Research Article

Global Solutions for an $m$-Component System of Activator-Inhibitor Type

S. Abdelmalek, 1,2 A. Gouadria, 3 and A. Youkana 3

1 Department of Mathematics, College of Sciences, Taibah University, Yanbu, Saudi Arabia
2 Department of Mathematics, University of Tebessa, 12002 Tebessa, Algeria
3 Department of Mathematics, University of Batna, 05000 Batna, Algeria

Correspondence should be addressed to S. Abdelmalek; sallllm@gmail.com

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This paper deals with a reaction-diffusion system with fractional reactions modeling $m$-substances into interaction following activator-inhibitor’s scheme. The existence of global solutions is obtained via a judicious Lyapunov functional that generalizes the one introduced by Masuda and Takahashi.

1. Introduction

In this paper, we are concerned with the existence of global solutions to a reaction-diffusion system with $m$ components generalizing the activator-inhibitor system:

$$
\begin{align*}
\partial_t u_1 - a_1 \Delta u_1 &= f_1(u) = \sigma - b_1 u_1 + \prod_{j=2}^{m} u_j^{p_{1j}}, \\
\partial_t u_i - a_i \Delta u_i &= f_i(u) = -b_i u_i + \prod_{j=2}^{m} u_j^{p_{ij}}, \\
&\quad i = 2, \ldots, m,
\end{align*}
$$

supplemented with Neumann boundary conditions

$$
\frac{\partial u_i}{\partial \eta} = 0, \quad \text{on } \partial \Omega \times \{t > 0\}, \quad i = 1, \ldots, m,
$$

and the positive initial data

$$
u_i (x, 0) = \varphi_i (x) \quad \text{on } \Omega, \quad i = 1, \ldots, m.
$$

Here $u = (u_1, u_2, \ldots, u_m)$, $\Omega$ is an open bounded domain of class $C^1$ in $\mathbb{R}^N$, with boundary $\partial \Omega$, and $\partial / \partial \eta$ denotes the outward normal derivative on $\partial \Omega$.

Throughout the paper, we make the following hypotheses:

The indexes $a_i$, $p_{ij}$ are nonnegative for all $i, j = 1, \ldots, m$, with $\sigma > 0$:

$$
0 < p_{11} - 1 < \max_{k=2,3,\ldots,m} \left\{ p_{k1} \min \left\{ 1, \frac{p_{1k}}{p_{kk} + 1}, \frac{p_{lj}}{p_{kj}} \right\} \right\}, \quad j = 2, \ldots, m, \quad j \neq k,
$$

we set $A_{ij} = (a_i + a_j) / (2 \sqrt{a_i a_j})$ for all $i, j = 1, \ldots, m$. Let $\alpha_i, i = 1, \ldots, m$, be positive constants such that

$$
\alpha_i > 2 \max \left\{ 1, \sum_{i=1}^{m} \frac{b_j}{b_i} \right\},
$$

$$
S'_r > 0, \quad l = 2, \ldots, m,
$$

where

$$
S'_r = S'_{r-1} \cdot S'_{r-1} - \left[ H'_r \right]^2, \quad r = 3, \ldots, l,
$$

$$
H'_r = \det_{1 \leq i, j \leq l} \left( A_{ij} \right)_{i \neq l, j \neq l} \prod_{k=1}^{m} (\det[k])^{2(p_{rk} - 1)}, \quad r = 3, \ldots, l - 1,
$$

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Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) (\( N = 1, 2, 3 \) in practice) with smooth boundary \( \partial \Omega \). \( a_1, a_2, \mu, \nu, \sigma > 0 \), and \( p, q, r \) and \( s \) are non-negative indexes with \( p > 1 \). Here \( u \) is the activator, and \( v \) is the inhibitor.

Global existence of solutions in \((0, \infty)\) was proved by Rothe [3], more than ten years after Gierer and Meinhardt’s original paper with special choice of the parameters: \( p = 2 \), \( q = 1 \), \( r = 2 \), \( s = 0 \), and \( N = 3 \). Masuda and Takahashi [4] were able to prove global estimates and bounds of the solution for Gierer and Meinhardt’s system in its general form. They proceeded by first proving lower bounds, then \( L^p \) bounds (for any \( p > 1 \)), then uniform estimates and bounds in appropriate Sobolev spaces. The key point is represented by the \( L^2 \) bounds, which are derived using in a subtle way the specific structure of the equations.

Li et al. [5] also studied the activator-inhibitor model.

