Research Article
Properties of a Class of \( p \)-Harmonic Functions

Elif Yaşar and Sibel Yalçın

Department of Mathematics, Faculty of Arts and Sciences, Uludag University, 16059 Bursa, Turkey

Correspondence should be addressed to Elif Yaşar; elifyasar@uludag.edu.tr

Received 21 January 2013; Accepted 21 May 2013

Academic Editor: Youyu Wang

Copyright © 2013 E. Yaşar and S. Yalçın. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

A \( p \) times continuously differentiable complex-valued function \( F = u + iv \) in a domain \( D \subseteq \mathbb{C} \) is \( p \)-harmonic if \( F \) satisfies the \( p \)-harmonic equation \( \Delta \cdots \Delta F = 0 \), where \( p \) is a positive integer. By using the generalized Salagean differential operator, we introduce a class of \( p \)-harmonic functions and investigate necessary and sufficient coefficient conditions, distortion bounds, extreme points, and convex combination of the class.

1. Introduction

A continuous complex-valued function \( f = u + iv \) in a domain \( D \subseteq \mathbb{C} \) is harmonic if both \( u \) and \( v \) are real harmonic in \( D \); that is, \( \Delta u = 0 \) and \( \Delta v = 0 \). Here \( \Delta \) represents the complex Laplacian operator

\[
\Delta = 4 \frac{\partial^2}{\partial z \partial \overline{z}} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.
\]  

In any simply connected domain \( D \) we can write \( f = h + \overline{g} \), where \( h \) and \( g \) are analytic in \( D \). We call \( h \) the analytic part and \( g \) the coanalytic part of \( f \). A necessary and sufficient condition for \( f \) to be locally univalent and sense preserving in \( D \) is that \( J_f = |f'|^2 - |f|^2 \geq 0 \) in \( D \). See [1, 2].

Denote by \( SH \) the class of functions \( f = h + \overline{g} \) that are harmonic, univalent, and sense preserving in the unit disk \( U = \{ z : |z| < 1 \} \) for which \( f(0) = f'(0) - 1 = 0 \). Then for \( f = h + \overline{g} \in SH \) we may express the analytic functions \( h \) and \( g \) as

\[
h(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad g(z) = \sum_{j=1}^{\infty} b_j z^j, \quad |b_1| < 1.
\]  

The properties of the class \( SH \) and its geometric subclasses have been investigated by many authors; see [1–6]). Note that \( SH \) reduces to the class \( S \) of normalized univalent functions in \( U \) if the coanalytic part of \( f \) is identically zero.

A \( p \) times continuously differentiable complex-valued function \( F = u + iv \) in a domain \( D \subseteq \mathbb{C} \) is \( p \)-harmonic if \( F \) satisfies the \( p \)-harmonic equation \( \Delta \cdots \Delta F = 0 \), where \( p \) is a positive integer. A function \( F \) is \( p \)-harmonic in a simply connected domain \( D \) if and only if \( F \) has the following representation:

\[
F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} f_{p-k+1}(z),
\]  

where \( f_{p-k+1}(z) = 0 \) in \( D \) for each \( k \in \{1, \ldots, p\} \). \( f_{p-k+1} \) has the form

\[
f_{p-k+1} = h_{p-k+1} + \overline{g}_{p-k+1},
\]  

where

\[
h_p(z) = z + \sum_{j=2}^{\infty} a_{j,p} z^j,
\]

\[
h_{p-k+1}(z) = \sum_{j=1}^{\infty} a_{j,p-k+1} z^j, \quad (k \geq 2),
\]  

\[
g_{p-k+1}(z) = \sum_{j=1}^{\infty} b_{j,p-k+1} z^j, \quad (k \geq 1).
\]  

Denote by \( SH_p \) the class of functions \( F \) of the form (3) that are harmonic, univalent, and sense-preserving in the unit disk. Apparently, if \( p = 1 \) and \( p = 2 \), \( F \) is harmonic and biharmonic, respectively.
Biharmonic functions have been studied by several authors, such as, [7–9]. Also, biharmonic functions arise in many physical situations, particularly, in fluid dynamics and elasticity problems. They have many important applications in engineering, biology, and medicine, such as in [10, 11].

