap \Omega_7) \quad \subseteq \text{Nil}(\Omega_0) \cap \Omega_7 \]
\[ = \text{span}_F\{x_j y^\lambda, x_i \xi \xi_j y^\lambda, \xi \xi_j y^\lambda | i, j \in M_1 \cup T, [i] = [j]\}. \]
(43)

Conversely, the assertion \( \text{span}_F\{x_j y^\lambda, x_i \xi \xi_j y^\lambda, \xi \xi_j y^\lambda | i, j \in M_1 \cup T, [i] = [j]\} \subseteq \text{Nil}(\Omega_0 \cap \Omega_7) \) follows from Lemmas 6 and 7.

(iii) Suppose \( f = y_1 x_1 y^\lambda + \sum_{i,j \in M_1} y_{ij} x_j y^\lambda \in \text{nil}(\Omega_0 \cap \Omega_7) \), where \( y, y_{ij} \in F \). If \( y = 0 \), then \( ad f(\xi \xi_j y^\lambda) = [f, \xi \xi_j y^\lambda] = y_1 \xi \xi_j y^\lambda \) and \( (ad f)^\lambda(\xi \xi_j y^\lambda) = y_1^\lambda \xi \xi_j y^\lambda \). A direct calculation shows that \( (ad f)^\lambda(\xi \xi_j y^\lambda) = y_1^\lambda \xi \xi_j y^\lambda \neq 0 \). It follows that \( f \) is not ad-nilpotent. Thus \( y = 0 \). Then \( f = \sum_{i,j \in M_1} y_{ij} x_j y^\lambda \in \text{nil}(\Omega_0 \cap \Omega_7) \) and \( \text{nil}(\Omega_0 \cap \Omega_7) \subseteq \text{span}_F\{x_j y^\lambda | i, j \in M_1\} \). Note that \( \text{span}_F\{x_j y^\lambda | i, j \in M_1\} \) is a subalgebra of \( \Omega \). Hence \( \text{Nil}(\Omega_0 \cap \Omega_7) \subseteq \text{span}_F\{x_j y^\lambda | i, j \in M_1\} \) and (iii) holds.

\[ \square \]

Let \( \rho \) be the corresponding representation with respect to \( \Omega \) module \( \Omega_{-1} \); that is, \( \rho(f) = \text{ad}_{\Omega_{-1}} f \), for all \( f \in \Omega \). It is easily seen that \( \rho(f) \) is faithful. For \( f \in \Omega_0 \), we denote by \( \rho(f) \) the matrix of \( \rho(f) \) relative to the fixed ordered \( F \)-basis:
\[ \{ x_2 y^\lambda, x_3 y^\lambda, \ldots, x_{n-1} y^\lambda, \xi_{r+1} y^\lambda, \ldots, \xi_s y^\lambda \}. \]
(44)

Denote by \( \mathfrak{g}(2n, q) \) the general linear Lie superalgebra of \( (2n + q) \times (2n + q) \) matrices over \( F \). Let \( e_\alpha \) denote the \( (s - 2) \times (s - 2) \) matrix whose \( (i, j) \)-entry is 1 and 0 elsewhere. Let \( E_{\alpha} \) denote the identity matrix of size \( n \times n \). Put \( G = (E_{\alpha})_0 \). Let \( \mathfrak{sp}(2n, F) \) be all the \( (2n) \times (2n) \) matrices set filled with \( A^T G + GA = 0 \). Put
\[ \mathfrak{sp}(2n, F) \]
(45)
Abstract and Applied Analysis

Set

\[ \mathcal{L} = \mathcal{W} \oplus \mathcal{E}_{s-2}. \]  

Lemma 10. (i) \( \rho(\Omega_0) = \mathcal{L} \).
(ii) If \( f \in \text{nil}(\Omega_0) \), then \( \rho(f) \) is a nilpotent matrix.

Proof. (i) Let \( i, j, k \in M_1 \cup T \). By computation, we have

\[ [x_i x_j, x_k y^\lambda] = [i] \delta_{ij} x_j y^\lambda + [j] \delta_{kj} x_k y^\lambda, \]
\[ [x_i, x_k y^\lambda] = (1 - \mu_k - \lambda) x_k y^\lambda. \]

Hence \( \rho(x_i x_j) = [i] e_j + [j] e_i \) and \( \rho(x_i) = (1 - \mu_k - \lambda)E_{s-2}. \) The other cases are treated similarly. Thus (i) holds.