Very recently, Bernasconi [6] considered the larger system:

\[
\partial_t a(x, t) = d_1 a_{xx} + \frac{a^2(x, t)}{h(x, t)} - \mu a(x, t) + \rho,
\]

\[
\partial_t h(x, t) = d_2 h_{xx} (x, t) + a^2(x, t) - \nu h(x, t) + \epsilon s(x, t),
\]

\[
\partial_t s(x, t) = d_3 s_{xx} (x, t) + a(x, t) - \kappa s(x, t),
\]

and Meinhardt et al. [7] proposed activator-inhibitor models to describe a theory of biological pattern:

\[
\partial_t a(x, t) = d_1 a_{xx} + \frac{a^2(x, t)}{h(x, t) s(x, t)} - \mu a(x, t) + \rho,
\]

\[
\partial_t h(x, t) = d_2 h_{xx} (x, t) + a^2(x, t) - \nu h(x, t) + \epsilon s(x, t),
\]

\[
\partial_t s(x, t) = d_3 s_{xx} (x, t) + a(x, t) - \kappa s(x, t),
\]

which is Gierer and Meinhardt’s system supplemented with a third equation, where \( a(x, t) \) is the activator, \( h(x, t) \) is the inhibitor, and \( s(x, t) \) is a source that acts as an inhomogeneous inhibitor.

Our paper generalizes the system in [5] to \( m \)-components.

2. Preliminary Observations and Notations

The usual norms in the spaces \( L^p(\Omega) \), \( L^{\infty}(\Omega) \), and \( C(\overline{\Omega}) \) are denoted, respectively, by the following:

\[
\|u\|_p = \frac{1}{|\Omega|} \int_{\Omega} |u(x)|^p \, dx,
\]

\[
\|u\|_{\infty} = \text{ess sup}_{x \in \Omega} |u(x)|,
\]

\[
\|u\|_{C(\overline{\Omega})} = \max_{x \in \overline{\Omega}} |u(x)|.
\]

and initial conditions

\[
u(x, 0) = \varphi_1 (x) > 0, \quad x \in \Omega, \quad (12)
\]

where \( \Omega \subset \mathbb{R}^N \) (\( N = 1, 2, 3 \) in practice) is a bounded domain with smooth boundary \( \partial \Omega \), \( a_1, a_2, \mu, \nu, \sigma > 0 \), and \( p, q, r \) and \( s \) are non-negative indexes with \( p > 1 \). Here \( u \) is the activator, and \( v \) is the inhibitor.

Before we prove our results, let us dwell a while on the existing literature concerning Gierer-Meinhardt’s type systems.

In 1972, following an ingenious idea of Turing [1], Gierer and Meinhardt [2] proposed a mathematical model for pattern formations of spatial tissue structures of hydra in morphogenesis. It is a system of reaction-diffusion equations of the form:

\[
\partial_t u - a_1 \Delta u = \sigma - \mu u + \frac{v^p}{v^{p+1}}, \quad x \in \Omega, \quad t > 0, \tag{10}
\]

\[
\partial_t v - a_2 \Delta v = -\nu v + \frac{u^q}{v^q}, \quad x \in \Omega, \quad t > 0,
\]

with Neumann boundary conditions

\[
\frac{\partial u}{\partial \eta} = 0, \quad x \in \partial \Omega, \quad t > 0, \tag{11}
\]

and initial conditions

\[
u(x, 0) = \varphi_1 (x) > 0, \quad x \in \Omega, \quad (12)
\]
It is well known that to prove global existence of solutions to (1)–(3), it suffices to derive a uniform estimate of \( \| f_i (u_1, u_2, \ldots, u_m) \|_p \), \( i = 1, \ldots, m \) on \([0; T_{\text{max}}] \) in the space \( L^p (\Omega) \) for some \( p > n / 2 \) (see Henry [8]).

Since \( f_1 \) are continuously differentiable on \( \mathbb{R}^m_+ \) for all \( i = 1, \ldots, m \), then for any initial data in \( C(\Omega) \), the system (1)–(3) admits a unique, classical solution \((u_1, u_2, \ldots, u_m)\) on \((0, T_{\text{max}}) \times \Omega\) with the alternative

(i) either \( T_{\text{max}} = \infty \);

(ii) or \( T_{\text{max}} < \infty \), and \( \lim_{t \to T_{\text{max}}} \sum_{i=1}^m \| u_i (t, \cdot) \|_\infty = \infty \).

Using the maximum principle, one derives the lower bounds of the components of the solution \( u \) of (1)–(3):

\[
 u_i (t, x) \geq e^{-h_1} \min \left( \varphi_i (x) \right) > 0, \quad i = 1, \ldots, m. \tag{16}
\]

Our aim is to construct a Lyapunov functional that allows us to obtain \( L^p \)-bounds on \( u_i \) leading to global existence.

### 3. Preparatory Lemmas

For the proof of Theorem 1, we need some preparatory lemmas whose proofs will be in the appendix.

