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Research Article

Global Solutions for an m-Component System of Activator-Inhibitor Type

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This paper deals with a reaction-diffusion system with fractional reactions modeling m-substances into interaction following activator-inhibitor's scheme. The existence of global solutions is obtained via a judicious Lyapunov functional that generalizes the one introduced by Masuda and Takahashi.

1. Introduction

In this paper, we are concerned with the existence of global solutions to a reaction-diffusion system with m components generalizing the activator-inhibitor system:

$$\begin{aligned} \partial_t u_1 - a_1 \Delta u_1 &= f_1(u) = \sigma - b_1 u_1 + \frac{u_1^{p_{11}}}{\prod_{j=2}^{m} u_j^{p_{1j}}}, \\ \partial_t u_i - a_i \Delta u_i &= f_i(u) = -b_i u_i + \frac{u_1^{p_{i1}}}{\prod_{j=2}^{m} u_j^{p_{ij}}}, \qquad x \in \Omega,\ t > 0, \\ & \qquad i = 2, \ldots, m, \end{aligned} \tag{1}$$

supplemented with Neumann boundary conditions

$$\frac{\partial u_i}{\partial \eta} = 0, \quad \text{on } \partial\Omega \times \{t > 0\},\ i = 1, \ldots, m, \tag{2}$$

and the positive initial data

$$u_i(x, 0) = \varphi_i(x) \quad \text{on } \Omega,\ i = 1, \ldots, m. \tag{3}$$

Here $u = (u_1, u_2, \ldots, u_m)$, Ω is an open bounded domain of class C^1 in $\mathbb{R}^N$, with boundary $\partial\Omega$, and $\partial/\partial\eta$ denotes the outward normal derivative on $\partial\Omega$.

Throughout the paper, we make the following hypotheses:

The indexes a_i, p_{ij} are nonnegative for all $i, j = 1, \ldots, m$, with $\sigma > 0$:

$$\begin{aligned} 0 &< p_{11} - 1 \\ &< \max_{k=2,3,\ldots,m} \left\{ p_{k1} \min \left\{ 1, \frac{p_{1k}}{p_{kk}+1}, \frac{p_{1j}}{p_{kj}}, \right. \right. \\ &\qquad \left. \left. j = 2, \ldots, m,\ j \neq k \right\} \right\}; \end{aligned} \tag{4}$$

we set $A_{ij} = (a_i + a_j)/(2\sqrt{a_i a_j})$ for all $i, j = 1, \ldots, m$. Let $\alpha_i, i = 1, \ldots, m$, be positive constants such that

$$\alpha_1 > 2 \max \left\{ 1, \sum_{i=1}^{m} \frac{b_i}{b_1} \right\}, \tag{5}$$

$$S_l^l > 0, \quad l = 2, \ldots, m, \tag{6}$$

where

$$S_l^r = S_{r-1}^{r-1} \cdot S_l^{r-1} - \left[H_l^{r-1} \right]^2, \quad r = 3, \ldots, l,$$

$$H_l^r = \det_{1 \le i, j \le l} \left(\left(a_{i,j} \right)_{\substack{i \neq l, \ldots r+1 \\ j \neq l-1, \ldots r}} \right) \prod_{k=1}^{k=r-2} (\det[k])^{2^{(r-k-2)}}, \qquad r = 3, \ldots, l-1,$$

$$S_l^2 = \alpha_1^2\alpha_l^2 a_1 a_l \left[\frac{1}{2\alpha_l} - A_{1l}^2\right],$$
$$H_l^2 = \alpha_1^2\alpha_2\alpha_l a_1 \sqrt{a_2 a_l}\left[\frac{\alpha_1 - 1}{\alpha_1}A_{2l} - A_{12}A_{1l}\right], \tag{7}$$

where $\det_{1\le i,j\le l}((a_{i,j})_{i\ne l,\dots,r+1, j\ne l-1,\dots,r})$ stands for the determinant of the r-square symmetric matrix obtained from the matrix $(a_{i,j})_{1\le i,j\le m}$ by removing the $(r+1)$th, $(r+2)$th, $\dots$, lth rows and the rth, $(r+1)$th, $\dots$, $(l-1)$th columns, where $\det[1],\dots,\det[m]$ are the minors of the matrix $(a_{ij})_{1\le i,j\le m}$. The elements of the matrix are as follows:

$$\begin{aligned} a_{11} &= a_1\alpha_1(\alpha_1 - 1), \\ a_{1i} &= -\alpha_1\alpha_i\frac{(a_1 + a_i)}{2}, \quad i = 2,\dots,m, \\ a_{ii} &= a_i\alpha_i(\alpha_i + 1), \quad i = 2,\dots,m, \\ a_{ij} &= \alpha_i\alpha_j\frac{(a_i + a_j)}{2}, \quad i,j = 2,\dots,m,\ i\ne j. \end{aligned} \tag{8}$$

The main result of the paper is the following.

Theorem 1. *Assume that condition* (4) *is satisfied. Let u be a solution of* (1)–(3) *with positive and bounded initial data, and let*

$$L(t) = \int_\Omega \frac{u_1^{\alpha_1}(t,x)}{\prod_{j=2}^m u_j^{\alpha_j}(t,x)}dx. \tag{9}$$

Then the functional L is uniformly bounded on the interval $[0,T^*]$, $T^* < T_{\max}$, where $T_{\max}(\|\varphi_1\|_\infty, \|\varphi_2\|_\infty,\dots,\|\varphi_m\|_\infty)$ denotes the eventual blow-up time.

