Research Article

Some Fixed Point Theorems in $b$-Metric Space Endowed with Graph

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We define some notions of contraction mappings in $b$-metric space endowed with a graph $G$ and subsequently establish some fixed point results for such classes of contractions. According to the applications of our results, we obtain fixed point theorems for cyclic operators and an existence theorem for the solution of an integral equation.

1. Introduction and Preliminaries

The study of $b$-metric spaces was initiated in some works of Bakhtin, Heinonen, Bourbaki, and Czerwik [1–4]. Afterwards, several articles which deal with fixed point theorems for single-valued and multivalued functions in $b$-metric space appeared [1–8].

Definition 1 (see [1, 4]). Let $X$ be a set, and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a $b$-metric on $X$, and the pair $(X, d)$ is called a $b$-metric space if, for all $x, y, z \in X$,

\begin{align*}
(d_1) & \quad d(x, y) = 0 \text{ if and only if } x = y, \\
(d_2) & \quad d(x, y) = d(y, x), \\
(d_3) & \quad d(x, z) \leq s[d(x, y) + d(y, z)].
\end{align*}

Note that the class of $b$-metric spaces contains the class of metric spaces.

Example 2. Let $X := l_p(\mathbb{R})$ with $0 < p < 1$, where $l_p(\mathbb{R}) := \{x = (x_n) \in \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$. Then, $d(x, y) = (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{1/p}$ is a $b$-metric on $X$ with $s = 2^{1/p}$.

A sequence $\{x_n\}$ in a $b$-metric space $X$ is said to be convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n \to \infty} x_n = x$.

A sequence $\{x_n\}$ in a $b$-metric space $X$ is said to be Cauchy if and only if $d(x_n, x_m) \to 0$ as $m, n \to \infty$. A $b$-metric space $(X, d)$ is complete if every Cauchy sequence in $X$ converges. In general, a $b$-metric is not continuous.

The famous Banach contraction principle [9] infers that every contraction on a complete metric space has a unique fixed point. Recently, Jachymski [10] introduced the notion of Banach $G$-contraction to generalize Banach contraction principle as follows. Let $(M, \delta)$ be a metric space, let $\Delta$ be the diagonal of the Cartesian product $M \times M$, and let $G$ be a directed graph such that the set $V(G)$ of its vertices coincides with $M$ and the set $E(G)$ of its edges contains all loops; that is, $E(G) \supseteq \Delta$. Assume that $G$ has no parallel edges. A mapping $f : M \to M$ is called a Banach $G$-contraction if (i) for all $x, y \in X ((x, y) \in E(G) \Rightarrow (fx, fy) \in E(G))$, (ii) $\exists \alpha < 1$ such that for all $x, y \in X, (x, y) \in E(G) \Rightarrow \delta(fx, fy) \leq \alpha \delta(x, y)$. A mapping $f : M \to M$ is known as Picard operator [11] if $f$ has a unique fixed point $x^*$ and $\lim_{n \to \infty} f^n x = x^*$ for all $x \in M$.

Various generalizations of Banach’s principle have been obtained by weakening contractive conditions. In this context, Matkowski [12] introduced class of $\varphi$-contractions in...
metric fixed point theory, and subsequently further study was developed in this setting by different authors when underlying space was taken to be a partially ordered set (see, e.g., [13,14]).

Let \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \). Consider the following properties:

(i) \( \varphi(t_1 + t_2) \leq \varphi(t_1) + \varphi(t_2) \), for all \( t_1, t_2 \in \mathbb{R}^+ \),
(ii) \( \varphi(t) < t \) for \( t > 0 \),
(iii) \( \varphi(0) = 0 \),
(iv) \( \lim_{n \to \infty} \varphi^n(t) = 0 \) for all \( t \geq 0 \),
(v) \( \sum_{n=0}^{\infty} \varphi^n(t) \) converges for all \( t > 0 \).

It is easily seen that (i) and (iv) imply (ii) and (i) and (ii) imply (iii).

We recall that a function \( \varphi \) satisfying (i), (ii), and (iv) is said to be a comparison function. A function \( \varphi \) satisfying (i), (ii), and (iv) is known as \( (\varphi) \) comparison function.

Any \((\varphi)\)-comparison function is a comparison function, but converse may not be true. For example, \( \varphi(t) = t/(1 + t) \). On the other hand, define \( \varphi(t) = t/(1 + 2t) \). Then, \( \varphi(t) = t/(1 + 2t) \).

Remark 10. Let \( \{x_n\} \) and \( \{y_n\} \) be equivalent sequences in \( X \).
(i) If \( \{x_n\} \) converges to \( x \), then \( \{y_n\} \) also converges to \( x \) and vice versa.
(ii) \( \{y_n\} \) is a Cauchy sequence whenever \( \{x_n\} \) is a Cauchy sequence and vice versa.

Now, we recollect some preliminaries from graph theory which we need for the sequel. Let \( G = (V(G), E(G)) \) be a directed graph. By letter \( G \), we denote the undirected graph obtained from \( G \) by ignoring the direction of edges.