For a function \( f \) in \( S \), differential operator \( D^p \) (\( n \in \mathbb{N}_0 \)) was introduced by Sălăgean [12]. Al-Oboudi [13] generalized \( D^p \) as follows:

\[
D^0_f (z) = f(z),
\]

\[
D^1_f (z) = (1 - \lambda) f(z) + \lambda zf'(z), \quad \lambda \geq 0, \tag{6}
\]

\[
D^n_f (z) = D^n (D^{n-1} f(z)) .
\]

When \( \lambda = 1 \), we get the Salagean differential operator.

For \( f(z) = h(z) + g(z) \) given by (2), Li and Liu [14] defined the following generalized Salagean operator \( D^n_f \) in \( SH \):

\[
D^n_f (z) = D^n h(z) + D^n g(z), \quad \lambda \geq 0, \tag{7}
\]

where

\[
D^n h(z) = z + \sum_{j=1}^{\infty} [1 + (k - 1)\lambda|^n a_j z^j, \tag{8}
\]

\[
D^n g(z) = \sum_{j=1}^{\infty} [1 + (k - 1)\lambda|^n b_j z^j.
\]

For a \( p \)-harmonic function \( F \) given by (3), we define the following operator:

\[
D^0_F (z) = F(z), \tag{9}
\]

\[
D^1_F (z) = (1 - \lambda) D^0_F (z) + \lambda \left[ z \left( D^0_F (z) \right)_z + \Xi \left( D^1_F (z) \right)_z \right], \quad \lambda \geq 0, \tag{10}
\]

\[
D^n_F (z) = D^n \left( D^{n-1} F(z) \right), \quad (n \in \mathbb{N}).
\]

If \( F \) is given by (3), then from (10) we see that

\[
D^n_F (z) = \sum_{k=1}^{p} |z|^{2(k-1)} \sum_{j=1}^{\infty} \left[ 1 + (j - 1)\lambda + 2 (k - 1)\lambda \right] a_{j,p-k+1} |z|^j
\]

\[+ \sum_{k=1}^{p} |z|^{2(k-1)} \sum_{j=1}^{\infty} \left[ 1 + (j - 1)\lambda + 2 (k - 1)\lambda \right] b_{j,p-k+1} |z|^j, \quad (a_{1,p} = 1, |b_{1,p}| < 1). \tag{11}
\]

When \( p = 1 \), we get the generalized Salagean operator for harmonic univalent functions defined by Li and Liu [14].

Denote by \( SH_p(n, \lambda, \alpha) \) the class of functions \( F \) of the form (3) which satisfy the condition

\[
\Re \left\{ \frac{D^{n+1} F(z)}{D^n F(z)} \right\} \geq \alpha, \quad 0 \leq \alpha < 1, \tag{12}
\]

where \( D^n F(z) \) is defined by (11).

We let the subclass \( \overline{SH}_p \) of \( SH_p \) consist of functions \( F \) of the form (3) which include \( h_{p-k+1} = h_{p-k+1} + g_{p-k+1} \), where

\[
h_p (z) = z - \sum_{j=2}^{\infty} |a_{j,p}| z^j, \tag{13}
\]

\[
g_{p-k+1} (z) = - \sum_{j=1}^{\infty} |b_{j,p-k+1}| z^j, \quad (k \geq 2),
\]

\[
g_{p-k+1} (z) = - \sum_{j=1}^{\infty} |b_{j,p-k+1}| z^j, \quad (k \geq 1). \tag{14}
\]

Define \( \overline{SH}_p(n, \lambda, \alpha) := SH_p(n, \lambda, \alpha) \cap \overline{SH}_p \).