(ii) As \( f \) is a nilpotent elements, \( \rho(f) \) is a nilpotent liner transformation. Then by the definition of \( \rho \), we see that \( \rho(f) \) is a nilpotent matrix. \( \square \)

Lemma 11. If \( f \in \text{nil}(\Omega_0 \cap \Omega_\gamma) \), \( f \neq 0 \), then there exists a \( z \in \Omega_0 \cap \Omega_\gamma \) such that \( [f, z] \neq \text{nil}(\Omega_0 \cap \Omega_\gamma) \).

Proof. By Lemma 9, we can assume that \( f = \sum_{i \in M_1} y_i x_i y^\lambda + \sum_{i \in T, j \in T} \beta_i x_j y^\lambda + \sum_{j \in T} \xi_j x_i y^\lambda \), where \( \gamma, \beta, \xi \in \mathcal{F} \).

Suppose \( \gamma \neq 0 \) for some \( i \in M_1 \) and \( z = x_i^2 \) or some \( j \in M_1 \) and \( x = x_i x_j y^\lambda \). Then every item of \( h \) does not contain \( x \). Then \( (ad[f, z])(x_i) = 4y_i y^\lambda x_i \).

where every item of \( h \) does not contain \( x \). Similarly, \( (ad[f, z])(x_i) = 4y_i y^\lambda x_i \), for all \( i \in M_1 \), and then \( [f, z] \) is not nilpotent.

Hence our assertion follows.

Lemma 12. \( \text{nil}(\Omega_1) \oplus \text{span}_F \{x_i x_j y^\lambda, x_i y^\lambda | i, j \in M_1 \cup T \} \) is ad-nilpotent element. Hence our assertion holds. \( \square \)

4. Filtration and Intrinsic Property

Lemma 13. \( \Omega_0 \cap \Omega_\gamma = \text{Nor}_{\Omega_\gamma}(\text{Nil}(\Omega_\gamma)) \) and \( \Omega_0 \cap \gamma \) is invariant.

Proof. Firstly, we prove the inclusion \( \Omega_0 \cap \Omega_\gamma \in \text{Nor}_{\Omega_\gamma}(\text{Nil}(\Omega_\gamma)) \). Lemma 6–9 show that \( \text{Nil}(\Omega_0 \cap \Omega_\gamma) \in \Omega_0 \cap \Omega_\gamma \), which combined with (iv) and (v) of Lemma 5, yield

\[ \text{Nil}(\Omega_\gamma) = \text{Nil}(\Omega_0 \cap \Omega_\gamma) = \text{Nil}(\Omega_0 \cap \Omega_\gamma) + \Omega_{\Omega_0} \cap \Omega_\gamma, \]
\[ [\Omega_0 \cap \Omega_\gamma, \text{Nil}(\Omega_0 \cap \Omega_\gamma)] \subseteq \text{Nil}(\Omega_0 \cap \Omega_\gamma), \]
\[ [\Omega_0 \cap \Omega_\gamma, \Omega_{\Omega_0} \cap \Omega_\gamma] \subseteq \Omega_{\Omega_0} \cap \Omega_\gamma. \]

Then

\[ [\Omega_0 \cap \Omega_\gamma, \text{Nil}(\Omega_\gamma)] = [\Omega_0 \cap \Omega_\gamma, \text{Nil}(\Omega_0 \cap \Omega_\gamma)] + [\Omega_0 \cap \Omega_\gamma, \Omega_{\Omega_0}] \]
\[ \subseteq \text{Nil}(\Omega_0 \cap \Omega_\gamma) \]
\[ + \Omega_{\Omega_0} \cap \Omega_\gamma = \text{Nil}(\Omega_\gamma), \]
that is, \( \Omega_0 \cap \Omega_\gamma \in \text{Nor}_{\Omega_\gamma}(\text{Nil}(\Omega_\gamma)) \).