Corollary 2. *Under the assumptions of Theorem 1 all solutions of* (1)–(3) *with positive initial data in $C(\overline{\Omega})$ are global. If in addition $b_1,\dots,b_m$, $\sigma > 0$, then u is uniformly bounded in $\overline{\Omega}\times[0,\infty)$.*

Before we prove our results, let us dwell a while on the existing literature concerning Gierer-Meinhardt's type systems.

In 1972, following an ingenious idea of Turing [1], Gierer and Meinhardt [2] proposed a mathematical model for pattern formations of spatial tissue structures of hydra in morphogenesis. It is a system of reaction-diffusion equations of the form:

$$\begin{aligned} \partial_t u - a_1\Delta u &= \sigma - \mu u + \frac{u^p}{v^q}, \\ \partial_t v - a_2\Delta v &= -\nu v + \frac{u^r}{v^s}, \end{aligned} \quad x\in\Omega,\ t>0, \tag{10}$$

with Neumann boundary conditions

$$\frac{\partial u}{\partial\eta} = \frac{\partial v}{\partial\eta} = 0, \quad x\in\partial\Omega,\ t>0, \tag{11}$$

and initial conditions

$$\begin{aligned} u(x,0) &= \varphi_1(x) > 0, \\ v(x,0) &= \varphi_2(x) > 0, \end{aligned} \quad x\in\Omega, \tag{12}$$

where $\Omega\subset\mathbb{R}^N$ ($N = 1,2,3$ in practice) is a bounded domain with smooth boundary $\partial\Omega$, $a_1, a_2, \mu, \nu, \sigma > 0$, and p, q, r and s are non negative windexes with $p > 1$. Here u is the activator, and v is the inhibitor.

Global existence of solutions in $(0,\infty)$ was proved by Rothe [3], more than ten years after Gierer and Meinhardt's original paper with special choice of the parameters: $p = 2$, $q = 1$, $r = 2$, $s = 0$, and $N = 3$. Masuda and Takahashi [4] were able to prove global estimates and bounds of the solution for Gierer and Meinhardt's system in its general form. They proceeded by first proving lower bounds, then L^p bounds (for any $p > 1$), then uniform estimates and bounds in appropriate Sobolev spaces. The key point is represented by the L^p bounds, which are derived using in a subtle way the specific structure of the equations.

Li et al. [5] also studied the activator-inhibitor model.

Very recently, Bernasconi [6] considered the larger system:

$$\begin{aligned} \partial_t a(x,t) &= d_a a_{xx} + \frac{a^2(x,t)}{h(x,t)} - \mu a(x,t) + \rho, \\ \partial_t h(x,t) &= d_h h_{xx}(x,t) + a^2(x,t) - \nu h(x,t) + \varepsilon s(x,t), \\ \partial_t s(x,t) &= d_s s_{xx}(x,t) + a(x,t) - \kappa s(x,t), \end{aligned} \tag{13}$$

and Meinhardt et al. [7] proposed activator-inhibitor models to describe a theory of biological pattern:

$$\begin{aligned} \partial_t a(x,t) &= d_a a_{xx} + \frac{a^2(x,t)}{h(x,t)s(x,t)} - \mu a(x,t) + \rho, \\ \partial_t h(x,t) &= d_h h_{xx}(x,t) + a^2(x,t) - \nu h(x,t), \\ \partial_t s(x,t) &= d_s s_{xx}(x,t) + a(x,t) - \kappa s(x,t), \end{aligned} \tag{14}$$

which is Gierer and Meinhardt's system supplemented with a third equation, where $a(x,t)$ is the activator, $h(x,t)$ is the inhibitor, and $s(x,t)$ is a source that acts as an inhomogeneous inhibitor.

Our paper generalizes the system in [5] to m-components.

2. Preliminary Observations and Notations

The usual norms in the spaces $L^p(\Omega)$, $L^\infty(\Omega)$, and $C(\overline{\Omega})$ are denoted, respectively, by the following:

$$\begin{aligned} \|u\|_p^p &= \frac{1}{|\Omega|}\int_\Omega |u(x)|^p dx, \\ \|u\|_\infty &= \operatorname*{ess\,sup}_{x\in\Omega}|u(x)|, \\ \|u\|_{C(\overline{\Omega})} &= \max_{x\in\overline{\Omega}}|u(x)|. \end{aligned} \tag{15}$$

It is well known that to prove global existence of solutions to (1)–(3), it suffices to derive a uniform estimate of $\|f_i(u_1, u_2, \ldots, u_m)\|_p$, $i = 1, \ldots, m$ on $[0; T_{\max})$ in the space $L^p(\Omega)$ for some $p > n/2$ (see Henry [8]).

Since the functions f_i are continuously differentiable on $\mathbb{R}_+^m$ for all $i = 1, \ldots, m$, then for any initial data in $C(\overline{\Omega})$, the system (1)–(3) admits a unique, classical solution $(u_1, u_2, \ldots, u_m)$ on $(0, T_{\max}) \times \Omega$ with the alternative

(i) either $T_{\max} = \infty$;

(ii) or $T_{\max} < \infty$, and $\lim_{t \nearrow T_{\max}} \sum_{i=1}^m \|u_i(t, \cdot)\|_\infty = \infty$.

Using the maximum principle, one derives the lower bounds of the components of the solution u of (1)–(3):

$$u_i(t, x) \geq e^{-b_i t} \min(\varphi_i(x)) > 0, \quad i = 1, \ldots, m. \tag{16}$$

Our aim is to construct a Lyapunov functional that allows us to obtain L^p-bounds on u_i leading to global existence.