The main object of the paper is to introduce a class of \( p \)-harmonic functions by using the generalized Salagean operator which was defined by Li and Liu [14]. We investigate necessary and sufficient coefficient conditions, extreme points, distortion bounds, and convex combination of the class.

2. Main Results

**Theorem 1.** Let \( F \) be a \( p \)-harmonic function given by (3). Furthermore, let

\[
\sum_{k=1}^{p} \sum_{j=1}^{\infty} \left[ 1 + (j - 1)\lambda + 2 (k - 1)\lambda - \alpha \right]
\]

\[\times \left[ 1 + (j - 1)\lambda + 2 (k - 1)\lambda \right] a_{j,p-k+1} + b_{j,p-k+1} \right] \]

\[\leq 2 (1 - \alpha), \tag{15}\]

where \( \lambda \geq 1, n \in \mathbb{N}, 0 \leq \alpha < 1, \) and \( a_{1,p} = 1 \). Then \( F \) is sense preserving, \( p \)-harmonic, and univalent in \( U \) and \( F \in \overline{SH}_p(n, \lambda, \alpha) \).

**Proof.** Suppose \( z_1, z_2 \in U \) and \( z_1 \neq z_2 \), so that \( |z_1| \leq |z_2| < 1: \)

\[
|F(z_1) - F(z_2)| \]

\[
\geq |f_p(z_1) - f_p(z_2)| \]

\[- \sum_{k=2}^{p} \left| |z_1|^{2(k-1)} f_{p-k+1} (z_1) - |z_2|^{2(k-1)} f_{p-k+1} (z_2) \right| \]

\[
\geq |z_1 - z_2| \left[ 1 - \sum_{j=2}^{\infty} \frac{|z_j - z_{j-1}|}{|z_1 - z_2|} \left| a_{j,p} \right| - \sum_{j=1}^{\infty} \frac{|z_j - z_{j-1}|}{|z_1 - z_2|} \left| b_{j,p} \right| \right].
\]
Using the fact that \( \text{Re} \omega \geq \alpha \) if and only if \( |1 - \alpha + \omega| \geq |1 + \alpha - \omega| \), it suffices to show that
\[
\left| (1 - \alpha) D^\alpha_F(z) + D^{\alpha+1}_F(z) \right| - \left| (1 + \alpha) D^\alpha_F(z) - D^{\alpha+1}_F(z) \right| \geq 0.
\]
Substituting for \( D^\alpha_F \) in (18), we obtain
\[
\left| (1 - \alpha) D^\alpha_F(z) + D^{\alpha+1}_F(z) \right| - \left| (1 + \alpha) D^\alpha_F(z) - D^{\alpha+1}_F(z) \right| \geq 2 (1 - \alpha) |z|
\]
(20)

This last expression is nonnegative by (15), and so the proof is complete.

**Theorem 2.** Let \( F \) be given by (13) and (14). Then \( F \in \overline{S}_H(p, n, \lambda, \alpha) \) if and only if

\[
\sum_{k=1}^{p} \sum_{j=1}^{\infty} |j + 2(k-1)| \left| [a_{j,k+1}] + [b_{j,k+1}] \right| \leq 2 (1 - \alpha),
\]

where \( \lambda \geq 1, n \in \mathbb{N}, 0 \leq \alpha < 1, \) and \( a_{1,p} = 1 \).

**Proof.** The “if” part follows from Theorem 1 upon noting that \( \overline{S}_H(p, n, \lambda, \alpha) \subset S_H(p, n, \lambda, \alpha) \). For the “only if” part, we show that \( F \notin \overline{S}_H(p, n, \lambda, \alpha) \) if the condition (20) does not hold.
Note that a necessary and sufficient condition for $F$ given by (13) and (14) to be in $SH_p(n, \lambda, \alpha)$ is that the condition (12) should be satisfied.