Let us consider the converse inclusion. Suppose that \( f = f_{-2} + f_{-1} \in \text{Nor}_{\Omega_\gamma}(\text{Nil}(\Omega_\gamma)) \), where \( f_{-2} \in \Omega_{-2} \) and \( f_{-1} \in \Omega_{-1} \). If \( f_{-2} \neq 0 \), then \( [f, x_i y^\lambda] = [f_{-2}, x_i y^\lambda] + [f_{-1}, x_i y^\lambda] = y_f x_i y^\lambda \neq 0 \) and \( f \neq \text{Nil}(\Omega_\gamma) \), where \( h \in \Omega_0 \cap \gamma \) and \( \gamma \in \mathcal{F} \), a contradiction. Consequently, \( f_{-2} = 0 \).

Now suppose \( f = f_{-1} + f_0 \), where \( f_{-1} = \sum_{i \in M_1} y_i x_i y^\lambda \) and \( f_0 \in \Omega_0 \cap \Omega_\gamma \). If \( y_{-1} \neq 0 \) and \( \gamma \neq 0 \), we have \( [f, x_i y^\lambda] = 2[1]y_j x_j y^\lambda \), a contradiction. Thus \( f_{-1} = 0 \) and \( f = f_0 \in \Omega_0 \cap \Omega_\gamma \). This proves the asserted inclusion.

By the proof above, we know that \( \Omega_0 \cap \Omega_\gamma \) is invariant. \( \square \)

Lemma 14. \( \Omega_0 \cap \Omega_\gamma = \{ f \in \text{nil}(\Omega_\gamma) \mid [f, \Omega_0 \cap \Omega_\gamma] \subseteq \text{nil}(\Omega_\gamma) \} \) and \( \Omega_0 \cap \Omega_\gamma \) is invariant.

Proof. Let \( \mathcal{M} = \{ f \in \text{nil}(\Omega_\gamma) \mid [f, \Omega_0 \cap \Omega_\gamma] \subseteq \text{nil}(\Omega_\gamma) \} \). Suppose \( f \in \text{nil}(\Omega_\gamma) \). By Lemma 5, \( f_{-2} = 0 \) and
Abstract and Applied Analysis

\( f^{-1} = 0 \). Then we can assume that \( f = f_0 + f_1 \in \mathcal{M} \), where
\( f_0 \in \Omega_0 \cap \Omega_1 \), \( f_1 \in \Omega_1 \cap \Omega_2 \). Let \( f_0 \neq 0 \). Clearly, \( f_0 \in \text{nil}(\Omega_0) \). Lemma 5(1) implies that \( f_0 \in \text{nil}(\Omega_0 \cap \Omega_2) \). According to Lemma 11, there exists \( z \in \Omega_0 \cap \Omega_2 \) such that \( f_0, z \notin \text{nil}(\Omega_0 \cap \Omega_2) \). Thus \( f_0, z \notin \text{nil}(\Omega_0 \cap \Omega_2) \) and \( f_1 \) is not nilpotent, contradicting the result of Lemma 5. Hence \( f_0 = 0 \) and \( f \in \Omega_1 \cap \Omega_2 \); that is, \( \mathcal{M} \subseteq \Omega_1 \cap \Omega_2 \). Conversely, \( \Omega_1 \cap \Omega_2 \cap \Omega_0 \cap \Omega_2 \subsetneq \Omega_1 \cap \Omega_2 \subsetneq \text{nil}(\Omega_0 \cap \Omega_2) \). It is obvious that \( \Omega_1 \cap \Omega_2 \subsetneq \mathcal{M} \) and the proof is complete.

Lemma 15. (i) \( \Omega_1 \cap \Omega_2 = \{ f \in \Omega_1 \mid [f, \Omega_2] \subseteq \Omega_0 \cap \Omega_2 \} \).
(ii) \( \Omega_0 \cap \Omega_2 = [\Omega_0, \Omega_1] \subsetneq \Omega_2 \).
(iii) \( \Omega_1 = \{ f \in \Omega_1 \mid [f, \Omega_1] \subseteq \Omega_0 \} \).