3. Preparatory Lemmas

For the proof of Theorem 1, we need some preparatory lemmas whose proofs will be in the appendix.

Lemma 3. *Assume that the constants q_{ij} satisfy*

$$\frac{q_{11} - 1}{q_{k1}} < \min\left\{1, \frac{q_{1k}}{q_{kk} + 1}, \frac{q_{1j}}{q_{kj}}, \ j = 2, \ldots, m, \ j \neq k\right\}. \tag{17}$$

Then for all $h_{i-1}, \alpha_i > 0$, $j, i = 1, \ldots, m$, there exist $C = C(h_{i-1}, \alpha_i) > 0$ and $\theta = \theta(\alpha_1) \in (0, 1)$, such that

$$\alpha_1 \frac{u_1^{q_{11}-1+\alpha_1}}{\prod_{j=2}^m u_j^{q_{1j}+\alpha_j}} \leq \alpha_k \frac{u_1^{q_{k1}+\alpha_1}}{u_k^{q_{kk}+1+\alpha_k} \prod_{j=2, j\neq k}^m u_j^{q_{kj}+\alpha_j}} + C\left(\frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}}\right)^\theta, \tag{18}$$

$u_1 \geq 0$, $u_i \geq h_{i-1}$, $i = 1, \ldots, m$, $k \in \{2, \ldots, m\}$.

Lemma 4 (see [9]). *Let $A = (a_{ij})_{1 \leq i, j \leq m}$. Then one has:*

$$K_m^m = \det[m] \cdot \prod_{k=1}^{k=m-2} (\det[k])^{2^{(m-k-2)}}, \quad m > 2, \qquad K_2^2 = \det[2], \tag{19}$$

where

$$K_m^l = K_{l-1}^{l-1} \cdot K_m^{l-1} - \left(H_m^{l-1}\right)^2, \quad l = 3, \ldots, m, \tag{20}$$

$$H_m^l = \det_{1 \leq i, j \leq m}\left(\left(a_{i,j}\right)_{\substack{i \neq (l+1), \ldots, m, \\ j \neq l, \ldots, (m-1)}}\right) \cdot \prod_{k=1}^{k=l-2} (\det[k])^{2^{(l-k-2)}}, \quad l = 3, \ldots, m-1, \tag{21}$$

$$K_m^2 = a_{11} a_{mm} - (a_{1m})^2, \tag{22}$$

$$H_m^2 = a_{11} a_{2m} - a_{12} a_{1m}. \tag{23}$$

Lemma 5. *Let $\alpha_1 > 2\max\{1, \sum_{i=1}^m b_i/b_1\}$. One has*

$$K_l^l > S_l^l, \quad l = 2, \ldots, m, \tag{24}$$

where

$$K_l^r = K_{r-1}^{r-1} \cdot K_l^{r-1} - \left[H_l^{r-1}\right]^2, \quad r = 3, \ldots, l,$$

$$H_l^r = \det_{1 \leq i, j \leq l}\left(\left(a_{i,j}\right)_{\substack{i \neq l, \ldots r+1 \\ j \neq l-1, \ldots r}}\right) \cdot \prod_{k=1}^{k=r-2} (\det[k])^{2^{(r-k-2)}}, \quad r = 3, \ldots, l-1, \tag{25}$$

$$K_l^2 = \alpha_1^2 \alpha_l^2 a_1 a_l \left[\frac{\alpha_1 - 1}{\alpha_1} \frac{\alpha_l + 1}{\alpha_l} - A_{1l}^2\right],$$

$$H_l^2 = \alpha_1^2 \alpha_2 \alpha_l a_1 \sqrt{a_2 a_l} \left[\frac{\alpha_1 - 1}{\alpha_1} A_{2l} - A_{12} A_{1l}\right].$$

Lemma 6 (see Masuda and Takahashi [4]). *Let $\mu, T > 0$ and let $f_j = f_j(t)$ be a nonnegative integrable function on $[0, T)$ and $0 < \theta_j < 1$ $(j = 1, \ldots, J)$. Let $W = W(t)$ be a positive function on $[0, T)$ satisfying the differential inequality*

$$\frac{dW(t)}{dt} \leq -\mu W(t) + \sum_{j=1}^J f_j(t) W^{\theta_j}(t), \quad 0 \leq t < T. \tag{26}$$

Then, one has

$$W(t) \leq \kappa, \quad 0 \leq t < T, \tag{27}$$

where κ is the maximal root of the algebraic equation:

$$x - \sum_{j=1}^J \left(\sup_{0<t<T} \int_0^t e^{-\mu(t-\xi)} f_j(\xi)\, d\xi\right) x^{\theta_j} = W(0). \tag{28}$$

4. Proofs

Proof of Theorem 1. Since u_1 satisfies $\partial_t u_1 - a_1 \Delta u_1 > 0$ on $\{u_1 < \sigma/b_1\}$, the maximum principle implies $u_1 \geq \delta := \min(\sigma/b_1, \min u_0(x)) > 0$.