## 2. Main Results

Throughout this section, let \( (X, d) \) be a \( b \)-metric space with coefficient \( s \geq 1 \), and \( \Delta \) is the diagonal of the Cartesian product \( X \times X \). \( G \) is a directed graph such that the set \( V(G) \) of its vertices coincides with \( X \), and the set \( E(G) \) of its edges contains all loops; that is, \( E(G) \subseteq \Delta \). Assume that \( G \) has no parallel edges. We assign to each edge having vertices \( x \) and \( y \) a unique element \( d(x, y) \).

### Definition 4.

One says that a mapping \( f : X \to X \) is a \( b \)-\((\varphi, G)\) contraction if for all \( x, y \in X \):

\[
(fx, fy) \in E(G) \quad \text{whenever} \quad (x, y) \in E(G);
\]

\[
d(fx, fy) \leq \varphi(d(x, y)) \quad \text{whenever} \quad (x, y) \in E(G),
\]

where \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) is a comparison function.

### Example 6.

Any constant mapping \( f : X \to X \) is a \( b \)-\((\varphi, G)\) contraction.

### Example 7.

Any self-mapping \( f \) on \( X \) is trivially a \( b \)-\((\varphi, G)\) contraction, where \( G_1 = (V(G), E(G)) = (X, \Delta) \).

### Remark 5.

Note that a Banach \( G \)-contraction is a \( b \)-\((\varphi, G)\) contraction.

### Example 8.

Let \( X = \mathbb{R} \), and define \( d : X \times X \to \mathbb{R} \) by \( d(x, y) = |x - y|^2 \). Then, \( d \) is a \( b \)-metric on \( X \) with \( s = 2 \). Further, \( fx = x/2 \), for all \( x \in X \). Then, \( f \) is a \( b \)-\((\varphi, G)\) contraction with \( \varphi(t) = t/4 \) and \( G_0 = (X, X \times X) \). Note that \( d \) is not a metric on \( X \).

### Definition 9.

Two sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) are said to be equivalent if \( \lim_{n \to \infty} d(x_n, y_n) = 0 \), and if each of them is a Cauchy sequence, then they are called Cauchy equivalent.

As a consequence of Definition 9, we get the following lemma.

### Remark 10.

Let \( \{x_n\} \) and \( \{y_n\} \) be equivalent sequences in \( X \).
(i) If \( \{x_n\} \) converges to \( x \), then \( \{y_n\} \) also converges to \( x \) and vice versa.
(ii) \( \{y_n\} \) is a Cauchy sequence whenever \( \{x_n\} \) is a Cauchy sequence and vice versa.
Proof. (i) By using (d2) (Definition 1) it can be easily proved.
(ii) Let \( x \in [x_0]_{\overline{G}} \). Then, there is a path \( x = x_0, z_1, \ldots, z_l = x \) between \( x \) and \( x_0 \). Since \( f \) is a \( b-(\varphi, G) \) contraction, then 
\[
(fz_{i-1}, fz_i) \in E(G) \quad \text{for all} \quad i = 1, 2, \ldots, l.
\]
Thus, \( fx \in [fz_0]_{\overline{G}} = [x_0]_{\overline{G}} \).
Suppose \( (x, y) \in E(\overline{G}) \). Then, \( (fx, fy) \in E(G) \), since \( f \) is a \( b-(\varphi, G) \) contraction. But \( [x_0]_{\overline{G}} \) is \( f \) invariant, so we conclude that \( (fx, fy) \in E(\overline{G}) \). Condition (2) is satisfied automatically as \( \overline{G} \) is a subgraph of \( G \).

From now on, we assume that coefficient of \( b \)-comparison function is at least as large as the coefficient of \( b \)-metric \( s \).

Lemma 12. Let \( f : X \to X \) be a \( b-(\varphi, G) \) contraction, where \( \varphi : [0, \infty) \to [0, \infty) \) is a \( b \)-comparison function. Then, given any \( x, y \in [x_i]_{\overline{G}} \), two sequences \( \{f^n x\} \) and \( \{f^n y\} \) are equivalent.

Proof. Let \( x \in X \), and let \( y \in [x_i]_{\overline{G}} \); there exists a path \( \{x_i\}_{i=0}^l \) in \( G \) from \( x \) to \( y \) with \( x_0 = x, x_1 = y, \) and \( (x_{i-1}, x_i) \in E(\overline{G}) \). From Proposition II, \( f \) is a \( b-(\varphi, \overline{G}) \) contraction. So,
\[
(f^n x_{i-1}, f^n x_i) \in E(\overline{G}) \quad \text{implies} \quad d \left( f^n x_{i-1}, f^n x_i \right) \leq \varphi \left( d \left( f^{n-1} x_{i-1}, f^{n-1} x_i \right) \right)
\]
for all \( n \in \mathbb{N} \) and \( i = 0, 1, 2, \ldots, l \). Hence,
\[
\begin{align*}
\sum_{i=1}^l s_i^n \varphi^n & \left( d \left( x_{i-1}, x_i \right) \right) \\
& \leq \varphi \left( d \left( f^{n-1} x_{i-1}, f^{n-1} x_i \right) \right)
\end{align*}
\]
We observe that \( \{f^n x_i\}_{i=0}^l \) is a path in \( G \) from \( f^n x \) to \( f^n y \).

