Research Article

Some Results on Fixed and Best Proximity Points of Multivalued Cyclic Self-Mappings with a Partial Order

M. De la Sen

Institute of Research and Development of Processes, University of Basque Country, Campus of Leioa (Bizkaia, Apatado) 644, 48080 Bilbao, Spain

Correspondence should be addressed to M. De la Sen; manuel.delasen@ehu.es

Received 17 October 2012; Revised 7 March 2013; Accepted 22 March 2013

Academic Editor: Abdul Latif

Copyright © 2013 M. De la Sen. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper is devoted to investigate the fixed points and best proximity points of multivalued cyclic self-mappings on a set of subsets of complete metric spaces endowed with a partial order under a generalized contractive condition involving a Hausdorff distance. The existence and uniqueness of fixed points of both the cyclic self-mapping and its associate composite self-mappings on each of the subsets are investigated, if the subsets in the cyclic disposal are nonempty, bounded and of nonempty convex intersection. The obtained results are extended to the existence of unique best proximity points in uniformly convex Banach spaces.

1. Introduction

Important attention is being devoted recently to the investigation of fixed points of self-mappings as well as to the investigation of associate relevant properties like, for instance, stability of the iterations [1–3] and existence and uniqueness of fixed points. On the other hand, the extension of those topics to the existence of either fixed points of multivalued self-mappings [1, 4–19], or common fixed points of several multivalued mappings or operators has received important attention; see, for example, [15–19] and references therein. This paper investigates some properties of fixed point and best proximity point results for multivalued cyclic self-mappings under a general contractive-type condition based on the Hausdorff metric between subsets of a metric space [4, 7–9] and which includes a particular case the contractive condition for contractive single-valued self-mappings, [1, 4–10] including the problems related to cyclic self-mappings, see for example, [7, 8, 11] and references therein. This includes strict contractive cyclic self-mappings and Meir-Keeler type cyclic contractions, [20, 21]. There is a rich background literature available on cyclic self-mappings and related fixed point and best proximity point results; see, for example, [22–30] and references therein. Some existing fixed point results on contractive single and multivalued self-mappings provided in [1, 4, 5, 9, 10, 31, 32] and references therein, under various types of contractive conditions, have been revisited and extended in [4]. There is also a wide sample of fixed point type results available on fixed points and asymptotic properties of the iterations for self-mappings satisfying a number of contractive-type conditions while being endowed with partial order conditions. See, for instance, [18, 19], and references therein. The main objective of this paper is the investigation of fixed point/best proximity point results for multivalued cyclic self-mappings in complete metric spaces, or uniformly convex Banach spaces. Such multivalued cyclic self-mappings satisfy a contractive-type condition, which is specified on the Hausdorff metric, for all pairs of elements in the union of the subsets defining the cyclic disposal which are subject to a partial order.

2. Properties of Distances and Fixed Points for Multivalued Cyclic Self-Mappings with a Partial Order

Assume that \((X, d)\) is a metric space for a set \(X\) endowed with some metric \(X \times X \to \mathbb{R}_+\) with \(\mathbb{R}_+ = \mathbb{R}_+ \cup \{0\}\). Let \(CL(X)\) be the family of all nonempty and closed subsets of the set \(X\). If \(A, B \in CL(X)\) then we can define \((CL(X), H)\) being the generalized hyperspace of \((X, d)\) equipped with the Hausdorff
metric $H : CL(X) \to R_0$, induced by the metric $d : X \times X \to R_0$:

$$H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\}$$  \hspace{1cm} (1)

for two sets $A \subseteq X$ and $B \subseteq X$ which are finite if both sets are bounded and zero if they have the same closure. The distance between $A \subseteq X$ and $B \subseteq X$ is

$$D = d(A, B) = \inf_{x \in A} d(x, y)$$

$$= \inf_{x \in B} d(x, B) = \max d(y, A).$$  \hspace{1cm} (2)

Denote by $P(X), B(X)$, and $CB(X)$ the sets of nonempty, and nonempty, bounded and nonempty, and bounded and closed sets of $X$, respectively. The following relations hold:

$$D \leq H(A, B) \leq \delta(A, B) = \sup_{x \in A} d(x, y)$$

$$\leq \delta(A, B) + \delta(B, C); \text{ } \forall A, B, C \in B(X),$$

$$[(A, B \in CB(X)) \land H(A, B) < \epsilon]$$

$$\implies \exists b \in B : d(a, b) < \epsilon, \forall a \in A$$  \hspace{1cm} (3)

and $\delta(A, B) = 0$ if and only if $A = B = \{x\}$. Consider also a self-mapping $T : \cup_{i \in I} A_i \to \cup_{i \in I} A_i$, where $A_i$ are nonempty closed sets of $X$; $\forall i \in \bar{I} = \{1, 2, \ldots, p\}$, subject to the constraints $T(A_i) \subseteq A_{i+1}$ such that $A_{i+1} \equiv A_i$ for any integer numbers $j \in \{1, p - 1\} \cap Z$ and $i \in I_n = Z_0 \cup \{0\}$ with $R_0 = R \cup \{0\}$. If $p \geq 2$ then $T : \cup_{i \in I} A_i \to \cup_{i \in I} A_i$ is a $p$-cyclic self-mapping. If $p = 1$ then $T : A_1 \to A_1$i, in particular, a self-mapping on $A_1$. We will also consider a partial order $\preceq$ on $X$ so that $(X, \preceq)$ is a partially ordered space and will assume, in general, that $T : \cup_{i \in I} A_i \to \cup_{i \in I} A_i$ is a $p$-cyclic self-mapping so that $A_i \ni x \to Tx(\neq \emptyset) \subset A_{i+1}; \forall i \in I, \forall x \in \cup_{i \in I} A_i$. The subsequent result does not assume a contractive condition for each iteration on adjacent subsets of the contractive mapping but a global contractive condition for the cyclic mapping for iterations on multiple strips of the $p$ subsets $A_i \subset X; i \in \bar{I}$. Therefore, the result that the distances between any two subsets being adjacent or not of [33] for nonexpansive self-mappings is not required.

If $T : \cup_{i \in I} A_i \to \cup_{i \in I} A_i$ is a $p$-cyclic self-mapping then the set $BP(A_i) \subset A_i$ will be said to be the set of best proximity points between $A_1$ to $A_{i+1}$ if $d(A_1, A_{i+1}) = D_i = d(z, y)$ for all $z \in A_1$ and some $y \in T z$. This concept generalizes that of best proximity points of subsets of single valued cyclic self-mappings which is established as follows. If $T : A_1 \cup A_2 \to A_1 \cup A_2$ is cyclic and single-valued then $x \in A_1$ and $Tx \in A_2$ are best proximity point if $d(A_1, A_2) = d(x, Tx)$, [33, 34]. The following result extends a previous one for the case of noncyclic multivalued self-mappings, [18, 19].

**Theorem 1.** Let $(X, \preceq)$ be a partially ordered space and $d : X \times X \to R_0$, with $(X, d)$ being a complete metric space. Let $A_i$ be a set of $(p \geq 2)$ nonempty, bounded, and closed subsets of $X; \forall i \in \bar{I}$, $A_i \in CB(X); \forall i \in \bar{I}$ with $D_i = d(A_i, A_{i+1}); \forall i \in \bar{I}$ and let $T : \cup_{i \in I} A_i \to \cup_{i \in I} A_i$ be a multivalued $p$-cyclic self-mapping on $\cup_{i \in I} A_i$, satisfying

(1) There exist $p$ real constants $k_i \in R_0^+$ satisfying $k = \prod_{i \in I} k_i \in [0, 1)$ such that the following condition holds:

$$H(Tx, Ty) \leq k_i d(x, y) + (1 - k_i) D_i$$

for any given $x \in A_i$ and $y \in A_{i+1}$ which fulfill $x \preceq y$, $\forall i \in \bar{I}$.