**Lemma 3.** Assume that the constants \( q_{ij} \) satisfy

\[
 q_{ii} - 1 \quad q_{kl} < \min \left( 1, \frac{q_{ik}}{q_{kk} + 1} q_{lj}, \quad j = 2, \ldots, m, \ j \neq k \right). \tag{17}
\]

Then for all \( h_{i-1}, \alpha_i > 0, j, i = 1, \ldots, m \), there exist \( C = C(h_{i-1}, \alpha_i) > 0 \) and \( \theta = \theta(\alpha_i) \in (0, 1) \), such that

\[
\alpha_i U_{ij}^{q_{ii} - 1 + \alpha_i} \leq \alpha_k \frac{U_{ij}^{q_{ii} - 1 + \alpha_i}}{U_{ij}^{q_{ii} - 1 + \alpha_i}} \prod_{j=2, j \neq k}^m U_{ij}^{q_{ij} \theta_{ij}} \tag{18}
\]

\[
+ \left( \frac{U_{ij}^{q_{ii} - 1 + \alpha_i}}{U_{ij}^{q_{ii} - 1 + \alpha_i}} \right)^\theta,
\]

\( u_i \geq 0, u_i \geq h_{i-1}, i = 1, \ldots, m, k \in \{2, \ldots, m\} \).

**Lemma 4** (see [9]). Let \( A = (a_{ij})_{1 \leq i, j \leq m} \). Then one has:

\[
K_m^l = \det [m] \prod_{k=1}^{m-l-2} (\det [k])^{2-\theta_{ij}}, \quad m > 2, \tag{19}
\]

where

\[
K_m^l = K_m^{l-1} \cdot K_m^{l-1} \cdot \left( H_m^{l-1} \right)^2, \quad l = 3, \ldots, m, \tag{20}
\]

\[
H_m^l = \det_{1 \leq i, j \leq m} \left( a_{ij} \right)_{i \neq l+1, j \neq l+1} \tag{21}
\]

\[
\cdot \prod_{k=1}^{m-l-2} (\det [k])^{q_{ij} - 1}, \quad l = 3, \ldots, m - 1,
\]

\[
K_m^l = a_{11} a_{mm} - (a_{1m})^2, \tag{22}
\]

\[
H_m^l = a_{11} a_{mm} - a_{12} a_{m2} - a_{22} a_{m2}. \tag{23}
\]

**Lemma 5.** Let \( \alpha_i > 2 \max \{1, \sum_{i=1}^m b_i / b_i \} \). One has

\[
K_m^l > S_l, \quad l = 2, \ldots, m, \tag{24}
\]

where

\[
K_m^l = K_m^{l-1} - \left( H_m^{l-1} \right)^2, \quad l = 3, \ldots, m, \tag{25}
\]

\[
H_m^l = \det_{1 \leq i, l \leq l} \left( a_{lj} \right)_{j \neq l} \tag{26}
\]

\[
\cdot \prod_{k=1}^{m-l-2} (\det [k])^{q_{ij} - 1}, \quad r = 3, \ldots, l - 1, \tag{27}
\]

\[
K_m^l = \alpha_1^2 \alpha_i^2 \alpha_1 \left[ \frac{\alpha_i - 1}{\alpha_i} + 1 \right] - A_{ij}^2, \tag{28}
\]

\[
H_m^l = \alpha_1^2 \alpha_2 \alpha_1 \left[ \frac{\alpha_i - 1}{\alpha_i} A_{ij}^2 \right] - A_{ij}^2 A_{ij}^2. \tag{29}
\]

**Lemma 6** (see Masuda and Takahashi [4]). Let \( \mu, T > 0 \) and let \( f_i (t) \) be a nonnegative integrable function on \([0, T] \) and \( \theta_j < 1 (j = 1, \ldots, J) \). Let \( W = W(t) \) be a positive function on \([0, T] \) satisfying the inequality

\[
\frac{dW(t)}{dt} \leq -\mu W(t) + \sum_{j=1}^J f_j (t) W^\theta_j (t), \quad 0 \leq t < T. \tag{30}
\]

Then, one has

\[
W(t) \leq \kappa, \quad 0 \leq t < T, \tag{31}
\]

where \( \kappa \) is the maximal root of the algebraic equation:

\[
x - \sum_{j=1}^J \left( \sup_{0 < t < T} \int_0^t e^{-\mu (t-s)} f_j (s) d\xi \right) x^{\theta_j} = W(0). \tag{32}
\]