Differentiating $L(t)$ with respect to t yields

$$\begin{aligned} L'(t) &= \int_\Omega \frac{d}{dt}\left(\frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}}\right)dx \\ &= \int_\Omega \left(\alpha_1 \frac{u_1^{\alpha_1-1}}{\prod_{j=2}^m u_j^{\alpha_j}}\partial_t u_1 \right. \\ &\qquad \left. - \sum_{i=2}^m \alpha_i \frac{u_1^{\alpha_1}}{u_i^{\alpha_i+1}\prod_{j=2,j\neq i}^m u_j^{\alpha_j}}\partial_t u_i\right)dx. \end{aligned} \tag{29}$$

Replacing $\partial_t u_i$, $i = 1, \ldots, m$, by its expression from (1), we get

$$\begin{aligned} L'(t) &= \int_\Omega \left(a_1\alpha_1 \frac{u_1^{\alpha_1-1}}{\prod_{j=2}^m u_j^{\alpha_j}}\Delta u_1 \right. \\ &\qquad - \sum_{i=2}^m \alpha_i a_i \frac{u_1^{\alpha_1}}{u_i^{\alpha_i+1}\prod_{j=2,j\neq i}^m u_j^{\alpha_j}}\Delta u_i \\ &\qquad - b_1\alpha \frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}} + \sum_{i=2}^m b_i\alpha_i \frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}} \\ &\qquad + \alpha_1 \frac{u_1^{p_{11}-1+\alpha_1}}{\prod_{j=2}^m u_j^{p_{1j}+\alpha_j}} \\ &\qquad - \sum_{i=2}^m \alpha_i \frac{u_1^{p_{i1}+\alpha_1}}{u_k^{p_{ii}+1+\alpha_i}\prod_{j=2,j\neq i}^m u_j^{p_{i,j}+\alpha_j}} \\ &\qquad \left. + \sigma\alpha_1 \frac{u_1^{\alpha_1-1}}{\prod_{j=2}^m u_j^{\alpha_j}}\right)dx \\ &:= I + J, \end{aligned} \tag{30}$$

where we have set

$$\begin{aligned} I &= a_1\alpha_1 \int_\Omega \frac{u_1^{\alpha_1-1}}{\prod_{j=2}^m u_j^{\alpha_j}}\Delta u_1\, dx \\ &\quad - \sum_{i=2}^m \alpha_i a_i \int_\Omega \frac{u_1^{\alpha_1}}{u_i^{\alpha_i+1}\prod_{j=2,j\neq i}^m u_j^{\alpha_j}}\Delta u_i\, dx, \\ J &= \left(-b_1\alpha_1 + \sum_{i=2}^m b_i\alpha_i\right)L(t) \\ &\quad + \alpha_1 \int_\Omega \frac{u_1^{p_{11}-1+\alpha_1}}{\prod_{j=2}^m u_j^{p_{1j}+\alpha_j}}dx \\ &\quad - \sum_{i=2}^m \alpha_i \int_\Omega \frac{u_1^{p_{i1}+\alpha_1}}{u_k^{p_{ii}+1+\alpha_i}\prod_{j=2,j\neq i}^m u_j^{p_{ij}+\alpha_j}}dx \\ &\quad + \sigma\alpha_1 \int_\Omega \frac{u_1^{\alpha_1-1}}{\prod_{j=2}^m u_j^{\alpha_j}}dx. \end{aligned} \tag{31, 32}$$

Estimation of I. We are going to show that $I \le 0$.

Using Green's formula, we obtain

$$\begin{aligned} I &= \int_\Omega \left(a_1\alpha_1 \left[-(\alpha_1 - 1)\frac{u_1^{\alpha_1-2}}{\prod_{j=2}^m u_j^{\alpha_j}}|\nabla u_1|^2 \right.\right. \\ &\qquad \left. + \sum_{i=2}^m \alpha_i \frac{u_1^{\alpha_1-1}}{u_i^{\alpha_i+1}\prod_{j=2,j\neq i}^m u_j^{\alpha_j}}\nabla u_1 \nabla u_i \right] \\ &\qquad + \sum_{i=2}^m a_i\ \alpha_i \\ &\qquad \cdot \left[\alpha_1 \frac{u_1^{\alpha_1-1}}{u_i^{\alpha_i+1}\prod_{j=2,j\neq i}^m u_j^{\alpha_j}}\nabla u_1 \nabla u_i \right. \\ &\qquad \left. - (\alpha_i + 1)\frac{u_1^{\alpha_1}}{u_i^{\alpha_i+2}\prod_{j=2,j\neq i}^m u_j^{\alpha_j}}|\nabla u_i|^2 \right. \\ &\qquad - \sum_{\substack{k=2\\k\neq i}}^m \alpha_k \\ &\qquad \times \frac{u_1^{\alpha_1}}{u_k^{\alpha_k+1}u_i^{\alpha_i+1}\prod_{j=2,j\neq i,j\neq k}^m u_j^{\alpha_j}} \\ &\qquad \left.\left. \times \nabla u_k \nabla u_i \right]\right)dx, \\ &= -\int_\Omega \left(\frac{u_1^{\alpha_1-2}}{\prod_{j=2}^m u_j^{\alpha_j+2}}(QT)\cdot T \right)dx, \end{aligned} \tag{33}$$

where $Q = (a_{i,j})_{1\le i,j\le m}$ is defined in (8) and

$$T = \left(\prod_{j=2}^m u_j\nabla u_1, \ldots, \prod_{\substack{j=1\\j\neq i}}^m u_j \nabla u_i, \ldots, \prod_{j=1}^{m-1} u_j \nabla u_m \right)^t. \tag{34}$$

The matrix Q is positive definite if and only if all its associated minor matrices $\Delta_1, \Delta_2, \ldots, \Delta_m$ are positive. To see this, we have the following.

(1) $\Delta_1 = a_1\alpha_1(\alpha_1 - 1) > 0$. Using (5), we get $\det[1] > 0$.