(2) If $d(x, y) < d_0$, for some given $d_0 \in R_0$, $y \in Tx$, and any given $x \in \cup_{i \in I} A_i$, then $x \preceq y$ with $y \in A_{i+1}$ if $x \in A_i$ for any given $i \in \bar{I}$.

(3) There are some $i \in \bar{I}$, some $x_i \in A_i$, and some $x_{i+1} \in TX_i \subset A_{i+1}$, such that $d(x_i, x_{i+1}) < d_0$ for some $d_0 > D_i$.

(4)

$$d_0 \geq \max \left\{ \max_{j \in I} \max (k_i \max_{j \in I} \mathcal{D}(D_{i+1})), \max_{j \in I} \max (k_i \mathcal{D}(D_{i+1})), \mathcal{D}(D_{i+1}) \right\}.$$  \hspace{1cm} (5)

Note that (6) implies that $d_{i,j} \leq \min(d_{i,j}, (d_{i,j} - 1 - k_i)D_i)/k_i$; $\forall j \in \bar{I}$. Then, the following properties hold.

(i) There is a partially ordered subsequence $S_i = \{x_{i+1,n+i}\}, n \in Z_0$, of the partially ordered sequence $S_i = \{x_{i+1}\}, n \in Z_0$, both of them of the first element $x_i$, with respect to the partial order $(X, \preceq)$, such that $x_{i+1,n+i} \in Q_{i+1}$ for $j \in I; \forall k \geq k_0, n_k \in Z_0$ for some $k_0 \in Z_0$, and the given $i \in \bar{I}$, where $Q_{i+1} \subset TX_i \subset A_{i+1}$, for any $j \in \bar{I} \cup \{0\}$ and the given $i \in \bar{I}$, are $p$ closed "quasi-proximity" sets in-between each pair of adjacent subsets of the multivalued $p$-cyclic self-mapping $T : \cup_{i \in I} A_i \to \cup_{i \in I} A_i$, such that

$$D_{i+1} \leq d(x_{i+1,n+i+1}, x_{i+1,n+i+1})$$

$$\leq k_i D + (1 - k_i) D = D,$$  \hspace{1cm} (6)

$$\forall j \in \bar{I} \cup \{0\}, \forall n \in Z_0,$$  \hspace{1cm} (7)

where $D = \min_{j \in I} D_{i,j}$ with $x_{i+1,n+i} \in TX_{i+1,j} \subset A_{i+1}$, $\forall n \in Z_0$, for the given $i \in \bar{I}$.

(ii) If $D_i = D_i \forall i \in \bar{I}$ then any partially ordered sequence $S_i = \{x_i\}$ of first element $x = x_i \in A_i$, fulfills

$$\exists \lim_{n \to \infty} d(x_{i+1,n+i+1}, x_{i+1,n+i+1}) = D_i$$

$$\forall j \in \bar{I} \cup \{0\}, \forall n \in Z_0, \text{and} x_{i+1,n+i+1} \in TX_{i+1,j} \subset A_{i+1}; \forall j \in \bar{I} \cup \{0\}, \forall n \in Z_0, \text{if} 0 \leq j \leq p - 1 \text{ and} x_{i+1,n+i+1} \in A_{i+1,j+1} \text{ if} 0 \leq j \leq p - 1, \forall n \in Z_0.
Let $B\mathcal{P}(A_j)$ be the set of best proximity points between $A_j$ and $A_{j+1}$; $\forall j \in \mathcal{P}$. Then, there is a sequence $\{x_n^{(j)}\} \subset B\mathcal{P}(A_j); \forall j \in \mathcal{P}$ such that the following limit exists:

$$
\lim_{n \to \infty} d \left( x_{np+j+1}, z_n^{(j)} \right) = D_j \quad \forall j \in \mathcal{P}
$$

with $x_{np+j+1} \in T x_{np+j}; \forall n \in \mathbb{Z}_+$

(iii) If assumption (3) is removed and (6) in assumption (4) is replaced by the stronger condition

$$
d_0 > \max \left( \max_{j \in \mathcal{P}} (D_j + \text{diam } (A_j)), \max_{j \in \mathcal{P}} \left( k_j \left( d_{0j} - D_j \right) + D_j \right) \right)
$$

then, properties (i)-(ii) hold for any $x \in \bigcup_{i \in \mathcal{P}} A_i$.

**Proof.** Let $x_i \in A_i$ for the given $i \in \mathcal{P}$ which satisfy assumption (3). Then, from such an assumption, there is $x_{i+1} \in T x_i$, which is also in $A_{i+1}$, since $T x \in A_{i+1}$ for any $x \in \bigcup_{i \in \mathcal{P}} A_i$, such that $d(x_i, x_{i+1}) < d_{0i} \leq d_0$. Thus, $x_i \preceq x_{i+1}$ from assumption (2), since $d_{0i} \leq d_0$. From (6) and assumptions (1)-(2) by considering the distance between adjacent subsets,

$$
D_j \leq H \left( T x_i, T \right)
$$

$$
\leq k_d \left( d(x_i, x_{i+1}) \right) + (1 - k_i) D_i
$$

$$
< k_d d_{0i} + (1 - k_i) D_i \leq d_{0i} \leq d_0,
$$

since $D_i < d_{0i} \leq (1/k_i)(k_i - 1)D_i + d_0$ from assumptions (3)-(4). From assumption (2) and (11), there is $x_{i+2} \in T x_{i+1} \subset A_{i+2}$ such that $x_{i+1} \leq x_{i+2}$, and then $d(x_{i+1}, x_{i+2}) < d_{0,i+1}$, and $d(x_{i+1}, x_{i+2}) \leq H(T x_i, T x_{i+1}) < d_0 \leq d_0$. Then, one gets from (11) and assumption (4):

$$
D_{i+1} \leq H \left( T x_{i+1}, T x_{i+2} \right)
$$

$$
\leq k_d \left( d(x_{i+1}, x_{i+2}) \right) + (1 - k_{i+1}) D_{i+1}
$$

$$
\leq k_d d_{0,i+1} + (1 - k_{i+1}) D_{i+1}
$$

$$
< k_d d_{0,i+1} + \left( 1 - k_{i+1} \right) \left[ d_{0} - k_{i+1} \left( d_{0,i+1} - D_{i+1} \right) \right]
$$

Again, from assumption (2), there is $x_{i+3} \in T x_{i+2}$ such that $x_i \preceq x_{i+1} \preceq x_{i+2} \preceq x_{i+3}$. Now, proceeding by complete induction with (12a) from $j = 0$ to $j = p - 1$, it follows that the existence of a partially ordered space $x_i \preceq x_{i+1} \preceq x_{i+2} \preceq \cdots \preceq x_{i+p-1}$ implies, from assumption (2), the existence of the partially ordered space $x_i \preceq x_{i+1} \preceq x_{i+2} \preceq \cdots \preceq x_{i+p-1}$ satisfying $x_{i+j} \in T x_{i+j-1} \subset A_{i+j}; \forall j \in \mathcal{P}$ with $A_{i+p-j} \equiv A_i; \forall k \in \mathcal{P}$. Also, proceeding recursively with (12b), one concludes, if $k = \prod_{i \in \mathcal{P}} [k_i]; \forall i \in \mathcal{P}$ and $D = \max_{i \in \mathcal{P}} D_i$, that there is a partially ordered sequence $x_i \preceq x_{i+1} \preceq x_{i+2} \preceq \cdots \preceq x_{i+p-1} \preceq x_{i+p}$ such that $x_{i+j} \in T x_{i+j-1} \subset T(A_{i+j-1}) \subset A_{i+j}; \forall j \in \mathcal{P}$, and

$$
d(x_{i+p+1}, x_{i+p}) \leq kd \left( x_i, x_{i+1} \right) + (1 - k) D
$$

so that there is a partially ordered sequence $x_i \preceq x_{i+1} \preceq x_{i+2} \preceq \cdots \preceq x_{i+p-1} \preceq x_{i+p}$ such that $x_{i+j} \in T x_{i+j-1} \subset T(A_{i+j}); \forall j \in \mathcal{P}$, and