### 4. Proofs

**Proof of Theorem 1.** Since \( u_1 \) satisfies \( \partial_t u_1 - a_1 \Delta u_1 > 0 \) on \( \{u_1 < \sigma / b_1\} \), the maximum principle implies \( u_1 \geq \delta := \min(\sigma / b_1, \min u_0(x)) > 0. \)
Differentiating $L(t)$ with respect to $t$ yields

$$L'(t) = \int_\Omega \frac{d}{dt} \left( \frac{u_i^{\alpha_i}}{\prod_{j=2}^{m} u_j^{\alpha_j}} \right) dx$$

$$= \int_\Omega \left( \alpha_i \frac{u_i^{\alpha_i-1}}{\prod_{j=2}^{m} u_j^{\alpha_j}} \partial_i u_i \right. + \left. - \sum_{i=2}^{m} \alpha_i \frac{u_i^{\alpha_i+1}}{\prod_{j=2}^{m} u_j^{\alpha_j}} \partial_i u_i \right) dx. \quad (29)$$

Replacing $\partial_i u_i, i = 1, \ldots, m$, by its expression from (1), we get

$$L'(t) = \int_\Omega \left( a_i \alpha_i \frac{u_i^{\alpha_i-1}}{\prod_{j=2}^{m} u_j^{\alpha_j}} \Delta u_i \right. + \left. - \sum_{i=2}^{m} \alpha_i \frac{u_i^{\alpha_i+1}}{\prod_{j=2}^{m} u_j^{\alpha_j}} \Delta u_i \right)$$

$$- b \alpha_1 \frac{u_1^{\alpha_1}}{\prod_{j=2}^{m} u_j^{\alpha_j}} + \sum_{i=2}^{m} b_i \alpha_i \frac{u_i^{\alpha_i}}{\prod_{j=2}^{m} u_j^{\alpha_j}}$$

$$+ \alpha_1 \frac{u_1^{p_1+1+\alpha_1}}{\prod_{j=2}^{m} u_j^{p_j+\alpha_j}}$$

$$- \sum_{i=2}^{m} \alpha_i \frac{u_i^{p_i+1+\alpha_i}}{\prod_{j=2}^{m} u_j^{p_j+\alpha_j}}$$

$$+ \sigma \alpha_i \frac{u_i^{\alpha_i-1}}{\prod_{j=2}^{m} u_j^{\alpha_j}} dx \quad (30)$$

$$:= I + J,$$ 

where we have set

$$I = a_1 \alpha_1 \int_\Omega \frac{u_1^{\alpha_1-2}}{\prod_{j=2}^{m} u_j^{\alpha_j}} \Delta u_1 dx$$

$$- \sum_{i=2}^{m} \alpha_i \alpha_i \int_\Omega \frac{u_i^{\alpha_i-1}}{\prod_{j=2}^{m} u_j^{\alpha_j}} \Delta u_i dx,$$

$$J = \left( -b \alpha_1 + \sum_{i=2}^{m} b_i \alpha_i \right) L(t)$$

$$+ \alpha_1 \int_\Omega \frac{u_1^{p_1+1+\alpha_1}}{\prod_{j=2}^{m} u_j^{p_j+\alpha_j}} dx.$$ 

$$- \sum_{i=2}^{m} \alpha_i \int_\Omega \frac{u_i^{p_i+1+\alpha_i}}{\prod_{j=2}^{m} u_j^{p_j+\alpha_j}} dx \quad (32)$$

**Estimation of $I$.** We are going to show that $I \leq 0$.

Using Green’s formula, we obtain

$$I = \int_\Omega \left( a_1 \alpha_1 \left[ -(\alpha_i - 1) \frac{u_i^{\alpha_i-2}}{\prod_{j=2}^{m} u_j^{\alpha_j}} |\nabla u_i|^2 + \sum_{i=2}^{m} \alpha_i \frac{u_i^{\alpha_i-1}}{\prod_{j=2}^{m} u_j^{\alpha_j}} |\nabla u_i|^2 \right] \right) dx,$$

$$= - \int_\Omega \left( \frac{u_1^{\alpha_1-2}}{\prod_{j=2}^{m} u_j^{\alpha_j}} (QT) \cdot T \right) dx,$$

where $Q = (a_{ij})_{1 \leq i, j \leq m}$ is defined in (8) and

$$T = \left( \prod_{j=2}^{m} u_j^1 \vee u_1^1, \ldots, \prod_{j=2}^{m} u_j^1 \vee u_j^1, \prod_{j=1}^{m-1} u_j \vee u_m \right)^t. \quad (31)$$

The matrix $Q$ is positive definite if and only if all its associated minor matrices $\Delta_1, \Delta_2, \ldots, \Delta_m$ are positive. To see this, we have the following.

1. $\Delta_1 = a_1 \alpha_1 (\alpha_1 - 1) > 0$. Using (5), we get $\text{det}[1] > 0$.
2. According to Lemma 4, we have

$$\text{det}[2] = k_2^2 = a_2^2 \alpha_2^2 \alpha_2 \left[ \frac{\alpha_1 - 1}{\alpha_1} \frac{\alpha_2}{\alpha_2} + A_2^2 \right]. \quad (35)$$

Using (6) and (24) for $l = 2$, we get $\text{det}[2] > 0$. 

