Letter to the Editor

A Note on the Semi-Inverse Method and a Variational Principle for the Generalized KdV-mKdV Equation

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Ji-Huan He systematically studied the inverse problem of calculus of variations. This note reveals that the semi-inverse method also works for a generalized KdV-mKdV equation with nonlinear terms of any orders.

1. Introduction

In [1], the semi-inverse method is systematically studied and many examples are given to show how to establish a variational formulation for a nonlinear equation. From the given examples, we found that it is difficult to find a variational principle for nonlinear evolution equations with nonlinear terms of any orders.

For example, consider the following generalized KdV-mKdV equation:

$$u_t + (\alpha + \beta u^p + \gamma u^{2p})u_x + u_{xxx} + \eta u_{xxxxx} + g(t)u = 0, \quad (1)$$

where α , β , γ , and η are constant coefficients, while p is a positive number. Equation (1) is an important model in plasma physics and solid state physics.

2. Variational Principle by He's Semi-Inverse Method

For (1), we introduce a potential function v defined as $u = v_x$; we have the following equation:

$$v_{xt} + (\alpha + \beta v_x^p + \gamma v_x^{2p})v_{xx} + v_{xxxx} + \eta v_{xxxxx} + g(t)v_x = 0. \quad (2)$$

In order to use the semi-inverse method [1–4] to establish a Lagrangian for (2), we first check some simple cases:

$$\begin{aligned} L &= -\frac{v_x v_t}{2} \quad \text{for } v_{xt} = 0, \\ L &= \frac{(v_{xx})^2}{2} \quad \text{for } v_{xxxx} = 0, \\ L &= -\frac{v_x^3}{6} \quad \text{for } \frac{(v_x^2)_x}{2} = v_x v_{xx} = 0, \\ L &= -\frac{v_x^n}{n(n-1)} \quad \text{for } v_x^{n-2} v_{xx} = 0. \end{aligned} \quad (3)$$

We can easily obtain a variational principle for (2) for $g(t) \equiv 0$, which is

$$J(v) = \iint \left\{ -\frac{1}{2} v_x v_t - \frac{1}{2} \alpha v_x^2 - \frac{\beta}{(p+2)(p+1)} v_x^{p+2} - \frac{\gamma}{(2p+2)(2p+1)} v_x^{2p+2} + v_{xx}^2 - \frac{\eta}{2} v_{xxx}^2 \right\} dx dt, \quad (4)$$

Now, according to the semi-inverse method [1–4], we construct a trial functional for (2):

$$J(v) = \iint \left\{ f(t) \left[-\frac{1}{2} v_x v_t - \frac{1}{2} \alpha v_x^2 - \frac{\beta}{(p+2)(p+1)} v_x^{p+2} - \frac{\gamma}{(2p+2)(2p+1)} v_x^{2p+2} + v_{xx}^2 - \frac{\eta}{2} v_{xxx}^2 \right] + F \right\} dx dt, \quad (5)$$

where F is an unknown function of u and/or its derivatives.

Making the trial-functional, (5), stationary with respect to v results in the following Euler-Lagrange equation:

$$\begin{aligned} & \frac{1}{2} (f v_x)_t + \frac{1}{2} (f v_t)_x + \alpha (f v_x)_x + \frac{\beta}{(p+1)} (f v_x^{p+1})_x \\ & + \frac{\gamma}{(2p+1)} (f v_x^{2p+1})_x + (f v_{xx})_{xx} - \eta (f v_{xxx})_{xxx} \\ & + \frac{\delta F}{\delta v} = 0, \end{aligned} \quad (6)$$

where $\delta F/\delta v$ is called variational differential with respect to v , defined as

$$\begin{aligned} \frac{\delta F}{\delta v} = & \frac{\partial F}{\partial v} - \frac{\partial}{\partial t} \left(\frac{\partial F}{\partial v_t} \right) - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial v_x} \right) + \frac{\partial^2}{\partial t^2} \left(\frac{\partial F}{\partial v_{tt}} \right) \\ & + \frac{\partial^2}{\partial x^2} \left(\frac{\partial F}{\partial v_{xx}} \right) + \dots \end{aligned} \quad (7)$$

We rewrite (6) in the form

$$\begin{aligned} & \frac{f_t}{2f} v_x + v_{xt} + \alpha v_{xx} + \beta v_x^p v_{xx} + \gamma v_x^{2p} v_{xx} + v_{xxxx} \\ & + \eta v_{xxxxx} + \frac{\delta F}{f \delta v} = 0. \end{aligned} \quad (8)$$

Comparison of (8) and (2) leads to the following results:

$$\frac{f_t}{2f} = g(t), \quad \frac{\delta F}{f \delta v} = 0, \quad (9)$$

from which we identify the unknown f and F as follows:

$$f(t) = e^{2 \int g(t) dt}, \quad F = 0. \quad (10)$$

We, therefore, obtain the following needed variational principle:

$$\begin{aligned} J(v) = & \iint \left\{ e^{2 \int g(t) dt} \left[-\frac{1}{2} v_x v_t - \alpha v_x^2 - \frac{\beta}{(p+2)(p+1)} v_x^{p+2} - \frac{\gamma}{(2p+2)(2p+1)} v_x^{2p+2} + v_{xx}^2 - \frac{\eta}{2} v_{xxx}^2 \right] \right. \\ & \left. + F \right\} dx dt. \end{aligned} \quad (11)$$

3. Conclusion

This note shows that the semi-inverse method in [1] works also for the present problem, and it is concluded that the semi-inverse method is a powerful mathematical tool to the construction of a variational formulation for a nonlinear equation; illustrating examples are available in [5–10].

The semi-inverse method can be extended to fractional calculus [11–14].

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