$$
D_{i+1} \leq d \left( x_{np+i+2}, x_{np+i+1} \right)
$$

$$
\leq H \left( T x_{np+i+1}, T x_{np+i} \right) + (1 - k_i) D_i
$$

$$
\leq \lim_{n \to \infty} \sup_{n \in \mathbb{Z}_+} \left[ k^n d \left( x_i, x_{i+1} \right) + (1 - k^n) D \right] = D;
$$

$$
\forall n \in \mathbb{Z}_+
$$

so that

$$
D_{i+j} \leq d \left( x_{np+i+j+1}, x_{np+i+j} \right)
$$

$$
\leq k_{i+j} D + (1 - k_{i+j}) D = D;
$$

$$
\forall j \in \mathcal{P} \cup \{0\}, \forall n \in \mathbb{Z}_0^+.
$$

Then, from (15), there are $p$ closed “quasi-proximity” sets $Q_i \subset A_i, Q_{i+1} \subset T x_i \subset A_{i+1}, Q_{i+p-1} \subset T^{p-1} x_i \subset A_{i+p-1} \subset A$, between each pair of adjacent subsets of the cyclic self-mapping $T: \bigcup_{i \in \mathcal{P}} A_i \to \bigcup_{i \in \mathcal{P}} A_i$ in view of (14), such
that there is a partially ordered subsequence \( \{x_{i+npj} \} \) of the partially ordered sequence \( x_i \leq x_{i+1} \leq x_{i+2} \leq \cdots \leq x_{i+p} \), being subject to \( x_{i+npj} \) in \( \mathbb{Q}+i \) for
\( \forall j \in \mathbb{P}^{-1} \cup \{0\}; \forall k \geq k_0 \), some \( k_0 \in \mathbb{Z}_{\geq 0} \). Thus, (7)
holds and then the property (i) has been proven. The relation
(8) of property (ii) for \( j = 0 \) is a direct consequence of property (i). From (8), it is also proven that the sequence \( S_i \) of first element \( x_i \in A_i \) in property (i) satisfies the following property \( \exists \lim_{n \to \infty} d(x_{i+npj+1}, x_{i+npj}) = D; \forall j \in \mathbb{P} \). Assume that, then, it follows that
\[
D \leq \liminf_{n \to \infty} d(x_{i+npj+1}, x_{i+npj}) \leq \limsup_{n \to \infty} d(x_{i+npj+1}, x_{i+npj}) \leq k_{jn} \lim_{n \to \infty} d(x_{i+npj+1}, x_{i+npj}) + (1 - k_j)D < k_jD + (1 - k_j)D = D,
\]
\( \forall j \in \mathbb{P}^{-1} \cup \{0\} \) and the given \( i \in \mathbb{P} \) so that, by using
completeness inductive, \( \exists \lim_{n \to \infty} d(x_{i+npj+1}, x_{i+npj}) = D \); \( \forall j \in \mathbb{P}^{-1} \cup \{0\} \) and the given \( i \in \mathbb{P} \) with \( \{x_{n}\} \) being partially ordered with respect to \((X, \leq)\), that is, \( x_{i+k}(\in T_{x_{i+k}} \leq x_{i+k+1}(\in T_{x_{i+k+1}}) ; \forall k \in \mathbb{Z}_+ \), and we can then reformulate the above limits of the distances as \( \exists \lim_{n \to \infty} d(x_{i+npj+1}, x_{i+npj}) = D; \forall j \in \mathbb{P}^{-1} \cup \{0\} \) for the given \( i \in \mathbb{P} \).

The remaining proof of property (ii) follows by contradiction. Suppose that the limit (9) does not exist for some sequence \( \{z^{(j)}_{n}\} \subset BP(A_j) \) for some \( j \in \mathbb{P} \). Since \( \{z^{(j)}_{n}\} \subset BP(A_j) \), \( \liminf_{n \to \infty} d(x_{npj+1}, z^{(j)}_{n}) < D \) is impossible in the case that \( \liminf_{n \to \infty} d(x_{npj+1}, z^{(j)}_{n}) \) would not exist for some \( j \in \mathbb{P} \). Then,
\[
\liminf_{n \to \infty} d(x_{npj+1}, z^{(j)}_{n}) = D
\]
for some \( z \in BP(A_j) \), \( y \in Tz \subset BP(A_j) \), since \( A_j \) and \( A_{j+1} \) are boundedly compact for all \( j \in \mathbb{P} \) since they are bound and closed and, then, compact. [7, 8]. This leads to a contradiction, since \( \{z^{(j)}_{n}\} \subset BP(A_j) \) and \( x_{npj+1} \in T_{npj+1} \subset \mathbb{P}^{-1} \cup \{0\} \). The property (ii) has been proven.

If assumption (3) is removed, while \( d_0 \) satisfies the stronger constraint (10), then there are infinitely many sequences \( S(x) \) for any arbitrary first element \( x \in \bigcup_{i \in \mathbb{P}} A_i \), in the partial order \( S \), of an iterated sequence through \( T \) for which property (i) and thus property (ii) both hold since \( d(x, y) = d(x, Tx) < d_0 \) from assumption (2). Hence, property (iii) follows so that the theorem has been fully proven.

Note that (5) is not guaranteed to be a cyclic contractive condition for each restricted map \( T : (\bigcup_{i \in \mathbb{P}} A_i) \rightarrow A_i \), since all the constants are not required to be less than one in (5), and furthermore, (5) and assumption (3) are fulfilled for some first element \( x_i \in A_i, x_{i+1} \in T_{x_i} \in A_{i+1} \) and some given \( i \in \mathbb{P} \) in the partial order \((X, d)\). Note also that sequences fulfilling the partial order of Theorem 1 can always be built through iterations with the multivalued \( p \)-self-mapping for any arbitrarily chosen \( A_i \) for any \( i \in \mathbb{P} \) from (6) characterizing assumption (4) of Theorem 1. The subsequent particular case of Theorem 1 applies when all the iterations between the cyclic disposal satisfy a cyclic contractive condition, that is, \( k_i < 1; \forall i \in \mathbb{P} \).

Note that Theorem 1(iii) also holds in the particular case that the partial order is a total order for all pairs in any Cartesian product \( A_i \times A_{i+1} \) of adjacent subsets \( A_i \subset X; \forall i \in \mathbb{P} \), since both elements of any ordered pair \((x, y) \in A_i \times A_{i+1}, y \in Tx; \forall i \in \mathbb{P} \) are comparable with respect to the partial order \( \leq \). Theorem 1(iii) establishes that any element in any subset \( A_i \subset X; \forall i \in \mathbb{P} \) is a first element of a nondecreasing (i.e., partially ordered) sequence with respect to the partial order \( \leq \) which fulfills properties (i)-(ii) of Theorem 1.

Theorem 2. In addition to assumptions (1)-(4) of Theorem 1, assume furthermore,

\( (6) \ D_j = D = 0; \forall j \in \mathbb{P} \) (i.e., \( \bigcap_{j \in \mathbb{P}} A_j \neq \emptyset \); (7) the limit \( x \) of any converging nondecreasing sequence \( \{x_n\}_{n \in \mathbb{N}_{\geq 0}} \) is comparable to each \( x_n; \forall n \in \mathbb{Z}_{\geq 0} \) in the partial order \((X, \leq)\), that is,
\[
\left[ x_n \leq x(\neq x_n) \right] \text{ for } x \in A_j, x_n \in A_j, \forall j \in \mathbb{P} \text{, } \forall n \in \mathbb{Z}_{\geq 0} \] \implies H(Tx, Tx_n) > k_jd(x, x_n).
\]

Then, there is a sequence \( \{x_{npj+i+j}\}_{n \in \mathbb{N}_{\geq 0}} \) satisfying \( x_{npj+i+j} \in T^{npj}x_i \) for some given initial element \( x = x_i \in A_i \) and some given \( i \in \mathbb{P} \) \( \forall j \in \mathbb{P}^{-1} \cup \{0\} \) which is non-decreasing and ordered with respect to the partial order \((X, \leq)\) and fulfills the following properties.