Using (d3) Definition 1 and (4), we have
\[
\begin{align*}
\sum_{i=1}^l s_i^n \varphi^n & \left( d \left( x_{i-1}, x_i \right) \right) \\
& \leq \sum_{i=1}^l s_i^n \varphi^n \left( d \left( x_{i-1}, x_i \right) \right)
\end{align*}
\]
Letting \( n \to \infty \), we obtain \( d(f^nx, f^ny) \to 0 \).

Proposition 13. Let \( f \) be a \( b-(\varphi, G) \) contraction, where \( \varphi : [0, \infty) \to [0, \infty) \) is a \( b \)-comparison function. Suppose that there is \( z_0 \) in \( X \) such that \( fz_0 \in [z_0]_{\overline{G}} \). Then, \( \{f^n z_0\} \) is a Cauchy sequence in \( X \).

Proof. Since \( fz_0 \in [z_0]_{\overline{G}} \), let \( \{y_i\}_{i=0}^l \) be a path from \( z_0 \) to \( fz_0 \). Then using the same arguments as in Lemma 12, we arrive at
\[
\begin{align*}
d \left( f^n z_0, f^{n+1} z_0 \right) & \leq \sum_{i=1}^r s_i \varphi_i^n \left( d \left( y_{i-1}, y_i \right) \right), \quad \forall n \in \mathbb{N}.
\end{align*}
\]
Example 16. Let $X = \{1/n : n \in \mathbb{N}\} \cup \{0\} \cup \mathbb{N}$ with respect to $b$-metric $d(x, y) = |x - y|^2$, and let $I$ be identity map on $X$. Consider a graph $G_2$ consisting of $V(G_2) = X$ and
\[
E(G_2) = \left\{ \left( \frac{1}{n}, \frac{1}{n+1} \right), \left( \frac{1}{n+1}, n \right), (n, 0), \right\},
\]
(12)
since $x_n = 1/n \to 0$ as $n \to \infty$. We note that $G_2$ is a $(C_G)$-graph, but $r(x_n, 0) = 2[(1/n) - (1/(n+1))]^2 + 2^3[(1/(n+1)) - n]^2 \to 0$ as $n \to \infty$. Thus, $G_2$ is not an $(H_G)$-graph.

Example 17. Let $X = \{1/n : n \in \mathbb{N}\} \cup \{\sqrt{5}/n : n \in \mathbb{N}\} \cup \{0\}$ with respect to $b$-metric $d(x, y) = |x - y|^2$, and let $I$ be identity map on $X$. Consider a graph $G_3$ consisting of $V(G_3) = X$ and
\[
E(G_3) = \left\{ \left( \frac{1}{n}, \frac{1}{n+1} \right), \left( \frac{1}{n+1}, \frac{\sqrt{5}}{n} \right), \left( \frac{\sqrt{5}}{n}, 0 \right); n \in \mathbb{N} \right\},
\]
(13)
since $x_n = 1/n \to 0$ as $n \to \infty$. Clearly $G_3$ is not a $(C_G)$-graph, but it is easy to verify that $G_3$ is an $(H_G)$-graph.

Above examples show that for a given $f$ notions, $(C_G)$-graph and $(H_G)$-graph are independent even if $f$ is identity map.

Theorem 18. Let $(X, d)$ be a complete $b$-metric space, and let $f$ be a $b$-$\{\phi, G\}$ contraction, where $\phi$ is $b$-comparison function. Assume that $d$ is continuous and there is $z_0 \in X$ for which $(z_0, f(z_0))$ is an edge in $G$. Then, the following assertions hold.

(1) If $G$ is a $(C_G)$-graph, then $f$ has a unique fixed point $t \in [z_0]_{G^{-}}$, and for any $y \in [z_0]_{G^{-}}$, $f^m y \to t$. Further, if $G$ is weakly connected, then $f$ is Picard operator.

(2) If $G$ is weakly connected $(H_G)$-graph, then $f$ has a unique fixed point $t \in X$ and for any $y \in X$, $f^m y \to t$.

Proof. (1) It follows from Proposition 13 that $\{f^n z_0\}$ is a Cauchy sequence in $X$. Since $X$ is complete, there exists $t \in X$ such that $f^n z_0 \to t$. Since $f^n z_0, f^{n+1} z_0 \in E(G)$, for all $n \in \mathbb{N}$, and $G$ is a $(C_G)$ graphs, there exists a subsequence $\{f^p z_0\}$ of $\{f^n z_0\}$ and $p \in \mathbb{N}$ such that $(f^p z_0, t) \in E(G)$ for all $k \geq p$. Observe that $(z_0, f^2 z_0, \ldots, f^p z_0, \ldots, f^n z_0, t)$ is a path in $G$. Therefore, $t \in [z_0]_{G^{-}}$. From (2), we get
\[
d(f^{n+1} z_0, t) \leq \phi(d(f^n z_0, t)) < d(f^n z_0, t) \quad \forall k \geq n_0.
\]
(14)
Letting $k \to \infty$, we obtain $\lim_{n \to \infty} f^{n+1} z_0 = f^m z_0 = t$, as $d$ is continuous. Since $\{f^p z_0\}$ is a subsequence of $\{f^n z_0\}$, we conclude that $f^n z_0 \to t$. Finally, if $y \in [z_0]_{G^{-}}$, it follows from Lemma 12 that $f^m y \to t$.