(2) According to Lemma 4, we have

$$\det[2] = K_2^2 = \alpha_1^2\alpha_2^2 a_1 a_2 \left[\frac{\alpha_1 - 1}{\alpha_1}\frac{\alpha_2 + 1}{\alpha_2} - A_{12}^2 \right]. \tag{35}$$

Using (6) and (24) for $l = 2$, we get $\det[2] > 0$.

(3) Again according to Lemma 4, we have

$$K_3^3 = \det[3]\det[1]. \tag{36}$$

But $\det[1] > 0$, thus $\operatorname{sign}(K_3^3) = \operatorname{sign}(\det[3])$. Using (6) and (24) for $l = 3$, we get $\det[3] > 0$.

(4) We suppose that $\det[k] > 0$, $k = 1, 2, \ldots, l-1$ and prove that $\det[l] > 0$; thus

$$\begin{gathered}\det[k] > 0, \quad k = 1, \ldots, (l-1) \\ \Longrightarrow \prod_{k=1}^{k=l-2} (\det[k])^{2^{(l-k-2)}} > 0.\end{gathered} \tag{37}$$

From Lemma 4,

$$K_l^l = \det[l] \cdot \prod_{k=1}^{k=l-2} (\det[k])^{2^{(l-k-2)}}. \tag{38}$$

This along with (37) yields

$$\operatorname{sign}\left(K_l^l\right) = \operatorname{sign}(\det[l]). \tag{39}$$

But from (6) and (24) $K_l^l > 0$; thus $\det[l] > 0$.

Consequently, we have $I \le 0$.

Estimation of J. We are going to estimate J by a function of $L(t)$.

According to the maximum principle, there exists C_0 depending on $\varphi_i(x)$, $i = 1, \ldots, m$, such that $u_i \ge C_0 > 0$, $i = 2, \ldots, m$. We then have

$$\begin{aligned}\frac{u_1^{\alpha_1-1}}{\prod_{j=2}^m u_j^{\alpha_j}} &= \left(\frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}}\right)^{(\alpha_1-1)/\alpha_1} \prod_{j=2}^m \left(\frac{1}{u_j}\right)^{\alpha_j/\alpha_1} \\ &\le \left(\frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}}\right)^{(\alpha_1-1)/\alpha_1} \left(\frac{1}{C_0}\right)^{\sum_{j=2}^m \alpha_j/\alpha_1},\end{aligned} \tag{40}$$

whereupon

$$\begin{gathered}\frac{u_1^{\alpha_1-1}}{\prod_{j=2}^m u_j^{\alpha_j}} \le C_2\left(\frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}}\right)^{(\alpha_1-1)/\alpha_1}, \\ \text{where } C_2 = \left(\frac{1}{C_0}\right)^{\sum_{j=2}^m \alpha_j/\alpha_1}.\end{gathered} \tag{41}$$

We have

$$\begin{aligned}J \le{}& \left(-b_1\alpha_1 + \sum_{i=2}^m b_i\alpha_i\right) L(t) \\ &+ \alpha_1 \int_\Omega \frac{u_1^{q_{11}-1+\alpha_1}}{\prod_{j=2}^m u_j^{q_{1j}+\alpha_j}}\, dx \\ &- \sum_{i=2}^m \alpha_i \int_\Omega \frac{u_1^{q_{i1}+\alpha_1}}{u_k^{q_{ii}+1+\alpha_i} \prod_{j=2, j\ne i}^m u_j^{q_{ij}+\alpha_j}}\, dx \\ &+ \sigma\alpha_1 \int_\Omega C_2\left(\frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}}\right)^{(\alpha_1-1)/\alpha_1} dx.\end{aligned} \tag{42}$$

Using Lemma 3, we obtain

$$\begin{aligned}J \le{}& \left(-b_1\alpha_1 + \sum_{i=2}^m b_i\alpha_i\right) L(t) \\ &+ \int_\Omega C\left(\frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}}\right)^\theta dx \\ &+ \sigma\alpha_1 \int_\Omega C_2\left(\frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}}\right)^{(\alpha_1-1)/\alpha_1} dx.\end{aligned} \tag{43}$$

Applying Hölder's inequality, we obtain

$$\begin{aligned}&\int_\Omega C\left(\frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}}\right)^\theta dx \\ &\quad\le \left(\int_\Omega \frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}} dx\right)^\theta C(\operatorname{meas}(\Omega))^{1-\theta}.\end{aligned} \tag{44}$$

So

$$\begin{gathered}\int_\Omega C\left(\frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}}\right)^\theta dx \le C_3 L^\theta(t), \\ C_3 = C(\operatorname{meas}(\Omega))^{1-\theta}.\end{gathered} \tag{45}$$

Also, we have

$$\begin{aligned}&\int_\Omega C_2\left(\frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}}\right)^{(\alpha_1-1)/\alpha_1} dx \\ &\quad\le \left(\int_\Omega \left(\frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}}\right) dx\right)^{(\alpha_1-1)/\alpha_1} \\ &\qquad\cdot \left(\int_\Omega (C_2)^{\alpha_1} dx\right)^{1/\alpha_1}.\end{aligned} \tag{46}$$

So

$$\begin{gathered}\int_\Omega C_2\left(\frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}}\right)^{(\alpha_1-1)/\alpha_1} dx \le C_4 L^{\frac{(\alpha_1-1)}{\alpha_1}}(t), \\ \text{where } C_4 = C_2(\operatorname{meas}(\Omega))^{1/\alpha_1}.\end{gathered} \tag{47}$$