\( (i) \ \exists \lim_{n \to \infty} d(x_{npj+i+j}, x_{npj+i+j+1}) = 0; j \in \mathbb{P}^{-1} \cup \{0\} \) and the given \( i \in \mathbb{P} \) with \( x_{npj+i+j} \in T_{npj+i+j} \subset \mathbb{P}^{-1} \cup \{0\} \), \( \forall n \in \mathbb{Z}_{\geq 0} \), and the sequence \( \{x_{npj+i+j}\}_{n \in \mathbb{N}_{\geq 0}} \) is a Cauchy sequence; \( j \in \mathbb{P}^{-1} \cup \{0\} \).

\( (ii) \) The sequence \( \{x_{npj+i+j}\}_{n \in \mathbb{N}_{\geq 0}} \) for any \( j \in \mathbb{P}^{-1} \cup \{0\} \) and the given \( i \in \mathbb{P} \) converge to a limit \( \bar{x} \in \bigcap_{i \in \mathbb{P}} A_j \), which is a fixed point of the composite self-mapping \( \bar{T}_j : A_j \rightarrow A_j \), where \( T_{n} = T_{n-1} \circ T_{n-2} \circ \cdots \circ T_{0} \) (p times) \( = T_{n} \mid A_j \) of domain \( A_j \), \forall j \in \mathbb{P} \) and also a fixed point of the self-mapping \( T : \bigcup_{i \in \mathbb{P}} A_i \rightarrow \bigcup_{i \in \mathbb{P}} A_i \), that is, \( \bar{x} \in \bigcap_{i \in \mathbb{P}} A_i \) and \( \bar{x} \in T^p \left( \subseteq \bigcap_{i \in \mathbb{P}} A_i \right) \); \( \forall j \in \mathbb{P} \).

\( (iii) \) If, in addition, \((X, d)\) is a convex metric space, what holds, in particular, if \( X \) is a Euclidean vector space and \( d : X \times X \rightarrow \mathbb{R}_{\geq 0} \) is the Euclidean metric, and
\[ \bigcap_{i \in \mathcal{P}} A_i \text{ is convex, then } x_i \in T \bigcap_{i \in \mathcal{P}}(\bigcap_{i \in \mathcal{P}} A_i) \text{ is the unique fixed point of } T : \bigcup_{i \in \mathcal{P}} A_i \to \bigcup_{i \in \mathcal{P}} A_i \text{ and } T_j : A_j \to A_j; \forall j \in \mathcal{P} \text{ and also the unique fixed point of } T^p : \bigcup_{i \in \mathcal{P}} A_i \to \bigcup_{i \in \mathcal{P}} A_i. \]

(iv) If assumption (4) of Theorem 1 is replaced by assumption (5) then properties (i)-(iii) hold for any \( x \in \bigcup_{i \in \mathcal{P}} A_i. \)

(v) If \( X \) is a Euclidean vector space then property (iii) holds also if the condition of \( (X, d) \) being a convex metric space is removed.

**Proof.** The property (i) follows from Theorem 1 when \( D_j = D = 0; \forall j \in \mathcal{P}. \) To address the proof of property (ii), it is first proven that \( \{x_{i+p} \}_{i \in \mathcal{Z}_n} \) is a Cauchy sequence in \( X; \forall j \in \mathcal{P}. \) Take \( m \in \mathcal{Z}_n \), so that one gets from (10) that

\[
d(x_{(n+m+1)p+i+j}, x_{(n+m)p+i+j}) \leq kd(x_{(n+m)p+i+j}, x_{(n+m-1)p+i+j}) \leq \cdots \leq k^m d(x_{np+i+j}, x_{(n-1)p+i+j})
\]

for some \( x = x_i \in A_i, \forall j \in \mathcal{P} \cap \{1 \} \) for any given \( i \in \mathcal{P} \) from assumption (1) of Theorem 1, where \( k = \prod_{i=1}^{|\mathcal{P}|} k_i. \) Then, \( d(x_{(n+m+1)p+i+j}, x_{(n+m)p+i+j}) \to 0 \) as \( m \to \infty; \forall j \in \mathcal{P} \cap \{1 \} \) and \( k \in \mathcal{P} \cap \{1 \} \). Thus, \( \{x_{np+i+j} \} \) is a Cauchy sequence to some \( x_{\infty+p+i+j} \in T\bigcup_{i \in \mathcal{P}} A_i \) \( \cap A_j; \forall j \in \mathcal{P} \cap \{1 \} \) and any given \( i \in \mathcal{P} \), such that \( x_{np+i+j} \leq x_{\infty+p+i+j}; \forall n \in \mathcal{Z}_n \), from (18), since all the elements of the generated non-decreasing sequence in the partial order \( (X, \preceq) \) are comparable from (18) and \((X, d)\) is complete. The property (i) has been proven.

To prove property (ii), assume that there are two distinct limits \( \bar{x} \in T\bigcup_{i \in \mathcal{P}} A_i \) and \( \bar{x}_j \in T\bigcup_{i \in \mathcal{P}} A_i \) for some distinct \( i, j \) in \( \mathcal{P}. \) Since the restricted composite self-mapping \( T_j \equiv T^p \mid A_j : A_j \to A_j \) is a cyclic contraction on the nonempty closed set \( A_j \) for any \( j \in \mathcal{P}, \) then we can built uniquely a restricted composite self-mapping \( T_{\ell} : A_j \to A_j \) defining the partially ordered sequence \( \{x_{np+i+j} \}, \) with first element \( x = x_i \in A_i, \) which converges to \( \bar{x} \) as \( n \to \infty, \) since such a restricted composite self-mapping satisfies also assumptions (1)-(2) of Theorem 1. Then, we can proceed in the same way, with \( T_{\ell} : A_j \to A_j \) generating \( \{x_{np+i+j} \} \) converging to \( \bar{x}_{\ell} \neq \bar{x}_j \) as \( n \to \infty \) for any \( j \in \mathcal{P}. \) Both such composite self-mappings are Lipschitz-continuous, since they are the contraction with the Lipschitz constant being the contractive constant \( k < 1, \) so that the limit of the distance can be permuted with the distance of the limits. Then, since \( \bar{x}_{\ell} = \bar{x}_{\ell} \bar{x}_j \) for \( \ell = i, j \) and since \( \bar{x}_{\ell} \) is a fixed point of \( T_{\ell} : A_j \to A_j \) for \( \ell = i, j, \not i \) in \( \mathcal{P}, \) the following contradiction holds to the existence of two distinct fixed points \( \bar{x}_j \in T_{\ell} \bar{x}_j, \bar{x}_j \in T_{\ell} \bar{x}_j \) for some \( \ell = i, j, \not i \in \mathcal{P}; \forall n \in \mathcal{Z}_n \)

\[
0 = H \left( \lim_{n \to \infty} T_{\ell}^n \bar{x}_j, \lim_{n \to \infty} T_{\ell}^n \bar{x}_j \right) = H \left( \lim_{n \to \infty} T_{\ell}^p \bar{x}_j, \lim_{n \to \infty} T_{\ell}^p \bar{x}_j \right) = \lim_{n \to \infty} H \left( T_{\ell}^p \bar{x}_j, T_{\ell}^p \bar{x}_j \right) \leq \lim_{n \to \infty} (k^n) d(\bar{x}_i, \bar{x}_j) = 0 \Rightarrow \bar{x}_i = \bar{x}_j \text{ for } i, j \not i \in \mathcal{P}. \]

(20)