(2) Let $G$ be weakly connected $(H_G)$-graph. From Proposition 13, $f^n z_0 \to t \in X$, and then $r(f^m z_0, t) \to 0$ as $n \to \infty$. Now, for each $n \in \mathbb{N}$, let $\{y^n_0\}; i = 0, 1, \ldots, M_n$ be a path from $f^n z_0$ to $t$ with $y_0 = t$ and $y^n_{0a} = f^n z_0$ in $G$; then
\[
d(t, f^m) \leq s[d(t, f^{n+1} z_0) + d(f^{n+1} z_0, f^m)]
\]
\[
\leq s \left[ d(t, f^{n+1} z_0) + \sum_{i=1}^{M_n} \phi(d(y^n_{i-1}, y^n_i)) \right]
\]
\[
\leq s \left[ d(t, f^{n+1} z_0) + \sum_{i=1}^{M_n} \phi(d(y^n_{i-1}, y^n_i)) \right]
\]
\[
\leq s \left[ d(t, f^{n+1} z_0) + r(f^n z_0, t) \right].
\]

Letting $n \to \infty$, the above inequality yields $f^m y \to t$. Let $y \in [z_0]_{G^{-}} := X$ be arbitrary; then from Lemma 12 and Remark 10, it is easily seen that $f^m y \to t$.

The following example shows that the condition of $(C_G)$ or $(H_G)$-graph in the hypothesis of Theorem 18 cannot be dropped.

Example 19. Let $X = [0, 1]$, let $d(x, y) = |x - y|^2$, and let $f x = x/2$ for all $x \in (0, 1]$ and $f(0, 1] = 1/2$. Then, $(X, d)$ is a complete $b$-metric space with $s = 2$. Further, $d$ is continuous, and $f$ is a $b$-$\{\phi, G\}$ contraction with $\phi(t) = t^2/4$, where $V(G_1) = X$ and $E(G_1) = \{(x, y) \in (0, 1) \times (0, 1); x \geq y\} \cup \{(0, 0), (1, 1)\}$. Note that $G_1$ is weakly connected, but $f$ has no fixed point in $[z_0]_{G_1}$. In general, observe that $G_1$ is not a $(C_G)$-graph because the sequence $f^n x = x/2^n \to 0$ for $x \in (0, 1]$ and $(f^m x, f^{m+1} x) \in E(G_1)$; $n \in \mathbb{N}$, but it does not contain any subsequence such that $(x_{n_0}, 0) \in E(G_1)$. Also, we note that for any fixed $x \in (0, 1], r(f^n x, 0) = 2[|x/2^n - 1|^2 + |0|^2] \to 0$ as $n \to \infty$.

Definition 20. Let $(X, d)$ be a $b$-metric space. A mapping $f : X \to X$ is called orbitally continuous if for all $x, y \in X$ and any sequence $\{k_n\}_{n \in \mathbb{N}}$ of positive integers, $f^{k_n} x \to y$ implies $f(k_n x) \to y$ as $n \to \infty$. A mapping $f : X \to X$ is called orbitally $G$-continuous if for all $x, y \in X$ and any sequence $\{k_n\}_{n \in \mathbb{N}}$ of positive integers, $f^{k_n} x \to y$ and $f(k_n x, f^{k_n} x) \in E(G)$ for all $n \in \mathbb{N}$ imply $f(k_n x) \to y$.

Theorem 21. Let $(X, d)$ be a complete $b$-metric space, and let $f$ be a $b$-$\{\phi, G\}$ contraction, where $\phi$ is a $b$-comparison function. Assume that $d$ is continuous, $f$ is orbitally $G$-continuous, and there is $z_0 \in X$ for which $(z_0, f(z_0))$ is an edge in $G$. Then, $f$ has a fixed point $t \in X$. Moreover, for any $y \in [z_0]_{G^{-}}$, $f^m y \to t$.

Proof. It follows from Proposition 13 that $\{f^n z_0\}$ is a Cauchy sequence in $(X, d)$. Since $X$ is complete, there exists $t \in X$ such that $\lim_{n \to \infty} f^n z_0 = t$. Since $(f^n z_0, f^{n+1} z_0) \in E(G)$ for all $n \in \mathbb{N}$, $f$ is orbitally $G$-continuous. Therefore, continuity of $d$ implies that $f(t) = t$. Let $y \in [z_0]_{G^{-}}$ be arbitrary; then it follows from Lemma 12 that $\lim_{n \to \infty} f^m y = t$. 

}\end{proof}
Slightly strengthening the continuity condition on \( f \), our next theorem deals with the graph \( G \) which may fail to have the property that there is \( z_0 \) in \( X \) for which \( (z_0, f_\infty z_0) \) is an edge in \( G \).

**Theorem 22.** Let \( (X, d) \) be a complete b-metric space, and let \( f \) be a \( b-(\varphi, G) \) contraction, where \( \varphi \) is a \( b \)-comparison function. Assume that \( f \) is continuous, \( f \) is orbitally continuous, and there is \( z_0 \) in \( X \) for which \( f_\infty z_0 \in [z_0]_G \). Then, for any \( y \in [z_0]_G \), \( f^n y \to t \in X \), where \( t \) is a fixed point of \( f \).