We then get

$$\begin{aligned}J \le{}& \left(-b_1\alpha_1 + \sum_{i=2}^m b_i\alpha_i\right) L(t) + C_3 L^\theta(t) \\ &+ \alpha_1\sigma C_4 L^{(\alpha_1-1)/\alpha_1}(t),\end{aligned} \tag{48}$$

which implies

$$J \le \left(-b_1\alpha_1 + \sum_{i=2}^{m} b_i\alpha_i\right) L(t) + C_5\left(L^{\theta}(t) + \alpha_1\sigma L^{(\alpha_1-1)/\alpha_1}(t)\right). \tag{49}$$

This yields the differential inequality:

$$L'(t) \le \left(-b_1\alpha_1 + \sum_{i=2}^{m} b_i\alpha_i\right) L(t) + C_5\left(L^{\theta}(t) + \alpha_1\sigma L^{(\alpha_1-1)/\alpha_1}(t)\right). \tag{50}$$

Thus under conditions (5), (6), and (8), we obtain $-b_1\alpha_1 + \sum_{i=2}^{m} b_i\alpha_i < 0$; using Lemma 6 we deduce that $L(t)$ is bounded on $(0, T_{\max})$; that is, $L(t) \le \gamma_1$, where γ_1 depends on $\varphi_i(x)$, $i = 1, \ldots, m$. □

Proof of Corollary 2 (L^{∞}-bounds). By Theorem 1, we have $u_1^{p_{i1}}/\prod_{j=2}^{m} u_j^{p_{ij}} \in L^{\infty}((0, T_{\max}), L^r(\Omega))$, $i = 2, \ldots, m$ for all $r > N/2$. By a simple argument relying on the variation-of-constants formula and the L^p-L^q-estimate (Proposition 48.4 see [10]), we deduce that u is uniformly bounded. Consequently, $T_{\max} = \infty$. □

Appendix

The purpose of this appendix is to prove the lemmas of Section 3 which have been used in the proof of Theorem 1.

Proof of Lemma 3. Inequality (18) is equivalent to

$$\alpha_1 \frac{u_1^{q_{11}-1}}{\prod_{j=2}^{m} u_j^{q_{1j}}} \le \alpha_k \frac{u_1^{q_{k1}}}{u_k^{q_{kk}+1}\prod_{j=2, j\ne k}^{m} u_j^{q_{kj}}} + C\left(\frac{u_1^{\alpha_1}}{\prod_{j=2}^{m} u_j^{\alpha_j}}\right)^{\theta-1}. \tag{A.1}$$

Let us set $\zeta = (\alpha_k u_1^{q_{k1}})/(u_k^{q_{kk}+1}\prod_{j=2, j\ne k}^{m} u_j^{q_{kj}})$.

Now, we write

$$\alpha_1 \frac{u_1^{q_{11}-1}}{\prod_{j=2}^{m} u_j^{q_{1j}}} = \alpha_1(\alpha_k)^{-(q_{11}-1)/q_{k1}}(\zeta)^{(q_{11}-1)/q_{k1}} \cdot \prod_{\substack{j=2\\ j\ne k}}^{m}(u_j)^{q_{kj}(q_{11}-1)/q_{k1}-q_{1j}} \cdot (u_k)^{(q_{kk}+1)(q_{11}-1)/q_{k1}-q_{1k}}. \tag{A.2}$$

For each ϵ such that $0 < \epsilon < \min\{1, q_{1k}/(q_{kk}+1), q_{1j}/q_{kj}, j = 2, \ldots, m, \text{and } j \ne k\} - (q_{11}-1)/q_{k1}$,

$$\begin{aligned}
&\alpha_1 \frac{u_1^{q_{11}-1}}{\prod_{j=2}^{m} u_j^{q_{1j}}} \\
&= \alpha_1(\alpha_k)^{-(q_{11}-1)/q_{k1}}(\zeta)^{(q_{11}-1)/q_{k1}+\epsilon} \times (\zeta)^{-\epsilon}\prod_{\substack{j=2\\ j\ne k}}^{m}(u_j)^{q_{kj}(q_{11}-1)/q_{k1}-q_{1j}} \times (u_k)^{(q_{kk}+1)(q_{11}-1)/q_{k1}-q_{1k}} \\
&= \alpha_1(\alpha_k)^{-(q_{11}-1)/q_{k1}-\epsilon}(\zeta)^{(q_{11}-1)/q_{k1}+\epsilon} \times \left(\frac{1}{u_1^{\alpha_1}}\right)^{q_{k1}\epsilon/\alpha_1} \times \prod_{\substack{j=2\\ j\ne k}}^{m}(u_j)^{q_{k,j}(q_{11}-1)/q_{k,1}-q_{1j}+\epsilon q_{kj}} \times (u_k)^{(q_{kk}+1)(q_{11}-1)/q_{k,1}-q_{1k}+\epsilon(q_{kk}+1)} \\
&\le \alpha_1(\alpha_k)^{-(q_{11}-1)/q_{k1}-\epsilon}(\zeta)^{(q_{11}-1)/q_{k1}+\epsilon} \times \left(\frac{1}{u_1^{\alpha_1}}\right)^{q_{k1}\epsilon/\alpha_1} \times \prod_{\substack{j=2\\ j\ne k}}^{m}(h_j)^{q_{kj}(q_{11}-1)/q_{k1}-q_{1j}+\epsilon q_{kj}} \\
&\quad \times (h_k)^{(q_{kk}+1)(q_{11}-1)/q_{k1}-q_{1k}+\epsilon(q_{kk}+1)} \prod_{j=2}^{m}\left(\frac{u_j}{h_j}\right)^{\alpha j q_{k1}\epsilon/\alpha_1} \\
&\le C_1(\zeta)^{(q_{11}-1)/q_{k1}+\epsilon}\left(\frac{\prod_{j=2}^{m} u_j^{\alpha j}}{u_1^{\alpha_1}}\right)^{q_{k1}\epsilon/\alpha_1},
\end{aligned} \tag{A.3}$$