Since any existing fixed point in \( A_j \cap A_j \) of \( T_j := T^p \mid A_j : A_j \to A_j \) for \( \ell = i, j, \not i \in \mathcal{P} \) is comparable in the partial order \( (X, \preceq) \) to any element of \( A_j \cap A_j \), \( \text{diam}(A_j \cap A_j) < d_0 \) and \( \bar{x}_j \in \bigcap \bar{x}_j \preceq A_j, \bar{x}_j \in \bigcap \bar{x}_j \preceq A_j; \forall i, j \not i \in \mathcal{P}. \) Assuming, with no loss in generality, that \( \bar{x}_j \preceq \bar{x}_j, \) one can build, from the assumptions of Theorem 1 and the current comparability assumption (7), a nondecreasing, converging and partially ordered sequence:

\[
z_1 = \bar{x}_j \preceq z_2 = z_3 \preceq \cdots \preceq z_n \cdots \preceq z = \lim_{n \to \infty} z_n
\]

(21)

such that

\[
z_{2n+1} \in T(2n+1)^p \preceq A_j, \quad z_{2n} \in T(2n+1)^j \preceq A_j, \forall n \in \mathcal{Z}_n.
\]

(22)

Thus, \( \lim_{n \to \infty} T_{\ell}^p \bar{x}_j \preceq A_j \) is \( \lim_{n \to \infty} T_{\ell}^p \bar{x}_j \preceq A_j \) for any \( i, j, \not i \in \mathcal{P}. \) Since, \( A_j \) and \( A_j \) are closed and nonempty for any distinct \( i, j \in \mathcal{P}, \) then \( z \in A_j \cap A_j. \) Since the pair \( (i, j) \) is arbitrary and the set \( \bigcap_1 A_j \) is nonempty and closed, then \( z \in T_{\ell} z \preceq \bigcap \bigcap_1 A_j \) for any \( j \in \mathcal{P}. \) Then, \( z \in T_{\ell} z \) is a fixed point of \( T_{\ell} : A_j \to A_j; \forall i \in \mathcal{P}. \) But, since \( \bar{x}_j \in T_{\ell} \bar{x}_j \) is a fixed point of \( T_{\ell} : A_j \to A_j, \) it cannot converge through an iterated sequence to any other distinct fixed point of the same self-mapping or to be distinct of it. Thus, \( x = \bar{x}_j = \bar{x}_j = \bar{x}_j \) is a fixed point of the restricted composite self-mapping of \( T_{\ell} : A_j \to A_j; \forall i \in \mathcal{P} \) which is the closed nonempty set \( \bigcap \bigcap_1 A_j \). Then, \( T_{\ell} \bar{x}_j = T_{\ell} \bar{x}_j = \text{Fix}(T_{\ell}) \subseteq \bigcap \bigcap_1 A_j; \forall n \in \mathcal{Z}_n. \) Thus, since \( \bar{x} \geq \bar{x}, \bar{x} \in T X, D_j = 0; \forall j \in \mathcal{P}, \) one gets from (10) that

\[
0 = d(\bar{x}, T^{p+1} \bar{x}) = H(T \bar{x}, T^{p+1} \bar{x}) \leq k \cdot d(\bar{x}, x, T^{p} \bar{x}) = 0,
\]

(23)

since \( \bar{x} \in T^{p+1} \bar{x} \) from \( H(T \bar{x}, T^{p+1} \bar{x}) = 0. \) Thus, \( \text{Fix}(T_{\ell}) \subseteq \text{Fix}(T_{\ell}) \subseteq T \bar{x}. \)

It remains to be proven that \( \bar{x} \in T X \subseteq \bigcap_1 A_j \) is the unique fixed point of \( T : \bigcup_1 A_j \to \bigcup_1 A_j; \) since \( d_0 > \text{diam}(A_j \cap A_j) \geq \text{diam}(\bigcap_1 A_j) \) and any existing fixed point in \( \bigcap_1 A_j \) is comparable, with respect to the partial order \( (X, \preceq) \), to any element of \( \bigcap_1 A_j. \) This is a consequence of...
assumption (2) of Theorem 1, so that \( d(\bar{x}_i, T^{np}_n x_{i+1}) \to 0 \) as \( n \to \infty \), and

\[
d(\bar{x}_i, T^{np}_n x_{i+1}) \leq H(T^{np}_n \bar{x}_i, T^{np}_n x_{i+1})
\leq k^n \left( \prod_{i=0}^{j-1} [k_i] \right) d(\bar{x}_i, T^{i-1} x_{i+1})
\leq d(\bar{x}_i, x_{i+1}); \quad \forall n \in Z_0^+.
\]

Thus, \( \bar{x}_i \in T x_i \Rightarrow \bar{x}_i \in T x_i \) as a result. In this sense, a Banach space \( (X, d) \) is complete.

3. The Main Result on Best Proximity Points for Nonintersecting Subsets

An "ad hoc" version of Theorem 2 will be obtained in this section for the case of nonintersecting subsets by proving the convergence to unique best proximity points within each subset \( A_i \), which are also unique fixed points of each of the composed self-mappings \( T_i : A_i \to A_i; \forall i \in \mathbb{P} \) extended in a natural way to the composite self-mapping \( T^p : \bigcup_{i \in \mathbb{P}} A_i \to \bigcup_{i \in \mathbb{P}} A_i \). In this sense, a Banach space endowed with the partial order \( \leq \) and that the subsets \( A_i; \forall i \in \mathbb{P} \) are convex but that their intersection is convex.

Remarks 1. (1) Note that the restricted composite multivalued self-mapping \( T_i : A_i \to A_i; \forall i \in \mathbb{P} \) is convex metric spaces under the Euclidean induced norm and that closed subsets of Euclidean spaces are convex metric spaces if and only if they are convex. This property is used in the proof of property (v) of Theorem 2. Finally, note that Theorem 2(v) holds independently of the metric (not necessarily for a norm-induced metric) and that properties (iii)-(iv) of Theorem 2 do not require that the subsets \( A_i \subset X \) for \( i \in \mathbb{P} \) are convex but that their intersection is convex.

3. The Main Result on Best Proximity Points for Nonintersecting Subsets

An "ad hoc" version of Theorem 2 will be obtained in this section for the case of nonintersecting subsets by proving the convergence to unique best proximity points within each subset \( A_i \), which are also unique fixed points of each of the composed self-mappings \( T_i : A_i \to A_i; \forall i \in \mathbb{P} \) extended in a natural way to the composite self-mapping \( T^p : \bigcup_{i \in \mathbb{P}} A_i \to \bigcup_{i \in \mathbb{P}} A_i \). In this sense, a Banach space endowed with the partial order \( \leq \) and that the subsets \( A_i; \forall i \in \mathbb{P} \) are convex but that their intersection is convex.

Remark 3. It is well known that a norm defines a metric. In this sense, a Banach space \( (X, \| \cdot \|) \) can be considered also a complete metric space \( (X, d) \) under the norm-induced metric. To practical effects, the induced metric is identical to the norm. The converse is not true in general since metric spaces are subject to less restrictive conditions than norms. However, under certain conditions, as for instance, if the metric is homogeneous and translation-invariant, then it can be considered as a norm in a natural way, say, a metric-induced norm. In this case, we can also consider the norm...
to be identical to the metric-induced norm. If \((X, d)\) is a complete metric space and \(X\) is a vector space and \(d : X \times X \to \mathbb{R}_0^+\) is a homogeneous and translation invariant metric, then \((X, \|\|)\) is also a Banach space under such a metric-induced norm \(d : X \times X \to \mathbb{R}_0^+\).

The next result is an "ad hoc" version for this paper of previous technical results. See Lemma 3.7, Lemma 3.8 and Theorem 3.10 in [33].