This is equivalent to $\text{Re}[A(z)/B(z)] \geq 0$, where

$$A(z) = (1 - \alpha) z - \sum_{j=2}^{\infty} (1 + (j - 1) \lambda - \alpha) [1 + (j - 1) \lambda]^n |a_{j,p}| z^j$$

$$B(z) = z - \sum_{j=1}^{\infty} (1 + (j - 1) \lambda - \alpha) [1 + (j - 1) \lambda]^n |b_{j,p}| z^j$$

The above condition must hold for all values of $z$, $|z| = r < 1$. Upon choosing the values of $z$ on the positive real axis, where $0 \leq z = r < 1$ we must have

$$F(z) = \sum_{k=1}^{\infty} \left( \sum_{j=1}^{\infty} [a_{j,p-k+1} + b_{j,p-k+1}] z^{j+2k-3} \right)$$

If the condition (20) does not hold, then the numerator in (22) is negative for $r$ is sufficiently close to 1. Hence there exist $z_0 = r_0$ in $(0, 1)$ for which the quotient in (22) is negative. This contradicts the required condition for $F \in SH_p(n, \lambda, \alpha)$ and so the proof is complete.

**Theorem 3.** Let $F$ be given by (13) and (14). Then $F \in SH_p(n, \lambda, \alpha)$ if and only if

$$F(z) = \sum_{k=1}^{\infty} \left( \sum_{j=1}^{\infty} [X_{j,p-k+1} h_{j,p-k+1}(z) + Y_{j,p-k+1} g_{j,p-k+1}(z)] \right)$$

where

$$h_{1,p}(z) = z,$n

$$h_{j,p}(z) = z - \frac{2(1 - \alpha)}{1 + (j - 1) \lambda - \alpha} [1 + (j - 1) \lambda]^n z^j, \quad (j \geq 2),$$

$$g_{j,p}(z) = z - \frac{2(1 - \alpha)}{1 + (j - 1) \lambda - \alpha} [1 + (j - 1) \lambda]^n z^j, \quad (j \geq 1),$$

$$h_{j,p-k+1}(z) = z - |z|^{2k-1} (2(1 - \alpha))$$

$$\times \left[ [1 + (j - 1) \lambda + 2(k - 1) \lambda - \alpha] \right]^{-1}$$

$$\times \left[ [1 + (j - 1) \lambda + 2(k - 1) \lambda - \alpha] \right]^{-1}.$$
Abstract and Applied Analysis

\[ g_{j,p-k+1}(z) = z - |z|^{2(k-1)}(2(1-\alpha)) \times \left[ \left(1 + (j-1)\lambda + 2(k-1)\lambda - \alpha \right) \times \left[1 + (j-1)\lambda + 2(k-1)\lambda\right]^{n} \right]^{-1}z^j, \]

\[ (j \geq 1, \ 2 \leq k \leq p), \]

and \( \sum_{j=1}^{p} \sum_{k=1}^{\infty} (X_{j,p-k} + Y_{j,p-k+1}) = 1, \ X_{j,p-k+1} \geq 0, \ Y_{j,p-k+1} \geq 0. \)

In particular, the extreme points of \( \text{SH}_p(n,\lambda,\alpha) \) are \( \{h_{j,p-k+1}(z)\} \) and \( \{g_{j,p-k+1}(z)\} \), where \( j \geq 1 \) and \( 1 \leq k \leq p \).

**Proof.** For functions \( F \) of the form (13) and (14) we have

\[ F(z) = \sum_{j=1}^{p} \sum_{k=1}^{\infty} \left( X_{j,p-k+1}h_{j,p-k+1}(z) + Y_{j,p-k+1}g_{j,p-k+1}(z) \right) \]

\[ = z - \sum_{j=2}^{\infty} (2(1-\alpha)) \left[ \left(1 + (j-1)\lambda - \alpha \right) \times \left[1 + (j-1)\lambda\right]^{n} \right]^{-1}X_{j,p}z^j \]

\[ - \sum_{j=1}^{\infty} (2(1-\alpha)) \left[ \left(1 + (j-1)\lambda - \alpha \right) \times \left[1 + (j-1)\lambda\right]^{n} \right]^{-1}Y_{j,p}z^j \]

\[ - \sum_{k=2}^{p} |z|^{2(k-1)} \times \sum_{j=1}^{\infty} (2(1-\alpha)) \left[ \left(1 + (j-1)\lambda + 2(k-1)\lambda - \alpha \right) \times \left[1 + (j-1)\lambda + 2(k-1)\lambda\right]^{n} \right]^{-1} \times \left( Y_{j,p-k+1}z^j + X_{j,p-k}z^j \right). \]