where

$$\begin{aligned}
C_1 &= \alpha_1(\alpha_k)^{-(q_{11}-1)/q_{k1}-\epsilon} \times \prod_{\substack{j=2\\ j\ne k}}^{m}(h_j)^{q_{kj}(q_{11}-1)/q_{k1}-q_{1,j}+\epsilon q_{k,j}-\alpha j q_{k1}\epsilon/\alpha_1} \\
&\quad \times (h_k)^{(q_{kk}+1)(q_{11}-1)/q_{k1}-q_{1k}+\epsilon(q_{kk}+1)-\alpha_k q_{k1}\epsilon/\alpha_1}.
\end{aligned} \tag{A.4}$$

Using Young's inequality for (A.3) with

$$C = C_1^{1+(q_{11}-1+q_{k1}\epsilon)/(q_{k1}-(q_{11}-1)-q_{k1}\epsilon)}, \qquad \theta = 1 - \frac{q_{k1}\epsilon}{\alpha_1(1-(q_{11}-1)/q_{k1}-\epsilon)}, \tag{A.5}$$

where ϵ is sufficiently small, we get inequality (18). □

Proof of Lemma 4. We prove this lemma by induction.

For $m = 2$, we have $K_2^2 = \det[2]$.

We consider the case $m = 3$.

By using the well-known Dodgson condensation [11] for the symmetric 3-square matrix:

$$\det[1]\det[3] = \det[2]\det_{1\le i,j\le 3}\left[\left(a_{i,j}\right)_{i\ne 2, j\ne 2}\right] - \left[\det_{1\le i,j\le 3}\left[\left(a_{i,j}\right)_{i\ne 3, j\ne 2}\right]\right]^2. \tag{A.6}$$

But

$$\det[2] = K_2^2,$$

$$\det_{1\le i,j\le 3}\left[\left(a_{i,j}\right)_{\substack{i\ne 2\\ j\ne 2}}\right] = a_{11}a_{33} - \left(a_{13}\right)^2 = K_3^2, \tag{A.7}$$

$$\det_{1\le i,j\le 3}\left[\left(a_{i,j}\right)_{\substack{i\ne 2\\ j\ne 3}}\right] = a_{11}a_{23} - a_{12}a_{13} = H_3^2.$$

So

$$\det[1]\det[3] = K_2^2\cdot K_3^2 - \left[H_3^2\right]^2. \tag{A.8}$$

Hence by using formula (20), formula (19) is correct for $m = 3$.

When $m \ge 4$, we suppose that formula (19) is correct for $(m-1), m-2, m-3, \ldots, 4$, and we prove it for m.

It is sufficient to prove that

$$K_m^{m-1} = \det_{1\le i,j\le m}\left(\left(a_{i,j}\right)_{\substack{i\ne m-1\\ j\ne m-1}}\right) \cdot \prod_{k=1}^{k=m-3}\left(\det[k]\right)^{2^{(m-k-3)}}. \tag{A.9}$$

By putting $l = m-1$ in formula (21), we get

$$H_m^{m-1} = \det_{1\le i,j\le m}\left(\left(a_{i,j}\right)_{\substack{i\ne m\\ j\ne m-1}}\right) \cdot \prod_{k=1}^{k=m-3}\left(\det[k]\right)^{2^{(m-k-3)}}. \tag{A.10}$$

From the mathematical induction proof, we have

$$K_{(m-1)}^{(m-1)} = \det[m-1] \cdot \prod_{k=1}^{k=m-3}\left(\det[k]\right)^{2^{(m-k-3)}}. \tag{A.11}$$

By putting $l = m$ in formula (20), we get

$$K_m^m = K_{m-1}^{m-1}\cdot K_m^{m-1} - \left(H_m^{m-1}\right)^2. \tag{A.12}$$

By replacing (A.9), (A.10), and (A.11) in (A.12), we obtain

$$K_m^m = \prod_{k=1}^{k=m-3}\left(\det[k]\right)^{2^{(m-k-2)}} \cdot \det[m-2]\cdot \det[m] = \det[m]\cdot\prod_{k=1}^{k=m-2}\left(\det[k]\right)^{2^{(m-k-2)}}, \tag{A.13}$$

and thus formula (19) is correct for m.

Now, we prove formula (A.9); we may generalize formula (A.9) as follows:

$$K_m^l = \det_{1\le i,j\le m}\left(\left(a_{ij}\right)_{\substack{i\ne m-1,\ldots l\\ j\ne m-1,\ldots l}}\right) \cdot \prod_{k=1}^{k=l-2}\left(\det[k]\right)^{2^{((l-2)-k)}}, \quad l = 3,\ldots,m-1. \tag{A.14}$$

Also, we prove formula (A.14) by induction. It is a second inductive proof included in the first one.

It is evident for $l = 2$.