**Lemma 4** (Lemma 3.7 and Lemma 3.8 of [33]). Let \(A_j \subseteq X\) for \(i \in \overline{p}\) be nonempty closed subsets of the vector space \(X\) of a uniformly convex Banach space \((X, \|\|)\) with norm-induced metric \(d : X \times X \to \mathbb{R}_0^+\) and \(d(A_i, A_{i+1}) = D_i\) and either \(A_i\) or \(A_{i+1}\) for \(i \in \overline{p}\) are, furthermore, convex (i.e., at least one of each two adjacent subsets is, in addition, convex). Consider sequences \(\{x_{np+1}\}_{n \in \mathbb{Z}_0^+} \subseteq A_i\) and \(\{z_{np+1}\}_{n \in \mathbb{Z}_0^+} \subseteq A_i\) and \(y_{np+1} \in Z_{n+1}\) satisfying

1. \(\|z_{np+1} - y_{np+1}\| \to D_i\) as \(n \to \infty\) for any given \(i \in \overline{p}\).
2. For every \(e \in R_j\), there is \(n_0 \in \mathbb{Z}_0^+\) such that \(\|x_{np+1} - z_{np+1}\| \leq D + e\) for all \(m, n \in \mathbb{Z}_0^+\) with \(m > n \geq n_0\) for any given \(i \in \overline{p}\).

Then, the following properties hold:

(i) For every \(e \in R_j\), there is \(n_0 \in \mathbb{Z}_0^+\) such that \(\|x_{np+1} - z_{np+1}\| \leq e\) for all \(m, n \in \mathbb{Z}_0^+\) with \(m > n \geq n_0\) for any given \(i \in \overline{p}\).

(ii) If \(\|z_{np+1} - y_{np+1}\| \to D_i\) as \(n \to \infty\) and \(\|x_{np+1} - y_{np+1}\| \to D_i\) as \(n \to \infty\) then \(\|x_{np+1} - z_{np+1}\| \to 0\) as \(n \to \infty\) for any given \(i \in \overline{p}\).

(iii) \(\{x_{np+1}\}_{n \in \mathbb{Z}_0^+}\) and \(\{z_{np+1}\}_{n \in \mathbb{Z}_0^+}\) are Cauchy sequences; \(\forall j \in \overline{p} - 1 \cup \{0\}\) and the given \(i \in \overline{p}\).

The proof of Lemma 4(i)-(ii) is supported by the nonemptiness, closeness, and convexity of the subsets \(A_j \subseteq X; j \in \overline{p}\) and the uniform convexity of the Banach space \((X, \|\|)\) [33]. The following main result for multivalued \(\rho\)-cyclic self-mappings is obtained from Theorem 1 and Lemma 4 while taking into account Remark 3.

**Theorem 5.** Let \(T : \bigcup_{i \in \overline{p}} A_i \to \bigcup_{i \in \overline{p}} A_i\) be a multivalued \((\geq 2)\)-cyclic self-mapping on \(\bigcup_{i \in \overline{p}} A_i\) with \(A_i \subseteq CB(X) \subseteq X\); \(\forall i \in \overline{p}\) being all nonempty and convex with \(D_i = d(A_i, A_{i+1})\); \(\forall i \in \overline{p}\). Assume the following:

1. Let \(X\) be a vector space and let \((X, d)\) be a convex complete metric space with \(d : X \times X \to \mathbb{R}_0^+\), being a homogeneous translation-invariant metric which induces a norm \(|\|\|\|\) on \(X\) such that \((X, \|\|)\) is a Banach space.
2. \((X, \|\|)\) is a uniformly convex Banach space with metric convexity.
3. The complete metric space \((X, d)\), equivalently, the Banach space \((X, \|\|)\), is endowed with a partial order \(\preceq\) defined by \((5)\) with \(x = x_i(\in A_i) \preceq y \in TX(\subseteq A_{i+1})\) for

any \((x, y) \in A_i \times A_{i+1}\) and some given \(i \in \overline{p}\) such that the resulting \((X, \preceq)\) partially ordered space is subject to assumptions (1)–(4) of Theorem 1 and assumption (7) of Theorem 2.

Then, the following properties hold.

(i) There are unique best proximity points \(\overline{x}_{i+1} \in TX_j \subseteq A_j\) with \(d(\overline{x}_j, x_{i+1}) = d(\overline{x}_j, TX_{i+1}) = D_j\) for each \(j \in \overline{p}\) which are also unique fixed points of each of the restricted composite self-mappings \(\overline{T}_j(\equiv T^p \mid A_j) : A_j \to A_j; \forall j \in \overline{p}\).

(ii) Take any \(x = x_i(\in A_j) \preceq y = x_{i+1} \in TX_j\) for any given \(i \in \overline{p}\) (i.e., \(x\) and \(y\) are partially ordered with respect to the partial ordered set \((X, \preceq)\) and consider the partially ordered sequences \(\{x_{np+1}\}\), being nondecreasing with respect to \(\preceq\) while satisfying \(x_{np+1} \in TX_{np+1}\); \(\forall j \in \overline{p}\) of first element subject to \(x = x_i(\in A_j) \preceq y = x_{i+1}\) for any given \(i \in \overline{p}\). Then, each of such sequences \(\{x_{np+1}\}\) converges to the unique best proximity point \(\overline{x}_j\) in \(A_j; \forall j \in \overline{p}\) which is also the unique fixed point of each of the restricted composite self-mapping \(\overline{T}_j : A_j \to A_j\) since \(\bigcap_{i \in \overline{p}} A_i \neq \emptyset\), then \(\overline{x} = \overline{x}_j \in \bigcap_{i \in \overline{p}} A_i\) is the unique fixed point of \(T : \bigcap_{i \in \overline{p}} A_i \to \bigcap_{i \in \overline{p}} A_i\), \(\overline{T}_j(\equiv T^p \mid A_j)\) and a fixed point of \(T^p : \bigcap_{i \in \overline{p}} A_i \to \bigcap_{i \in \overline{p}} A_i; \forall j \in \overline{p}\).

(iii) If assumption (4) of Theorem 1 is replaced by its assumption (5), then the convergence to the above unique best proximity points holds for partially ordered sequences of first element \(x \in \bigcup_{i \in \overline{p}} A_i\).

Proof. Note from the various hypothesis the uniformly convex Banach space \((X, \|\|)\) possesses the metric convexity property with respect to the norm metric \(|\|\|\|\) while it is endowed with a partial order \(\preceq\) under assumptions (1)–(4) of Theorem 1. From property (ii) of Theorem 1, (8), the nonemptiness and closeness of the subsets \(A_j \subseteq X; \forall i \in \overline{p}\), and Lemma 4(i)-(ii), it follows that

\[
\exists \lim_{n \to \infty} d(x_{np+1}, x_{np+1}) = D_{i+1} ;
\]

\[
\exists \lim_{n \to \infty} d(x_{(n+1)p+i+1}, x_{np+1}) = 0 ;
\]

\(\forall j \in \overline{p} - 1 \cup \{0\}\) and the given \(i \in \overline{p}\).

\[
\overline{x}_{i+1} \in TX_{i+1} \subseteq A_{i+1} \text{ for } x_{np+1} \in TX_{np+1+1} \subseteq A_{i+1} \text{ and the given } i \in \overline{p} \text{ are partially ordered with respect to the partial order } \preceq; \text{ from Theorem 1, of first element } x_{i+1} = x_i \text{ generated from the iteration } x_{np+1} \in TX_{np+1}; \forall j \in \overline{p} - 1 \cup \{0\} \text{ and the given } i \in \overline{p}\) are all Cauchy sequences. Since \((X, d) \equiv (X, \|\|)\) is complete, it follows that

\[
x_{np+1} \to \overline{x}_{i+1} \in TX_{np+1} \subseteq A_{i+1} \text{ for } x_{np+1} \in TX_{np+1+1} \subseteq A_{i+1} \text{ and the given } i \in \overline{p}\) as \(n \to \infty\). \(\forall j \in \overline{p} - 1 \cup \{0\}\) and the given \(i \in \overline{p}\) since \(A_j \subseteq X\).
is nonempty, bounded and closed; ∀j ∈ ℙ and the given
i ∈ ℙ. Thus, one gets from (26), since A_j ⊆ X is nonempty,
bounded and closed, and then boundedly compact, and also
approximatively compact with respect to A_j−1 [8, 35], that:

\[ D_{i+j} ≤ d(ξ_{np+1+j}, ξ_{np+i+j}) \]
\[ → d(ξ_{i+j}, ξ_{i+j+1}) = D_{i+j} \]  (27)
\[ = d(ξ_{i+j}, Tξ_{i+j}) \]  as \( n → ∞ \);