Then

\[ \sum_{j=2}^{\infty} \frac{(1 + (j-1)\lambda - \alpha)(1 + (j-1)\lambda)^n}{2(1-\alpha)} \times \left( \frac{2(1-\alpha)}{(1 + (j-1)\lambda - \alpha)(1 + (j-1)\lambda)^n}X_{j,p} \right) \]

\[ + \sum_{j=1}^{\infty} \frac{(1 + (j-1)\lambda - \alpha)(1 + (j-1)\lambda)^n}{2(1-\alpha)} \times \left( \frac{2(1-\alpha)}{(1 + (j-1)\lambda - \alpha)(1 + (j-1)\lambda)^n}Y_{j,p} \right) \]

\[ + \sum_{k=2}^{p} \sum_{j=1}^{\infty} \left( [1 + (j-1)\lambda + 2(k-1)\lambda - \alpha] \times [1 + (j-1)\lambda + 2(k-1)\lambda]^{n} \right) (2(1-\alpha))^{-1} \times \left( [1 + (j-1)\lambda + 2(k-1)\lambda]^{n} \right)^{-1} \]

\[ \times \left( X_{j,p-k+1} + Y_{j,p-k+1} \right) \]

\[ = \sum_{j=1}^{p} X_{j,p} + \sum_{j=1}^{p} Y_{j,p} \]

\[ + \sum_{k=2}^{p} \sum_{j=1}^{\infty} \left( X_{j,p-k+1} + Y_{j,p-k+1} \right) = 1 - X_{1,p} \leq 1, \]

and so \( F \in \text{SH}_p(n,\lambda,\alpha) \). Conversely, if \( F \in \text{SH}_p(n,\lambda,\alpha) \), then

\[ |a_{j,p}| \leq \frac{2(1-\alpha)}{(1 + (j-1)\lambda - \alpha)(1 + (j-1)\lambda)^n}, \quad (j \geq 2), \]

\[ |a_{j,p-k+1}| \leq \frac{2(1-\alpha)}{(1 + (j-1)\lambda + 2(k-1)\lambda - \alpha) \times [1 + (j-1)\lambda + 2(k-1)\lambda]^{n}} \times \left( [1 + (j-1)\lambda + 2(k-1)\lambda]^{n} \right)^{-1}, \quad (j \geq 1, \ 2 \leq k \leq p), \]

\[ |b_{j,p}| \leq \frac{2(1-\alpha)}{(1 + (j-1)\lambda - \alpha)(1 + (j-1)\lambda)^n}, \quad (j \geq 2), \]