Proof. (i) Put \( \mathcal{A} = \{ f \in \Omega_1 \mid [f, \Omega_2] \subseteq \Omega_0 \cap \Omega_2 \} \). Suppose \( f = f_1 + f_0 \), where \( f_0 = \sum_{i \in \mathbb{M}, i \notin T} y_i x_i^k \), \( f_1 \in \Omega_1 \cap \Omega_2 \), \( f_0 \in \Omega_0 \cap \Omega_2 \), \( f_0 \neq 0 \), and \( y_i \neq 0 \) for some \( i \in T \).

Thus \( f_0 = 0 \) and \( f = f_1 \in \Omega_1 \cap \Omega_2 \). Hence \( f_0 \notin \mathcal{A} \). Let \( f_0 = 0 \) and \( f \in \mathcal{A} \).

Conversely, let \( f \in \mathcal{A} \). Then \( f \in \Omega_1 \cap \Omega_2 \).

(ii) We first prove the inclusion \( \Omega_0 \cap \Omega_2 \subsetneq [\Omega_0, \Omega_1] \cap \Omega_2 \).

Let \( x^k y^\lambda \in \Omega_0 \cap \Omega_2 \), where \( \lambda \in \mathbb{F} \). Noting that \( k \) does not contain \( x_1 \), \( \lambda \) contains all monomial expressions occurring in \( f_1 \). Then we may assume that
\[
 f = \sum_{\lambda \notin \mathbb{F}} x^k y^\lambda + \sum_{\lambda \in \mathbb{F}} k^p x_1^{k', y^\lambda}\xi^\mu, \tag{53}
\]
where \( \chi, \omega \in \mathbb{F} \). For any \( j \in \mathbb{M} \), we have
\[
 0 = \sum_{\lambda \notin \mathbb{F}} x^k y^\lambda + \sum_{\lambda \in \mathbb{F}} k^p x_1^{k', y^\lambda}\xi^\mu, \tag{54}
\]
where \( h \) is the sum of summand that the exponent of \( x_1 \) is less than \( t \). Since \( k \lambda \) is linear independence, \( x^{k', y^\lambda}\xi^\mu = 0 \); that is, \( k_i = 0 \). For any \( j \in T \), we get
\[
 0 = \sum_{\lambda \in \mathbb{F}} x^k y^\lambda\xi^\mu + h, \tag{55}
\]
where \( h \) is the sum of summand that the exponent of \( x_1 \) is less than \( t \). Thus \( \sum_{\lambda \in \mathbb{F}} x^k y^\lambda\xi^\mu = 0 \); that is, \( k_i = 0 \).

Theorem 16. \( \Omega_0 \cap \Omega_2 = \Omega_0 \cap \Omega_2 \) is transitive.

Proof. Assume the contrary. Suppose that there exists a nonzero \( f \neq 0 \) such that \( [f, \Omega_1] \neq 0 \), where \( l \in \mathbb{N} \). Let \( t \) be the maximal exponent of \( x_1 \) of all monomial expressions occurring in \( f \). Then we may assume that
\[
 f = \sum_{k \in \mathbb{M}} x^k y^\lambda + \sum_{\lambda \in \mathbb{F}} k^p x_1^{k', y^\lambda}\xi^\mu, \tag{56}
\]
where \( \chi, \omega \in \mathbb{F} \). For any \( j \in \mathbb{M} \), we have
\[
 0 = \sum_{k \in \mathbb{M}} x^k y^\lambda + \sum_{\lambda \in \mathbb{F}} k^p x_1^{k', y^\lambda}\xi^\mu + h, \tag{57}
\]
where \( h \) is the sum of summand that the exponent of \( x_1 \) is less than \( t \). Thus \( \sum_{\lambda \in \mathbb{F}} x^k y^\lambda\xi^\mu = 0 \); that is, \( k_i = 0 \).

Theorem 17. Suppose that \( \Omega_0 \cap \Omega_2 \) is transitive.