Abstract and Applied Analysis
(3) Again according to Lemma 4, we have
\[ K_3^3 = \det[3] \det[1]. \] (36)
But \( \det[1] > 0 \), thus \( \text{sign}(K_3^3) = \text{sign}(\det[3]). \)

Using (6) and (24) for \( l = 3 \), we get \( \det[3] > 0 \).

(4) We suppose that \( \det[k] > 0, k = 1,2,\ldots,l-1 \) and prove that \( \det[l] > 0 \); thus
\[ \det [k] > 0, \quad k = 1, \ldots, (l - 1) \]
\[ \implies \prod_{k=1}^{k_{l-2}} (\det[k])^{(j_{l-k-2})} > 0. \] (37)

From Lemma 4,
\[ K_l^i = \det [l] \prod_{k=1}^{k_{l-2}} (\det[k])^{(j_{l-k-2})}. \] (38)

This along with (37) yields
\[ \text{sign} (K_l^i) = \text{sign} (\det [l]). \] (39)

But from (6) and (24) \( K_l^i > 0 \); thus \( \det[l] > 0 \).

Consequently, we have \( I \leq 0 \).

**Estimation of J.** We are going to estimate \( J \) by a function of \( L(t) \).

According to the maximum principle, there exists \( C_0 \) depending on \( \varphi_i(x), i = 1, \ldots, m \), such that \( u_i \geq C_0 > 0, i = 2, \ldots, m \). We then have
\[
\frac{u_1^{\alpha_i-1}}{\prod_{j=2}^{m} u_j^{\alpha_j}} \left( \frac{u_1^{\alpha_i}}{\prod_{j=2}^{m} u_j^{\alpha_j}} \right)^{(\alpha_i-1)/\alpha_i} \leq C_2 \left( \frac{\sum_{j=2}^{m} u_j^{\alpha_j}}{\prod_{j=2}^{m} u_j^{\alpha_j}} \right)^{\alpha_i/\alpha_j},
\] (40)

whereupon
\[
\frac{u_1^{\alpha_i-1}}{\prod_{j=2}^{m} u_j^{\alpha_j}} \leq C_2 \left( \frac{\sum_{j=2}^{m} u_j^{\alpha_j}}{\prod_{j=2}^{m} u_j^{\alpha_j}} \right)^{\alpha_i/\alpha_j},
\] (41)

where \( C_2 = \left( \frac{1}{C_0} \right)^{\sum_{j=2}^{m} \alpha_j/\alpha_i}. \)

We have
\[
J \leq \left( -b_1 \alpha_1 + \sum_{i=2}^{m} b_i \alpha_i \right) L(t)
\]
\[ + \alpha_1 \int_{\Omega} \left( \frac{u_1^{\alpha_i-1} u_i^{\alpha_i}}{\prod_{j=2}^{m} u_j^{\alpha_j}} \right)^{(\alpha_i-1)/\alpha_i} \frac{dx}{\prod_{j=2}^{m} u_j^{\alpha_j}}
\]
\[ + \sum_{i=2}^{m} \alpha_i \int_{\Omega} \left( \frac{u_1^{\alpha_i-1} u_i^{\alpha_i}}{\prod_{j=2}^{m} u_j^{\alpha_j}} \right)^{(\alpha_i-1)/\alpha_i} \frac{dx}{\prod_{j=2}^{m} u_j^{\alpha_j}}
\]
\[ + \sigma \alpha_1 \int_{\Omega} C_2 \left( \frac{u_1^{\alpha_i}}{\prod_{j=2}^{m} u_j^{\alpha_j}} \right)^{(\alpha_i-1)/\alpha_i} \frac{dx}{\prod_{j=2}^{m} u_j^{\alpha_j}}. \]

Using Lemma 3, we obtain
\[
J \leq \left( -b_1 \alpha_1 + \sum_{i=2}^{m} b_i \alpha_i \right) L(t)
\]
\[ + \int_{\Omega} C \left( \frac{u_1^{\alpha_i}}{\prod_{j=2}^{m} u_j^{\alpha_j}} \right)^{\theta} \frac{dx}{\prod_{j=2}^{m} u_j^{\alpha_j}}
\]
\[ + \sigma \alpha_1 \int_{\Omega} C_2 \left( \frac{u_1^{\alpha_i}}{\prod_{j=2}^{m} u_j^{\alpha_j}} \right)^{(\alpha_i-1)/\alpha_i} \frac{dx}{\prod_{j=2}^{m} u_j^{\alpha_j}}. \]