*Proof.* It follows from Proposition 13 that \( \{f^n z_0\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete, there exists \( t \in X \) such that \( f^n z_0 \to t \). Since \( f \) is orbitally continuous, then \( \lim_{n \to \infty} f^n z_0 = f t \) which yields \( f t = t \). Let \( y \in [z_0]_G \) be arbitrary; then from Lemma 12, \( \lim_{n \to \infty} f^n y = t \).

**Remark 23.** In addition to the hypothesis of Theorems 21 and 22, if we assume that \( G \) is weakly connected, then \( f \) will become Picard operator [11] on \( X \).

**Remark 24.** Theorem 18 generalizes/extends claims 4 \(^0\) and 5 \(^0\) of [10, Theorem 3.2] and [7, Theorem 4(1)]. Theorem 21 generalizes claims 2 \(^0\) and 3 \(^0\) of [10, Theorem 3.3]. Theorem 22 generalizes claims 2 \(^0\) and 3 \(^0\) of [10, Theorem 3.4], and thus generalizes extends results of Nieto and Rodriguez-López [24, Theorems 2.1 and 2.3], Petrusel and Rus [11, Theorem 4.3], and Ran and Reurings [25, Theorem 2.1]. We mention here that Theorem 18 cannot be improved using comparison function instead of \( b \)-comparison function (see, Gwóźdź-Łukawska and Jachymski [26, Example 2]).

We observe that Theorem 22 can be used to extend famous fixed point theorem of Edelstein to the case of b-metric space. We need to define notion of \( e \)-chainable property for b-metric space.

**Definition 25.** A b-metric space \( (X, d) \) is said to be \( e \)-chainable, for some \( \varepsilon > 0 \), if for \( x, y \in X \) there exist \( x_i \in X; i = 0, 1, 2, \ldots, l \) with \( x_0 = x, x_l = y \) such that \( d(x_{i-1}, x_i) < \varepsilon \) for \( i = 1, 2, \ldots, l \).

**Corollary 26.** Let \( (X, d) \) be a complete \( e \)-chainable b-metric space. Assume that \( d \) is continuous, and there exists a \( b \)-comparison function \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) such that \( f : X \to X \) satisfying
\[
d(x, y) < \varepsilon \quad \text{implies} \quad d(f x, f y) \leq \varphi(d(x, y)), \quad (16)
\]
for all \( x, y \in X \). Then, \( f \) is a Picard operator.

*Proof.* Consider a graph \( G \) consisting of \( V(G) := X \) and \( (x, y) \in E(G) \) if and only if there exists \( i \) such that \( d(x, y) < \varepsilon \). Since \( G \) is weakly connected, \( G \) is weakly connected graph. Let \( (x, y) \in E(G) \), and from (16), we have
\[
d(f x, f y) \leq \varphi(d(x, y)) < d(x, y) < \varepsilon. \quad (17)
\]
Then, \( d(f x, f y) \in E(G) \). Therefore, in view of (16), \( f \) is \( b-(\varphi, G) \) contraction. Further, (16) implies that \( f \) is continuous. Now, the conclusion follows by using Theorem 22.

Now, we establish a fixed point theorem using a general contractive condition.

**Theorem 27.** Let \( (X, d) \) be a complete b-metric space, and let \( G \) be a \( (C_j) \)-graph in \( X \times X \) such that \( V(G) = X \) and \( f : X \to X \) is an edge-preserving mapping. Assume that \( d \) is continuous and there exist \( \alpha, \beta, \gamma > 0 \) with \( s \alpha + (s + 1) \beta + s(s + 1) \gamma < 1 \) and for all \((x, y) \in E(G)\)
\[
d(f x, f y) \leq \alpha d(x, y) + \beta d(x, f x) + \gamma d(y, f y) + \lim_{n \to \infty} d^n y \to t \in X, \quad \text{S \in G is a (C_f)-graph, there exists a subsequence \( \{f^n z_0\} \) converges to \( t \in X \). Since \( G \) is a \( (C_j) \)-graph, there exists a subsequence \( \{f^n z_0\} \) such that
\[
G(\alpha, \beta, \gamma > 0 \) with \( s \alpha + (s + 1) \beta + s(s + 1) \gamma < 1 \) and for all \((x, y) \in E(G)\)
\[
d(f x, f y) \leq \alpha d(x, y) + \beta d(x, f x) + \gamma d(y, f y) + \lim_{n \to \infty} d^n y \to t \in X, \quad \text{S \in G is a (C_f)-graph, there exists a subsequence \( \{f^n z_0\} \) converges to \( t \in X \). Since \( G \) is a \( (C_j) \)-graph, there exists a subsequence \( \{f^n z_0\} \) such that
\[
G(\alpha, \beta, \gamma > 0 \) with \( s \alpha + (s + 1) \beta + s(s + 1) \gamma < 1 \) and for all \((x, y) \in E(G)\)
\[(f^n z_0, t) \in E(G)\] for all \(k \geq p\). Using (18) for all \(k \geq p\), we have
\[
d(f^{n+1} z_0, ft) 
\leq ad(f^n z_0, t) + \beta \left[ d(f^n z_0, f^{n+1} z_0) + d(t, ft) \right] 
+ \gamma \left[ d(f^n z_0, ft) + d(t, f^{n+1} z_0) \right].
\]
(23)
Since the \(b\)-metric \(d\) is continuous and \(\beta + \gamma < 1\), letting \(k \to \infty\) inequality (23) yields \(ft = t\). Also note that \((z_0, f z_0, f^2 z_0, \ldots, f^n z_0, \ldots, f^r, t)\) is a path in \(G\) and hence in \(G\), therefore \(t \in [z_0]_G\).