\( ∀ j ∈ ℙ \setminus \{0 \} \) and the given \( i ∈ ℙ \), where \( ξ_{i+j+1} ∈ \{T^pζ_i \}_{i ∈ ℙ} \setminus \{0 \} \) and the given \( i ∈ ℙ \).
Since all the subsets \( A_j \subseteq X \); \( ∀ j ∈ ℙ \) are nonempty, closed,
and boundedly compact; \( ∀ j ∈ ℙ \) then \( ξ_j \in A_j \) is a best
proximity point in \( A_j \) of \( T : Σ_{i∈ℙ} A_i → Σ_{i∈ℙ} A_i \), and it is
also a fixed point of the restricted composite self-mapping
\( \tilde{T}_j : U_{i∈ℙ} A_i \rightarrow U_{i∈ℙ} A_i \) \( A_j \); \( ∀ j ∈ ℙ \). Thus, there are
Cauchy, then convergent since \( (X, d) \) is complete, sequences
\( \{x_{np+j}\}_{i ∈ ℙ} \) with respect to first elements \( x_{i+j} ∈ Tξ_{i+j} \);
\( ∀ j ∈ ℙ \setminus \{0 \} \) and the given \( i ∈ ℙ \), each being convergent
| to \( x \), where \( x \) is the first element of
\( \{x_{np+i}\}_{i ∈ ℙ} \) \( A_i \), which consists of partially ordered elements
with respect to the partial order \( ≤ \) such that

\[ x_{i+j} ≤ \cdots ≤ x_{(n−1)p+j} \in \{T^pζ_i \}_{i ∈ ℙ} \setminus \{0 \} \]
\[ ≤ x_{(np)+j} ≤ \cdots ≤ x_{i+j} \]
\[ = \lim_{n→∞} x_{np+i+j} \]  (28)

with \( \{x_{np+i+j}\} \subseteq A_i \); \( j ∈ ℙ \setminus \{0 \} \), \( ∀ i ∈ ℙ \) for the
given \( i ∈ ℙ \). But \( ξ_{i+j} \in A_{i+j} \); \( ∀ j ∈ ℙ \setminus \{0 \} \) and the given
\( i ∈ ℙ \), is a fixed point of the restricted composite self-mapping
\( \tilde{T}_j : U_{i∈ℙ} A_i \rightarrow U_{i∈ℙ} A_i \) \( | A_j, \forall j ∈ ℙ \). A fixed
point of the composite self-mapping \( \tilde{T} : U_{i∈ℙ} A_i \rightarrow U_{i∈ℙ} A_i \)
from Lemma 4(ii) to which the partially ordered sequences of first
\( x \) is convergent. It is also a best proximity point in \( A_i \) of the self-mapping
\( T : U_{i∈ℙ} A_i \rightarrow U_{i∈ℙ} A_i \) \( A_i \); \( ∀ i ∈ ℙ \) from Lemma 4(iii) and the second part of Lemma 4(ii). Then,
\( \bar{x}_j \in T\bar{x}_j \subseteq T^p\bar{x}_j \) \( ∀ j ∈ ℙ \). The uniqueness property
of each of those \( p \) best proximity points \( \bar{x}_j \in T\bar{x}_j \) in each of
the subsets \( A_j \subseteq X \) follows from their uniqueness as fixed
points of the restricted self-mappings \( \tilde{T}_j : U_{i∈ℙ} A_i \rightarrow U_{i∈ℙ} A_i \);
\( A_j \); \( ∀ j ∈ ℙ \), from Theorem 2, since \( (X, d) \) is a convex metric
space and the subsets \( A_j \subseteq X \) are convex; \( ∀ j ∈ ℙ \). On the other hand,
it turns out that if all the subsets have nonempty intersection, such an
intersection is convex so that the best proximity points are all identical
and the unique fixed point of \( T : U_{i∈ℙ} A_i \rightarrow U_{i∈ℙ} A_i \) and \( T^p : U_{i∈ℙ} A_i \rightarrow U_{i∈ℙ} A_i \) from
Theorem 2, this leads to the proofs of (i)–(iii).

Remarks 2. (1) Theorem 5 proves the uniqueness of the best
proximity points for any partially ordered sets with first elements
in any of the subsets of the multivalued \( p \)-cyclic
self-mapping on \( U_{i∈ℙ} A_i \) satisfying assumptions (1)–(4) of
Theorem 1 as it was commented, in Section 2 concerning such
a theorem, the given \( A_j ⊆ X \) for some \( i ∈ ℙ \) to select the first
two elements of the partial order can be chosen arbitrarily
by construction from (6), namely, from assumption (4) of
Theorem 1.

(2) The value of the individual contractive constants being
less than, equal to, or larger than one for each pair of adjacent
subsets is irrelevant in Theorem 5 provided that its product is
less than one. Note also that Theorem 5 holds also if the
distances between each pair of adjacent subsets are not
necessarily identical.

(3) Note also that, for Euclidean metric, the convexity of \( X \)
keeps as hypothesis for the uniqueness of the best proximity
points of the multivalued self-mapping, since although the
subsets \( A_j \) of \( X, i ∈ ℙ \) are convex, the existence of points
belonging to such subsets guaranteeing the equality in the
triangle inequality for the metric would not be otherwise
guaranteed, since such sets are disjoint and pair-wise disjoint.

(4) It can be observed that the metric convexity of the
space \( (X, d) \) cannot be relaxed to that of \( (\cap_{i∈ℙ} A_i, d) \), since
the subsets \( A_j \subseteq X \) do not necessarily intersect.

(5) Note that the results of Sections 2 and 3 obtained from
the contractive condition (5) also hold for multivalued self-
mappings \( T : U_{i∈ℙ} A_i \rightarrow U_{i∈ℙ} A_i \) which are not cyclic; that
is, \( T(A_j) ∩ \bar{A}_{i+j} ≠ ∅ \) for some \( i ∈ ℙ \) but fulfill the condition
\( T_x \cap A_{i+j} ≠ ∅ \forall x \in A_j, ∀i ∈ ℙ \).

4. Example

Consider two bounded and closed real subsets \( A_1 = [ε, M] \);
\( A_2 = [−M, −ε] = −A_1 \) for nonnegative positive real
constants \( ε, M \) with \( ε ≤ M \) under the Euclidean metric so that
\( D = d(A_1, A_2) = 2ε \).

Consider also a scalar discrete dynamic system of state
\( x_k \) operating at each state value under \( N \) tentatively feedback
controls \( u^{(i)}_k = x^{(i)}_k \); \( ∀ i ∈ I_k \), where the indexing set of
tentative states at the \( (k+1)\)-th sampling point is defined by

\[ I_{k+1} = I_k \times N \]
\[ = I_0 \times N \times \cdots \times N \]  (29)
\[ = \{1\} \times N^k, \forall k ∈ Z_0^+, \] where “\( x \)” stands for the Cartesian product of sets, \( x^{(i)}_0 = x^{(i)}_0 = x \)

\[ = x \in [−M, −ε] ∪ [ε, M] \) is the initial point of an iteration
through a self-mapping \( T \) from \([−M, −ε] ∪ [ε, M] \) to itself and
\( I_0 = \{1\} \). Then the discrete state trajectory takes values in
alternated points at \( A_1 \) and \( A_2 \) from the initial state condition
\( x \) such that \( x^{(i)}_k \in T^{(i)}_x = \{x^{(i)}_{k+1}, \ldots, x^{(i)}_{k+N}\}; \forall k ∈ Z_0^+ \)