Applying Hölder’s inequality, we obtain
\[
\int_{\Omega} C \left( \frac{u_1^{\alpha_i}}{\prod_{j=2}^{m} u_j^{\alpha_j}} \right)^{\theta} \frac{dx}{\prod_{j=2}^{m} u_j^{\alpha_j}} \leq C_3 \left( \text{meas}(\Omega) \right)^{1-\theta}. \]

So
\[
\int_{\Omega} C \left( \frac{u_1^{\alpha_i}}{\prod_{j=2}^{m} u_j^{\alpha_j}} \right)^{\theta} \frac{dx}{\prod_{j=2}^{m} u_j^{\alpha_j}} \leq C_4 L^{\theta} (t),
\]
\[ C_3 = C(\text{meas}(\Omega))^{1-\theta}. \]

Also, we have
\[
\int_{\Omega} C_2 \left( \frac{u_1^{\alpha_i}}{\prod_{j=2}^{m} u_j^{\alpha_j}} \right)^{\alpha_i/\alpha_j} \frac{dx}{\prod_{j=2}^{m} u_j^{\alpha_j}} \leq C_4 L^{\theta} (t), \]
\[ \text{where } C_4 = C_2(\text{meas}(\Omega))^{1/\alpha_i}. \]

We then get
\[
J \leq \left( -b_1 \alpha_1 + \sum_{i=2}^{m} b_i \alpha_i \right) L(t) + C_4 L^{\theta} (t)
\]
\[ + \alpha_1 \sigma_4 L^{(\alpha_i-1)/\alpha_i} (t), \]
}\]
which implies
\[ J \leq \left( -b_1\alpha_1 + \sum_{i=2}^{m} b_i\alpha_i \right) L(t) + C_5 \left( Lt^\alpha \right) . \tag{49} \]

This yields the differential inequality:
\[ L'(t) \leq \left( -b_1\alpha_1 + \sum_{i=2}^{m} b_i\alpha_i \right) L(t) + C_5 \left( Lt^\alpha \right). \tag{50} \]

Thus under conditions (5), (6), and (8), we obtain \(-b_1\alpha_1 + \sum_{i=2}^{m} b_i\alpha_i < 0\); using Lemma 6 we deduce that \(L(t)\) is bounded on \((0, T_{\text{max}}]\) that is, \(L(t) \leq \gamma_1\), where \(\gamma_1\) depends on \(\varphi_i(x)\), \(i = 1, \ldots, m\).

\[ \square \]

**Proof of Corollary 2 (L^\infty -bounds).** By Theorem 1, we have \(u_1^{q_1}/\prod_{j=2}^{m} u_j^{p_j} \in L^\infty((0, T_{\text{max}}], L^r(\Omega)), r > N/2\). By a simple argument relying on the variation-of-constants formula and the \(L^p-L^q\)-estimate (Proposition 48.4 see [10]), we deduce that \(u\) is uniformly bounded. Consequently, \(T_{\text{max}} = \infty\).

\[ \square \]

**Appendix**

The purpose of this appendix is to prove the lemmas of Appendix which imply
\[ J \leq \left( -b_1\alpha_1 + \sum_{i=2}^{m} b_i\alpha_i \right) L(t) + C_5 \left( Lt^\alpha \right) . \tag{49} \]

For each \(\varepsilon\) such that \(0 < \varepsilon < \min\{1, q_{1k}/(q_{kk} + 1), q_{11}/q_{kk}, j = 2, \ldots, m\text{, and } j \neq k\}\), \(- (q_{11} - 1)/q_{kk}\),
\[ \alpha_1 \frac{u_1^{q_{11} - 1}}{\prod_{j=2}^{m} u_j^{q_j}} = \alpha_1 \left( \frac{u_1}{\prod_{j=2}^{m} u_j} \right)^{q_1} \times \left( \frac{h_1}{u_1^{q_1}} \right)^{q_{11} - 1}/q_{kk} - q_{11} \cdot \varepsilon \left( \frac{h_1}{u_1^{q_1}} \right)^{q_{11} - 1}/q_{kk} - q_{11} \cdot \varepsilon \right) \tag{A.3} \]

\[ \times \left( \frac{1}{\alpha_1} \right) u_1^{q_1}/u_1^{q_1} \times \prod_{j=2}^{m} \left( h_1 \right)^{q_j/q_{kk} - q_{11} - q_{kk} + \varepsilon q_{kk} - q_{11} - q_{kk} \cdot \varepsilon} \times \left( \prod_{j=2}^{m} \left( \frac{u_j}{h_1} \right)^{q_j/q_{kk} - q_{11} - q_{kk} + \varepsilon q_{kk} - q_{11} - q_{kk} \cdot \varepsilon} \times \left( h_1 \right)^{q_{11} - 1}/q_{kk} + \varepsilon q_{kk} - q_{11} - q_{kk} \cdot \varepsilon} \right) \tag{A.4} \]