\[ |b_{j,p-k+1}| \leq \frac{2(1-\alpha)}{(1 + (j-1)\lambda + 2(k-1)\lambda - \alpha) \times [1 + (j-1)\lambda + 2(k-1)\lambda]^{n}} \times \left( [1 + (j-1)\lambda + 2(k-1)\lambda]^{n} \right)^{-1}, \quad (j \geq 1, \ 2 \leq k \leq p). \]
Set
\[
X_{j,p} = \left( (1 + (j-1)\lambda - \alpha) \times [1 + (j-1)\lambda]^n \right) (2(1-\alpha))^{-1} \left| a_{j,p} \right|, \quad (j \geq 2),
\]
\[
Y_{j,p} = \left( (1 + (j-1)\lambda - \alpha) [1 + (j-1)\lambda]^n \right) (2(1-\alpha))^{-1} \left| b_{j,p} \right|, \quad (j \geq 1),
\]
\[
X_{j,p-k+1} = \left( (1 + (j-1)\lambda + 2(k-1)\lambda - \alpha) \times [1 + (j-1)\lambda + 2(k-1)\lambda]^n \right) (2(1-\alpha))^{-1} \left| a_{j,p-k+1} \right|, \quad (j \geq 1, \quad 2 \leq k \leq p),
\]
\[
Y_{j,p-k+1} = \left( (1 + (j-1)\lambda + 2(k-1)\lambda - \alpha) [1 + (j-1)\lambda + 2(k-1)\lambda]^n \right) (2(1-\alpha))^{-1} \left| b_{j,p-k+1} \right|, \quad (j \geq 1, \quad 2 \leq k \leq p),
\]
\[
X_{1,p} = 1 - \sum_{j=2}^{\infty} X_{j,p} - \sum_{j=1}^{\infty} Y_{j,p} - \sum_{k=2}^{p} \sum_{j=1}^{\infty} [X_{j,p-k+1} + Y_{j,p-k+1}],
\]
where \( X_{1,p} \geq 0 \). Then, as required, we obtain
\[
F(z) = \sum_{k=1}^{p} \left( X_{j,p-k+1} h_{j,p-k+1}(z) + Y_{j,p-k+1} g_{j,p-k+1}(z) \right).
\]

Theorem 4. Let \( F \in \mathcal{SH}_{p}(n, \lambda, \alpha) \). Then for \( |z| = r < 1 \) we have
\[
|F(z)| \leq \left( 1 - |h_{1,p}| + \sum_{k=2}^{p} \left( |a_{1,p-k+1}| + |b_{1,p-k+1}| \right) \right) r + \left( \sum_{k=2}^{p} \left( |a_{1,p-k+1}| + |b_{1,p-k+1}| \right) \right)^2 r^2
\]
\[
\leq \left( 1 - |h_{1,p}| + \sum_{k=2}^{p} \left( |a_{1,p-k+1}| + |b_{1,p-k+1}| \right) \right) r + \left( \sum_{k=2}^{p} \left( |a_{1,p-k+1}| + |b_{1,p-k+1}| \right) \right)^2 r^2
\]
\[
\leq \left( 1 + \frac{2(1-\alpha)}{(1+\lambda-\alpha)[1+\lambda]^n} \right) r + \left( \sum_{k=2}^{p} \frac{[1+2(k-1)\lambda-\alpha][1+2(k-1)\lambda]^n}{(1+\lambda-\alpha)[1+\lambda]^n} \right) r^2.
\]

Proof. We only prove the right-hand inequality. The proof for the left-hand inequality is similar and will be omitted.

\[
|F(z)| \leq \left( 1 - \frac{2(1-\alpha)}{(1+\lambda-\alpha)[1+\lambda]^n} \right) r + \left( \sum_{k=2}^{p} \frac{[1+2(k-1)\lambda-\alpha][1+2(k-1)\lambda]^n}{(1+\lambda-\alpha)[1+\lambda]^n} \right) r^2.
\]

(30)

Proof. We only prove the right-hand inequality. The proof for the left-hand inequality is similar and will be omitted.

\[
|F(z)| \leq \left( 1 - \frac{2(1-\alpha)}{(1+\lambda-\alpha)[1+\lambda]^n} \right) r + \left( \sum_{k=2}^{p} \frac{[1+2(k-1)\lambda-\alpha][1+2(k-1)\lambda]^n}{(1+\lambda-\alpha)[1+\lambda]^n} \right) r^2.
\]

(31)
The following covering result follows from the left-hand inequality in Theorem 4.