We note that Theorem 27 does not guarantee the uniqueness of fixed point, but this can be accomplished under some assumptions as in the following theorem.

**Theorem 28.** In addition to the hypothesis of Theorem 27, one further assumes that \(a + 2s\beta + 2\gamma < 1\) for the same set of \(a, \beta, \gamma \geq 0\) and for any two fixed points \(t_1, t_2 \in X\) such that \((t_1, z)\) and \((t_2, z) \in E(G)\). Then, \(f\) has a unique fixed point.

**Proof.** Let \(t_1\) and \(t_2\) be two fixed points of \(f\), and then there exists \(z \in X\) such that \((t_1, z)\) and \((t_2, z) \in E(G)\). By induction, we have \((t_1, f^n z), (t_2, f^n z) \in E(G)\) for all \(n = 0, 1, \ldots\). From (18), we have
\[
d(t_1, f^n z) \leq ad(t_1, f^{n-1} z) + \beta d(f^{n-1} z, f^n z) 
+ \gamma \left[ d(f^{n-1} z, t_1) + d(t_1, f^n z) \right] 
\leq ad(t_1, f^{n-1} z) + \beta \delta \left[ d(f^{n-1} z, t_1) + d(t_1, f^n z) \right] 
+ \gamma \left[ d(t_1, f^n z) + d(f^{n-1} z, t_1) \right].
\]
(24)
On rearranging,
\[
d(t_1, f^n z) \leq \left[ \frac{a + s \beta + \gamma}{1 - s \beta - \gamma} \right] d(t_1, f^{n-1} z), \quad \forall n = 1, 2, \ldots
\]
(25)
Continuing recursively, (25) gives
\[
d(t_1, f^n z) \leq \left[ \frac{a + s \beta + \gamma}{1 - s \beta - \gamma} \right]^n d(t_1, z).
\]
(26)
Since \([a + s \beta + \gamma]/(1 - s \beta - \gamma) < 1\], then \(\lim_{n \to \infty} d(t_1, f^n z) = 0\). Similarly, one can show that \(\lim_{n \to \infty} d(t_2, f^n z) = 0\), which by using (d3) Definition 1 infers that \(d(t_1, t_2) = 0\).

Suppose that \((X, \preceq)\) is a partially ordered set. Consider graph \(G_2\) consisting of \(E(G_2) = \{(x, y) \in X \times X : x \preceq y \text{ or } y \preceq x\}\), and \(V(G_2)\) coincides with \(X\). We note that if a self-mapping \(f\) is monotone with respect to the order, then, for graph \(G_2\), it is obvious that \(f\) is edge-preserving, or equivalently we can say that \(f\) maps comparable elements onto comparable elements.

Following corollaries are the direct consequences of Theorem 28.

**Corollary 29.** Let \((X, d)\) be a complete metric space, where \(X\) is partially ordered set with respect to \(\preceq\). Let \(f : X \to X\) be nondecreasing (or nonincreasing) with respect to \(\preceq\). Assume that there exists \(\alpha, \beta, \gamma \geq 0\) with \(\alpha + 2s\beta + 2\gamma < 1\) such that
\[
d(fx, fy) \leq ad(x, y) + \beta [d(x, fx) + d(y, fy)] 
+ \gamma [d(x, fy) + d(y, fx)],
\]
for all comparable \(x, y \in X\). If the following conditions hold:

(i) there exists \(x_0 \in X\) such that \(x_0 \preceq fx_0\),
(ii) for nondecreasing (or nonincreasing) sequence \(\{x_n\} \to x \in X\), there exists a subsequence \(\{x_{n_k}\}\) such that \(x_{n_k} \preceq x\) for all \(k\).

Then, \(f\) has a fixed point. Moreover, if for all \(x, y \in X\) there exists \(z \in X\) such that \(x \preceq z\) and \(y \preceq z\), then the fixed point is unique.

**Corollary 30.** Let \((X, d)\) be a complete metric space, where \(X\) is partially ordered set with respect to \(\preceq\). Let \(f : X \to X\) be nondecreasing (or nonincreasing) with respect to \(\preceq\). Assume that there exists a constant \(0 < c < 1/2\) such that
\[
d(fx, fy) \leq c [d(x, fx) + d(y, fy)],
\]
for all comparable \(x, y \in X\). If the following conditions hold:

(i) there exists \(x_0 \in X\) such that \(x_0 \preceq fx_0\),
(ii) for nondecreasing (or nonincreasing) sequence \(\{x_n\} \to x \in X\), there exists a subsequence \(\{x_{n_k}\}\) such that \(x_{n_k} \preceq x\) for all \(k\).

Then, \(f\) has a fixed point. Moreover, if for all \(x, y \in X\) there exists \(z \in X\) such that \(x \preceq z\) and \(y \preceq z\), then the fixed point is unique.