where
\[ C_1 = \alpha_1 \left( \frac{u_1}{\prod_{j=2}^{m} u_j} \right)^{q_1} \times \left( \frac{h_1}{u_1^{q_1}} \right)^{q_{11} - 1}/q_{kk} - q_{11} \cdot \varepsilon \right) \times \left( \frac{1}{\alpha_1} \right) u_1^{q_1}/u_1^{q_1} \times \prod_{j=2}^{m} \left( h_1 \right)^{q_j/q_{kk} - q_{11} - q_{kk} + \varepsilon q_{kk} - q_{11} - q_{kk} \cdot \varepsilon} \times \left( h_1 \right)^{q_{11} - 1}/q_{kk} + \varepsilon q_{kk} - q_{11} - q_{kk} \cdot \varepsilon} \right) \tag{A.5} \]

Using Young's inequality for (A.3) with
\[ C = C_1 \left( \frac{q_{11} - 1}{q_{kk} + q_{kk} - q_{11} - q_{kk} + \varepsilon q_{kk} - q_{11} - q_{kk} \cdot \varepsilon} \right) \tag{A.5} \]

where \(\varepsilon\) is sufficiently small, we get inequality (18).

\[ \square \]

**Proof of Lemma 4.** We prove this lemma by induction.

For \(m = 2\), we have \(K^2 = \det[2]\).

We consider the case \(m = 3\).
By using the well-known Dodgson condensation [11] for the symmetric 3-square matrix:

\[
\begin{align*}
\det[1] \det[3] &= \det[2] \left[ \det_{1 \leq i,j \leq 3} \left[ (a_{i,j})_{i \neq 2,j \neq 2} \right] \right] \\
&\quad - \left[ \det_{1 \leq i,j \leq 3} \left[ (a_{i,j})_{i \neq 3,j \neq 2} \right] \right]^2.
\end{align*}
\] (A.6)

But

\[
\det[2] = K^2_2,
\]

\[
\det_{1 \leq i,j \leq 3} \left[ (a_{i,j})_{i \neq 2,j \neq 2} \right] = a_{13}^2 = K^2_3,
\] (A.7)

\[
\det_{1 \leq i,j \leq 3} \left[ (a_{i,j})_{i \neq 3,j \neq 2} \right] = a_{12}a_{23} - a_{13}a_{21} = H^2_3.
\]

So

\[
\] (A.8)

Hence by using formula (20), formula (19) is correct for \( m = 3 \).

When \( m \geq 4 \), we suppose that formula (19) is correct for \( m-1 \), \( m-2 \), \( m-3 \), \ldots, \( 4 \), and we prove it for \( m \).

It is sufficient to prove that

\[
K^{m-1}_m = \det_{1 \leq i,j \leq m} \left( (a_{i,j})_{i \neq m-1,j \neq m-1} \right) \prod_{k=1}^{m-3} (\det[k])^{(m-k-3)}.
\] (A.9)

By putting \( l = m-1 \) in formula (21), we get

\[
H^{m-1}_m = \det_{1 \leq i,j \leq m} \left( (a_{i,j})_{i \neq m,j \neq m-1} \right) \prod_{k=1}^{m-3} (\det[k])^{(m-k-3)}.
\] (A.10)

From the mathematical induction proof, we have

\[
K^{(m-1)}_{m-1} = \det [m-1] \prod_{k=1}^{m-3} (\det[k])^{(m-k-3)}.
\] (A.11)

By putting \( l = m \) in formula (20), we get

\[
K^m_m = K^{m-1}_m \cdot K^{m-1}_m - [H^{m-1}_m]^2.
\] (A.12)

By replacing (A.9), (A.10), and (A.11) in (A.12), we obtain

\[
K^m_m = \prod_{k=1}^{m-3} (\det[k])^{(m-k-2)} \cdot \det [m-2] \cdot \det [m] \cdot (\det[k])^{(m-k-2)},
\] (A.13)

\[
= \det [m] \prod_{k=1}^{m-2} (\det[k])^{(m-k-2)},
\]

and thus formula (19) is correct for \( m \).

Now, we prove formula (A.9); we may generalize formula (A.9) as follows:

\[
K^l_m = \det_{1 \leq i,j \leq m} \left( (a_{i,j})_{i \neq m-l,j \neq m-l} \right) \prod_{k=l-2}^{l-1} (\det[k])^{(m-l-k)},
\] (A.14)

\[
l = 3, \ldots, m - 1.
\]

Also, we prove formula (A.14) by induction. It is a second inductive proof included in the first one.

It is evident for \( l = 2 \).