Proof. As the isomorphism \( \varphi \) is an even mapping, we have \( \varphi(\Omega_0) = \Omega_0 \) and \( \varphi(\text{nil}(\Omega_0)) = \text{nil}(\Omega_0) \). Hence \( \varphi(\text{nil}(\Omega_0)) = \text{nil}(\Omega_0) \) for all \( \lambda \). By Lemmas 13 and 14, we obtain
\[
 \varphi(\Omega_0 \cap \Omega_2) = \Omega_0 \cap \Omega_2, \tag{58}
\]
\[
 \varphi(\Omega_0) = \Omega_0, \tag{59}
\]
Now by virtue of Lemma 15, we get
\[
 \varphi(\Omega_0 \cap \Omega_2) = \Omega_0 \cap \Omega_2, \tag{60}
\]
\[
 \varphi(\Omega_0) = \Omega_0, \tag{61}
\]
\[
 \varphi(\Omega_0) = \Omega_0, \tag{62}
\]
\[
 \varphi(\Omega_0) = \Omega_0, \tag{63}
\]
\[
 \varphi(\Omega_0) = \Omega_0, \tag{64}
\]
It follows that \( \varphi(\Omega_{[0]}) = \Omega_{[0]}' \). Because \( \Omega_{[i]} \leq \Omega_{[-1]} \), we have

\[
\varphi(\Omega_{[-1]}) = \varphi(\Omega_{[-1]} \cap \Omega_{[-1]} + \Omega_{[-1]} \cap \Omega_{[-1]}) = \varphi(\Omega_{[-1]} + \Omega_{[-1]} \cap \Omega_{[-1]}) = \Omega_{[-1]}' \cap \Omega_{[-1]}' = \Omega_{[-1]}'.
\]

Since \( \Omega \) is transitive by the lemma above, we have \( \Omega_{[i+1]} = \{ f \in \Omega_{[i]} \mid [f,\Omega_{[i-1]}] \subseteq \Omega_{[i]} \} \) for all \( i \geq 0 \). It is easy to show that \( \varphi(\Omega_{[i]}) = \Omega_{[i]}' \) by induction on \( i \).

**Theorem 18.** Suppose that \( \phi \) is an automorphism of \( \Omega \). Then \( \phi(\Omega_{[i]}) = \Omega_{[i]}' \) for all \( i \geq -2 \), that is, the filtration of \( \Omega \) is invariant under the automorphism group of \( \Omega \).

**Proof.** This is a direct consequence of Theorem 17.

**Theorem 19.** Let \( \Omega(r, m, q) \) and \( \Omega'(r', m', q') \) be the Lie superalgebras of \( \Omega \)-type. Then \( \Omega(r, m, q) \cong \Omega'(r', m', q') \) if and only if \( r = r', m = m', q = q' \).

**Proof.** The sufficient condition is obvious. We will prove the necessary condition. Assume that \( \phi : \Omega(r, m, q) \to \Omega'(r', m', q') \) is an isomorphism of Lie superalgebra. According to Theorem 17, we have

\[
\varphi\left(\Omega(r, m, q)_{[-2]}\right) = \Omega'(r', m', q')_{[-2]},
\]

\[
\varphi\left(\Omega(r, m, q)_{[-1]}\right) = \Omega'(r', m', q')_{[-1]}.
\]

Then \( \phi \) induces an isomorphism of \( \mathbb{Z}_2 \)-graded spaces

\[
\frac{\Omega(r, m, q)_{[-2]}}{\Omega(r, m, q)_{[-1]}} \to \frac{\Omega'(r', m', q')_{[-2]}}{\Omega'(r', m', q')_{[-1]}},
\]

It is easy to see that

\[
\Omega(r, m, q)_{[-2]} \cong \Omega(r, m, q)_{[-2]},
\]

\[
\Omega'(r', m', q')_{[-2]} \cong \Omega'(r', m', q')_{[-2]}.
\]

We conclude that \( m = m' \) by the dimension comparison. Similarly, we also obtain an isomorphism of \( \mathbb{Z}_2 \)-graded spaces:

\[
\frac{\Omega(r, m, q)_{[-1]}}{\Omega(r, m, q)_{[0]}} \cong \frac{\Omega'(r', m', q')_{[-1]}}{\Omega'(r', m', q')_{[0]}},
\]

and the quotient space \( \Omega_{[-1]}\omega_{[0]} \) is isomorphic to \( \Omega_{-1} \). A comparison of dimensions shows that \( r + q = r' + q' \). Note that \( \varphi(\Omega(r, m, q)_{\alpha}) = \Omega'(r', m', q')_{\alpha} \), where \( \alpha \in \mathbb{Z}_2 \). It follows that \( r = r' \) and \( q = q' \).

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