**Corollary 31.** Let \((X, d)\) be a complete metric space, where \(X\) is partially ordered set with respect to \(\preceq\). Let \(f : X \to X\) be nondecreasing (or nonincreasing) with respect to \(\preceq\). Assume that there exists a constant \(0 < c < 1/2\) such that
\[
d(fx, fy) \leq c [d(x, fx) + d(y, fx)],
\]
for all comparable \(x, y \in X\). If the following conditions hold:

(i) there exists \(x_0 \in X\) such that \(x_0 \preceq fx_0\),
(ii) for nondecreasing (or nonincreasing) sequence \(\{x_n\} \to x \in X\), there exists a subsequence \(\{x_{n_k}\}\) such that \(x_{n_k} \preceq x\) for all \(k\).

Then, \(f\) has a fixed point. Moreover, if for all \(x, y \in X\) there exists \(z \in X\) such that \(x \preceq z\) and \(y \preceq z\), then the fixed point is unique.

**Remark 32.** We note that in Theorem 27, the condition “there is \(z_0 \in X\) for which \((z_0, fz_0)\) is an edge in \(G\)” yields \(f^n z_0 \to t\), where \(t \in X\) is a fixed point of \(f\). Consider a graph \(G := (X, X \times X)\). For such graph under the assumptions of Theorems 27 and 28, it infers that \(f\) is a Picard operator. Thus, many standard fixed point theorems can be easily deduced from Theorem 28 as follows.
Corollary 33 (Hardy and Rogers [27]). Let \((X, d)\) be a complete metric space, and let \(f : X \to X\). Suppose that there exists constants \(\alpha, \beta, \gamma \geq 0\) such that
\[
d(fx, fy) \leq \alpha d(x, y) + \beta [d(x, fx) + d(y, fy)] + \gamma [d(x, fy) + d(y, fx)],
\]
for all \(x, y \in X\), where \(\alpha + 2\beta + 2\gamma < 1\); then \(f\) has a unique fixed point in \(X\).

Corollary 34 (Kannan [28]). Let \((X, d)\) be a complete metric space, and let \(f : X \to X\). Suppose that there exists a constants \(c \in (0, 1)\) such that
\[
d(fx, fy) \leq c [d(x, fx) + d(y, fy)],
\]
for all \(x, y \in X\), where \(0 < c < 1/2\); then \(f\) is a Picard operator.

Corollary 35 (Chatterjea [29]). Let \((X, d)\) be a complete metric space, and let \(f : X \to X\). Suppose that there exists constants \(c \in (0, 1)\) such that
\[
d(fx, fy) \leq c [d(x, fx) + d(y, fx)],
\]
for all \(x, y \in X\), where \(0 < c < 1/2\); then \(f\) is a Picard operator.

3. Applications

Let \(X\) be a nonempty set, let \(m\) be a positive integer, \(\{X_i\}_{i=1}^m\) be nonempty closed subsets of \(X\), and let \(f : \bigcup_{i=1}^m X_i \to \bigcup_{i=1}^m X_i\). Then, \(X := \bigcup_{i=1}^m X_i\) is known as cyclic representation of \(X\) w.r.t. \(f\) if
\[
f(X_i) \subset X_2, \ldots, f(X_{m-1}) \subset X_m, \quad f(X_m) \subset X_1,
\]
and operator \(f\) is known as cyclic operator [30].

Theorem 36. Let \((X, d)\) be a complete b-metric space such that \(d\) is continuous functional on \(X \times X\). Let \(m\) be positive integer, let \(\{X_i\}_{i=1}^m\) be nonempty closed subsets of \(X\), and let \(f : \bigcup_{i=1}^m X_i \to \bigcup_{i=1}^m X_i\) be an operator. Then, \(X := \bigcup_{i=1}^m X_i\) is known as cyclic representation of \(X\) w.r.t. \(f\) if
\[
f(X_i) \subset X_2, \ldots, f(X_{m-1}) \subset X_m, \quad f(X_m) \subset X_1,
\]
and operator \(f\) is known as cyclic operator [30].

Remark 37 (see [31, Theorem 2.1(1)]). It can be deduced from Theorem 36 if \((X, d)\) is a metric space.

On the same lines as in proof of Theorem 36, we obtain the following consequence of Theorem 28.

Theorem 38. Let \((X, d)\) be a complete b-metric space such that \(d\) is continuous functional on \(X \times X\). Let \(m\) be positive integer, let \(\{X_i\}_{i=1}^m\) be nonempty closed subsets of \(X\), let \(Y := \bigcup_{i=1}^m X_i\), and let \(f : Y \to Y\). Suppose that
\[
(i) \bigcup_{i=1}^m X_i \text{ is cyclic representation of } Y \text{ w.r.t. } f,
(ii) \text{ there exist } \alpha, \beta, \gamma \geq 0 \text{ with } \alpha + s(s+1)\beta + s(s+1)\gamma < 1 \text{ such that}
\]
\[
d(fx, fy) \leq \alpha d(x, y) + \beta [d(x, fx) + d(y, fy)] + \gamma [d(x, fy) + d(y, fx)],
\]
whenever \(x \in X_i, y \in X_{i+1}\), where \(X_{m+1} = X_1\).

Then, \(f\) has a unique fixed point \(t \in \bigcap_{i=1}^m X_i\).