\[ x^{(i)}_{k+1} = \begin{cases} x^{(i)}_{k+1} \quad \text{if } \|x^{(i)}_{k+1}\| < M, \forall i ∈ I_k, \forall k ∈ Z_0^+, \\
M \text{ sgn} x^{(i)}_{k+1} \quad \text{if } \|x^{(i)}_{k+1}\| > M, \forall i ∈ I_k, \forall k ∈ Z_0^+, \\
\epsilon \text{ sgn} x^{(i)}_{k+1} \quad \text{if } \|x^{(i)}_{k+1}\| = M, \forall i ∈ I_k, \forall k ∈ Z_0^+. \end{cases} \]  (30)
where
\[ x^{(i,k+1)}_{k+1} = a_k x_k^{(i)} + b_k u_k^{(i)} = (a_k - b_k K^{(i,k)}) x_k^{(i)}, \quad \forall i_k \in I_k, \forall k \in \mathbb{Z}_+ \]  \hfill (31)

with \( a_k \neq 0 \) and \( b_k > 0 \) being nonzero real numbers under a sequence of controllers of gains
\[ K^{(i,k)}_k = \frac{\rho_k^{(i)}}{b_k}, \quad \rho_k^{(i)} \in [0, \rho]; \quad \forall i_k \in I_k, \forall k \in \mathbb{Z}_+ \]  \hfill (32)

for some nonnegative real constant \( \rho \) so that, after replacing (32) into (31), this leads to the controlled closed-loop trajectory sequence given by (30) subject to \( x^{(i,k+1)}_{k+1} = -\rho_k^{(i)} x_k^{(i)}; \quad \forall i_k \in I_k \) with \( x^{(i,k+1)}_{k+1} \leq \min(\max(\rho_k^{(i)}), |x_k^{(i)}|, \varepsilon), M); \forall i_k \in I_k, \forall k \in \mathbb{Z}_+ \). Assume that (5) holds for \( x \preceq y \) so that, for some \( y \in Tx \) and some \( z \in Ty \), one gets
\[ d(z, y) \leq H(Tx, Ty) \leq Kd(x, y) + 2(1 - K) \varepsilon \]  \hfill (33)

for some \( i_k \in I_k; \forall k \in \mathbb{Z}_+ \). The above constraints guarantee the generation of an iterated nondecreasing partially ordered sequence \( S(x) = \left\{ x = x_0^{(i)}, x_1^{(i)}, \ldots, x_n^{(i)}, \ldots \right\} \) by the self-mapping \( T \) on \([-M, -\varepsilon] \cup [\varepsilon, M] \) with respect to the partial order “\( \preceq \)” under a sequence of control gains satisfying \( 0 < K^{(i,k)}_k = 0 \preceq (\rho_k^{(i)} + a_k)/b_k \leq 1/(|b_k|)((d_0/M) - |a_k - 1|) \) and \( |a_k - 1| < 2\varepsilon/M \leq d_0/M; \forall k \in \mathbb{Z}_+ \).

It follows from Theorem 5 that such a sequence has two subsequences, built by keeping the same order with alternate terms of \( S(x) \), converging, respectively, to the unique best proximity points, \( x^*_+ = \varepsilon \in A_1 \) and \( x^*_- = -\varepsilon \in A_2 \) of the self-mapping. If \( \varepsilon = 0 \) both such sequences and \( S(x) \) converge to the resulting unique fixed point of the 2-cyclic composite self-mapping \( T \) on \([-M, -\varepsilon] \cup [\varepsilon, M] \). If the considered subsets are \( A_1 = [-\varepsilon, M]\); \( A_2 = [-M, \varepsilon] = -A_1 \), the first term of the chain of inequalities (34) is valid with the replacement \( 2\varepsilon \to 0 \) while the terms \( 2(1 - K)\varepsilon \) are all zeroed in such a chain of inequalities. Thus, (34) holds if there is a sequence of controller gains satisfying
\[ 0 < K^{(i,k)}_k = \frac{\rho_k^{(i)} + a_k}{b_k} \leq \frac{1}{|b_k|} \left( \frac{d_0}{KM} - |a_k - 1| \right) \]  \hfill (36)

under sufficiency-type constraints \( d_0 > KM \) if \( |a_k| < d_0/KM - 1; \forall k \in \mathbb{Z}_+ \) or \( d_0 < KM \) and \( a_k > 1 - d_0/KM \) if \( a_k \in [0, 1]; \forall k \in \mathbb{Z}_+ \). Thus, \( S(x) = \{x = x_0^{(i)}, x_1^{(i)}, \ldots, x_n^{(i)}, \ldots \} \) converges to \( x^* = 0 \) which is the a unique fixed point of \( T \) on \([-M, -\varepsilon] \cup [\varepsilon, M] \).

for some \( K \in (0, 1) \). Note also from (30)-(31) that (5) is guaranteed under the set of constraints below:
\[ 2\varepsilon \leq d(x^{(i,k+1)}, x^{(i,k+1)}_{k+1}) \leq d(x^{(i,k+1)}_{k+1}, x^{(i,k+1)}_{k+1}) \]  \hfill (34)

where
\[ x^{(i,k+1)}_{k+1} = a_k x_k^{(i)} + b_k u_k^{(i)} = (a_k - b_k K^{(i,k)}) x_k^{(i)}, \quad \forall i_k \in I_k, \forall k \in \mathbb{Z}_+ \]  \hfill (31)
The following extension of the example is direct. Assume that (31) is replaced by the second-order uncoupled discrete dynamic system:

\[
\bar{x}_{k+1}^{(i_k)} = (x_{1k+1}^{(i_k)}, x_{2k+1}^{(i_k)})^T = A_k x_k^{(i_k)} + B_k u_k^{(i_k)} = (A_k - B_k K_k^{(i_k)}) x_k^{(i_k)}; \quad \forall i_k \in I_k, \forall k \in Z_{0+},
\]

where the superscript \(T\) stands for transposition and \(x_{k+1}^{(i_k)} = (x_{1k+1}, x_{2k+1})^T\), subject to the tentative scalar feedback controls \(u_k^{(i_k)} = -K_k^{(i_k)} x_k^{(i_k)}\), where \(A_k = \text{Diag}(a_{1k}, a_{2k})\) is a sequence of real \(2 \times 2\) diagonal matrices, \(B_k = (b_{1k}, b_{2k})^T\) is a sequence of two dimensional real column vectors, \(K_k^{(i_k)}\) are sequences of two-dimensional real row vectors. The set of inequalities (34) are replaced by parallel ones by using the maxima for norms/distances of the two state vector components. The resulting inequalities are obtained from the initial basic constraints:

\[
2\epsilon \leq d(x_{k+1}^{(i_k)}, x_{k+1}^{(i_k)}) \leq d(x_{k+1}^{(i_k)}, x_{k+1}^{(i_k)}) = \max_{1 \leq j \leq 2} \|K_k^{(i_k)} x_k^{(i_k)}\|
\]

for some \(i_k \in I_k, \forall k \in Z_{0+}\). To keep the validity of (30) for each state component, the constraints (35a) and (35b) are replaced by

\[
d_0 > \max_{1 \leq j \leq 2} \left(\min_{1 \leq j \leq 2} \left(\frac{d_0}{M - a_{jk} - |a_{jk} - 1|}\right), \frac{K[a_{jk+1} - b_{jk} K_j^{(i_k)} - 1]}{a_{jk+1} - b_{jk} K_j^{(i_k)} - 1}\right) \quad \text{if } \epsilon > 0
\]

\[
0 < K_k^{(i_k)} \leq \left(\min_{1 \leq j \leq 2} \left(\frac{d_0}{M - a_{jk} - |a_{jk} - 1|}\right), \frac{K[a_{jk+1} - b_{jk} K_j^{(i_k)} - 1]}{a_{jk+1} - b_{jk} K_j^{(i_k)} - 1}\right) \quad \text{if } \epsilon = 0
\]

for some \(i_k \in I_k, \forall k \in Z_{0+}\).

Acknowledgments

The author is very grateful to the Spanish Government for its support of this research through Grant DPI2012-30651 and to the Basque Government by its support of this research through Grants IT378-10 and SAIOTEK S-PB120U015. He is also grateful to UPV/EHU for its financial support through UFI 2011/07 and to the referees for their useful comments.

References


Submit your manuscripts at
http://www.hindawi.com