**Corollary 5.** Let $F$ of the form (13) and (14) be so that $F \in \mathcal{SH}_p(n, \lambda, \alpha)$. Then
\[
\left\{ \omega : |\omega| < \frac{(1 + \lambda - \alpha) [1 + \lambda]^{\nu} - (1 - \alpha)}{(1 + \lambda - \alpha) [1 + \lambda]^{\nu}} + \frac{(1 - \alpha) - (1 + \lambda - \alpha) [1 + \lambda]^n}{(1 + \lambda - \alpha) [1 + \lambda]^n} |b_{j,p}| \right. \\
+ \sum_{k=2}^{p} \left[ (1 + 2 (k - 1) \lambda - \alpha) [1 + 2 (k - 1) \lambda]^n \right. \\
- (1 + \lambda - \alpha) [1 + \lambda]^n \left( 1 + \lambda - \alpha \right) \left. [1 + \lambda]^{-1} \right] \\
\times \left[ |a_{i,p-k+1}| + |b_{i,p-k+1}| \right] \subset F(U).
\] (32)

**Theorem 6.** The class $\mathcal{SH}_p(n, \lambda, \alpha)$ is closed under convex combinations.

**Proof.** Let $F_i \in \mathcal{SH}_p(n, \lambda, \alpha)$ for $i = 1, 2, \ldots$, where $F_i$ is given by
\[
F_i(z) = z - \sum_{j=1}^{\infty} |a_{i,j,p}| z^j - \sum_{j=1}^{\infty} |b_{i,j,p}| \bar{z}^j \\
- \sum_{k=2}^{p} \left[ (1 + 2 (k - 1) \lambda - \alpha) [1 + 2 (k - 1) \lambda]^n \right. \\
- (1 + \lambda - \alpha) [1 + \lambda]^n \left( 1 + \lambda - \alpha \right) \left. [1 + \lambda]^{-1} \right] \\
\times \left[ |a_{i,j,p-k+1}| + |b_{i,j,p-k+1}| \right] (\bar{z}) \subset F_i(U).
\] (33)

Then by (20),
\[
\sum_{k=1}^{p} \sum_{i=1}^{\infty} \left[ (1 + (j - 1) \lambda + 2 (k - 1) \lambda - \alpha) \right. \\
\times \left[ 1 + (j - 1) \lambda + 2 (k - 1) \lambda \right]^{\nu} \\
\times \left. (2 (1 - \alpha))^{-1} \left[ |a_{i,j,p-k+1}| + |b_{i,j,p-k+1}| \right] \right] \leq 1.
\] (34)

For $\sum_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1$, the convex combination of $F_i$ may be written as
\[
\sum_{i=1}^{\infty} t_i F_i(z) \\
= z - \sum_{j=1}^{\infty} \left( \sum_{i=1}^{\infty} t_i |a_{i,j,p}| z^j + |b_{i,j,p}| \bar{z}^j \right) \\
- \sum_{k=2}^{p} \left[ (1 + 2 (k - 1) \lambda - \alpha) [1 + 2 (k - 1) \lambda]^n \right. \\
- (1 + \lambda - \alpha) [1 + \lambda]^n \left( 1 + \lambda - \alpha \right) \left. [1 + \lambda]^{-1} \right] \\
\times \left[ |a_{i,j,p-k+1}| + |b_{i,j,p-k+1}| \right] (\bar{z}) \right).
\] (35)

Then by (34),
\[
\sum_{i=1}^{\infty} \sum_{k=1}^{p} \sum_{j=1}^{\infty} \left[ (1 + (j - 1) \lambda + 2 (k - 1) \lambda - \alpha) \right. \\
\times \left[ 1 + (j - 1) \lambda + 2 (k - 1) \lambda \right]^{\nu} \\
\times \left. (2 (1 - \alpha))^{-1} \left[ |a_{i,j,p-k+1}| + |b_{i,j,p-k+1}| \right] \right] \leq 1.
\] (36)

This is the condition required by (20) and so $\sum_{i=1}^{\infty} t_i F_i(z) \in \mathcal{SH}_p(n, \lambda, \alpha)$.

**References**


Submit your manuscripts at http://www.hindawi.com