For \( l = 3 \), formula (20) will be:

\[
K^3_m = K^2_2 \cdot K^2_3 - [H^2_3]^2.
\] (A.15)

Since we already know that

\[
K^2_2 = \det[2],
\]

\[
K^2_m = \det_{1 \leq i,j \leq m} \left( (a_{i,j})_{i \neq m-1,j \neq m-1} \right),
\] (A.16)

\[
H^2_m = \det_{1 \leq i,j \leq m} \left( (a_{i,j})_{i \neq m,j \neq m-1} \right),
\]

simple substitution of these three formulas in the formula (A.15) followed by the application of the modified well-known Dodgson condensation which has been modified in [11] will lead to formula (A.14) for \( l = 3 \), directly.

When \( l \geq 4 \), we suppose that formula (A.14) is correct for \( l-1 \), and we prove it for \( l \).

Formula (20) for \( l-1 \) reads

\[
K^l_{m-1} = K^{l-1}_{m-1} \cdot K^{l-1}_{m-1} - [H^{l-1}_{m-1}]^2.
\] (A.17)

According to the first induction, we have

\[
K^{(l-1)}_{m-1} = \det [l-1] \prod_{k=1}^{m-3} (\det[k])^{(m-k-3)}.
\] (A.18)

According to the second induction, we have

\[
K^{l}_{m-1} = \det_{1 \leq i,j \leq m} \left( (a_{i,j})_{i \neq m-1,j \neq m-1} \right) \prod_{k=1}^{l-3} (\det[k])^{(m-l-k)}.
\] (A.19)

According to formula (21), we have:

\[
H^{l}_{m-1} = \det_{1 \leq i,j \leq m} \left( (a_{i,j})_{i \neq m,j \neq m-1} \right) \prod_{k=1}^{l-3} (\det[k])^{(m-l-k)}.
\] (A.20)
By replacing (A.18), (A.19), and (A.20) in (A.17) and by using the well-known Dodgson condensation, we obtain formula (A.14) for \( l \). Therefore, the second inductive proof is finished and consequently the first one.

**Proof of Lemma 5.** We prove this lemma by induction:

\[
K^l_i > S^l_i, \quad l = 2, \ldots, m. \tag{A.21}
\]

For \( l = 2 \), we have

\[
K^2_i = a_1^2a_2^2a_3^2a_4^2 \frac{\alpha_1 - 1}{2\alpha_2} = \frac{1}{2\alpha_2} - A_{12}^2.
\]

Because

\[
\frac{\alpha_1 - 1}{\alpha_2} > \frac{1}{2\alpha_2}. \tag{A.23}
\]

Assuming \( l \geq 3 \), we suppose (24) is true for \((l - 1), l - 2, l - 3, \ldots, 3\), and we prove it for \( l \). Hence, we aim to prove

\[
K^l_i > S^l_i, \quad K^{l-1}_i > S^{l-1}_i, \quad K^{l-2}_i > S^{l-2}_i, \ldots, \tag{A.24}
\]

\[
K^{l-1}_i > S^{l-1}_i \implies K^l_i > S^l_i. \tag{A.25}
\]

Recall that

\[
K^l_i = K^{l-1}_i K^{l-1}_i - [H^{l-1}_i]^2. \tag{A.26}
\]

It is then sufficient to prove

\[
K^{l-1}_i > S^{l-1}_i, \tag{A.27}
\]

which will satisfy the inequality

\[
K^l_i = K^{l-1}_i K^{l-1}_i - [H^{l-1}_i]^2 > S^{l-1}_i S^{l-1}_i - [H^{l-1}_i]^2 = S^l_i. \tag{A.28}
\]

In order to prove (A.26), we first generalize it in the form

\[
K^r_i > S^r_i, \quad r = 2, \ldots, l - 1. \tag{A.29}
\]

This can be proven by mathematical induction. It is a secondary inductive proof inside the primary one. For \( r = 2 \), it is evident that

\[
K^2_i > S^2_i. \tag{A.30}
\]

For \( r = 3 \), the formula

\[
K^3_i = K^2_i K^2_i - [H^2_i]^2 > S^2_i S^2_i - [H^2_i]^2 = S^3_i \tag{A.31}
\]

is evident too.

When \( r \geq 4 \), we suppose formula (A.28) true for \( l = 2 \):

\[
K^{l-2}_i > S^{l-2}_i \tag{A.32}
\]

and we prove it for \( l = 1 \):

\[
K^{l-1}_i > S^{l-1}_i. \tag{A.33}
\]

We have

\[
K^{l-1}_i = K^{l-2}_i K^{l-2}_i - [H^{l-2}_i]^2 > S^{l-2}_i S^{l-2}_i - [H^{l-2}_i]^2 = S^{l-1}_i. \tag{A.34}
\]

Accordingly, we have

\[
K^l_i > S^l_i. \tag{A.35}
\]

This finishes the proof.

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**References**