Remark 39. [32, Theorem 7] and [33, Theorem 3.1] can be deduced from Theorem 38, but it does not ensure \(f\) to be a Picard operator.

Remark 40. We note that in proof [32, Theorem 7], the author's argument prove that is valid only if the terms of sequence \(\{x_n\}\) are Picard iterations; otherwise it is void. For example, let \(Y = \bigcup_{i=1}^m X_i\) where \(X_1 = \{1/n : n \text{ odd}\} \cup \{0\}, X_2 = \{1/n : n \text{ even}\} \cup \{0\}, X_3 = \{0\}\), and define \(f : Y \to Y\) as \(fx = x/2; x \in Y \setminus X_3\) and \(fx = 0; x \in X_3\). We see that (33) is satisfied and \(1/n \to 0\), but \(X_3\) does not contain infinitely many terms of \(\{x_n\}\). In the following, we give the corrected argument to prove that is valid.

Let \(X_1 \to x \in X\) such that \((x_n, x_{n+1}) \in E(G)\) for all \(n \geq 1\). Keeping in mind construction of \(G\), there exists at least one pair of closed sets \(\{X_j, X_{j+1}\}\) such that both sets contain infinitely many terms of sequence \(\{x_n\}\), since \(X_i\)'s are closed so that \(x \in X_j \cap X_{j+1}\) for some \(j \in \{1, \ldots, m\}\), and thus one can easily extract a subsequence such that \((x_{n_k}, x) \in E(G)\) holds for \(k \geq 1\).

Now, we establish an existence theorem for the solution of an integral equation as a consequence of our Theorem 18.

Theorem 41. Consider the integral equation
\[
x(t) = \int_a^b k(t, s, x(s)) \, ds + g(t), \quad t \in [a, b],
\]
where \(k : [a, b] \times [a, b] \times \mathbb{R}^n \to \mathbb{R}^n\) and \(g : [a, b] \to \mathbb{R}^n\) is continuous. Assume that
\[
(i) \int_a^b k(t, s, x(s)) \, ds \text{ is nondecreasing for each } t, s \in [a, b],
(ii) \text{ there exists a } (c)\text{-comparison function } \varphi : \mathbb{R}^n \to \mathbb{R}^n\text{ and a continuous function } p : [a, b] \times [a, b] \to \mathbb{R}^n\text{ such that}
\]
\[
d(x(t), x(s)) \leq \varphi(d(x(t), x(s))) + p(t, s),
\]
and conclude that \(G\) is weakly connected \((C_j)\)-graph, and thus conclusion follows from Theorem 18. \(\square\)
such that $|k(t, s, x(s)) - k(t, s, y(s))| \leq p(t, s)\psi(|x(s) - y(s)|)$ for each $t, s \in [a, b]$ and $x \leq y$ (i.e., $x(t) \leq y(t)$ for all $t \in [a, b]$).

(iii) $\sup_{t \in [a,b]} \int_{a}^{b} p(t, s)ds \leq 1$,

(iv) there exists $x_0 \in C([a, b], \mathbb{R}^n)$ such that $x_0(t) \leq \int_{a}^{b} k(t, s, x_0(s))ds + g(t)$ for all $t \in [a, b]$.

Then, the integral equation (35) has a unique solution in the set $\{x \in C([a, b], \mathbb{R}^n) : x(t) \leq x_0(t) \text{ or } x(t) \geq x_0(t) \text{ for all } t \in [a, b]\}$.

**Proof.** Let $(X, \| \cdot \|_{\infty})$, where $X = C([a, b], \mathbb{R}^n)$, and define a mapping $T : C([a, b], \mathbb{R}^n) \to C([a, b], \mathbb{R}^n)$ by

$$Tx(t) = \int_{a}^{b} k(t, s, x(s))ds + g(t), \quad t \in [a, b]. \tag{36}$$

Consider a graph $G$ consisting of $V(G) := X$ and $E(G) = \{(x, y) \in X \times X : x(t) \leq y(t) \text{ for all } t \in [a, b]\}$. From property (i), we observe that the mapping $T$ is nondecreasing, thus $T$ preserves edges. Furthermore, $G$ is a $(C)$-graph; that is, for every nondecreasing sequence $\{x_n\} \subset X$ which converges to some $x \in X$; then $x_n(t) \leq x(t)$ for all $t \in [a, b]$. Now, for every $x, y \in X$ with $(x, y) \in E(G)$, we have

$$\left|Tx(t) - Ty(t)\right| \leq \int_{a}^{b} \left|k(t, s, x(s)) - k(t, s, y(s))\right|ds \leq \int_{a}^{b} p(t, s)\psi\left(|x(s) - y(s)|\right)ds \leq \psi\left(\|x - y\|_{\infty}\right)\int_{a}^{b} p(t, s)ds.$$

Hence, $d(Tx, Ty) \leq \varphi(d(x, y))$. From (iv), we have $(x_0, Tx_0) \in E(G)$, so that $\{x_0\} \subseteq \{x \in C([a, b], \mathbb{R}^n) : x(t) \leq x_0(t) \text{ or } x(t) \geq x_0(t) \text{ for all } t \in [a, b]\}$. The conclusion follows from Theorem 18.

Note that Theorem 41 specifies region of solution which invokes the novelty of our result.

**References**


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