For $l = 3$, formula (20) will be:

$$K_m^3 = K_2^2\cdot K_m^2 - \left[H_m^2\right]^2. \tag{A.15}$$

Since we already know that

$$K_2^2 = \det[2],$$

$$K_m^2 = \det_{1\le i,j\le m}\left(\left(a_{i,j}\right)_{\substack{i\ne m-1,\ldots 2\\ j\ne m-1,\ldots 2}}\right), \tag{A.16}$$

$$H_m^2 = \det_{1\le i,j\le m}\left(\left(a_{i,j}\right)_{\substack{i\ne m-1,\ldots,2\\ j\ne m,\ldots,3}}\right),$$

simple substitution of these three formulas in the formula (A.15) followed by the application of the modified well-known Dodgson condensation which has been modified in [11] will lead to formula (A.14) for $l = 3$. directly.

When $l \ge 4$, we suppose that formula (A.14) is correct for $l-1$, and we prove it for l.

Formula (20) for $l-1$ reads

$$K_m^l = K_{l-1}^{l-1}\cdot K_m^{l-1} - \left[H_m^{l-1}\right]^2. \tag{A.17}$$

According to the first induction, we have

$$K_{(l-1)}^{(l-1)} = \det[l-1]\prod_{k=1}^{k=l-3}\left(\det[k]\right)^{2^{(l-k-3)}}. \tag{A.18}$$

According to the second induction, we have

$$K_m^{l-1} = \det_{1\le i,j\le m}\left(\left(a_{i,j}\right)_{\substack{i\ne m-1,\ldots,l-1\\ j\ne m-1,\ldots,l-1}}\right) \cdot \prod_{k=1}^{k=(l-3)}\left(\det[k]\right)^{2^{((l-3)-k)}}. \tag{A.19}$$

According to formula (21), we have:

$$H_m^{l-1} = \det_{1\le i,j\le m}\left(\left(a_{i,j}\right)_{\substack{i\ne m,\ldots,l\\ j\ne m-1,\ldots,l-1}}\right) \cdot \prod_{k=1}^{k=l-3}\left(\det[k]\right)^{2^{((l-3)-k)}}. \tag{A.20}$$

By replacing (A.18), (A.19), and (A.20) in (A.17) and by using the well-known Dodgson condensation, we obtain formula (A.14) for l. Therefore, the second inductive proof is finished and consequently the first one. □

Proof of Lemma 5. We prove this lemma by induction:

$$K_l^l > S_l^l, \quad l = 2, \ldots, m. \tag{A.21}$$

For $l = 2$, we have

$$\begin{aligned} K_2^2 &= \alpha_1^2\alpha_2^2 a_1 a_2 \left[\frac{\alpha_1 - 1}{\alpha_1}\frac{\alpha_2 + 1}{\alpha_2} - A_{12}^2\right] \\ &> \alpha_1^2\alpha_2^2 a_1 a_2 \left[\frac{1}{2\alpha_2} - A_{12}^2\right] \\ &= S_2^2. \end{aligned} \tag{A.22}$$

Because

$$\alpha_1 > 2, \text{ then } \frac{\alpha_1 - 1}{\alpha_1}\frac{\alpha_2 + 1}{\alpha_2} > \frac{1}{2\alpha_2}. \tag{A.23}$$

Assuming $l \geq 3$, we suppose (24) is true for $(l-1), l-2, l-3, \ldots, 3$, and we prove it for l. Hence, we aim to prove

$$\begin{gathered} K_2^2 > S_2^2, \qquad K_3^3 > S_3^3, \qquad K_4^4 > S_4^4, \ldots, \\ K_{l-1}^{l-1} > S_{l-1}^{l-1} \Longrightarrow K_l^l > S_l^l. \end{gathered} \tag{A.24}$$

Recall that

$$K_l^l = K_{l-1}^{l-1}K_l^{l-1} - \left[H_l^{l-1}\right]^2. \tag{A.25}$$

It is then sufficient to prove

$$K_l^{l-1} > S_l^{l-1}, \tag{A.26}$$

which will satisfy the inequality

$$\begin{aligned} K_l^l &= K_{l-1}^{l-1}K_l^{l-1} - \left[H_l^{l-1}\right]^2 \\ &> S_{l-1}^{l-1}S_l^{l-1} - \left[H_l^{l-1}\right]^2 = S_l^l. \end{aligned} \tag{A.27}$$

In order to prove (A.26), we first generalize it in the form

$$K_l^r > S_l^r, \quad r = 2, \ldots, l-1. \tag{A.28}$$

This can be proven by mathematical induction. It is a secondary inductive proof inside the primary one. For $r = 2$, it is evident that

$$K_l^2 > S_l^2. \tag{A.29}$$

For $r = 3$, the formula

$$K_l^3 = K_2^2K_l^2 - \left[H_l^2\right]^2 > S_2^2S_l^2 - \left[H_l^2\right]^2 = S_l^3 \tag{A.30}$$

is evident too.

When $r \geq 4$, we suppose formula (A.28) true for $l-2$:

$$K_l^{l-2} > S_l^{l-2} \tag{A.31}$$

and we prove it for $l-1$:

$$K_l^{l-1} > S_l^{l-1}. \tag{A.32}$$

We have

$$\begin{aligned} K_l^{l-1} &= K_{l-2}^{l-2}K_l^{l-2} - \left[H_l^{l-2}\right]^2 \\ &> S_{l-2}^{l-2}S_l^{l-2} - \left[H_l^{l-2}\right]^2 \\ &= S_l^{l-1}. \end{aligned} \tag{A.33}$$

Then

$$K_l^{l-1} > S_l^{l-1}. \tag{A.34}$$

Accordingly, we have

$$K_l^l > S_l^l. \tag{A.35}$$

This finishes the proof. □

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