Research Article

New Generalization of f -Best Simultaneous Approximation in Topological Vector Spaces

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Let K be a nonempty subset of a Hausdorff topological vector space X , and let f be a real-valued continuous function on X . If for each $x = (x_1, x_2, \dots, x_n) \in X^n$, there exists $k_0 \in K$ such that $F_K(x) = \sum_{i=1}^n f(x_i - k_0) = \inf\{\sum_{i=1}^n f(x_i - k) : k \in K\}$, then K is called f -simultaneously proximal and k_0 is called f -best simultaneous approximation for x in K . In this paper, we study the problem of f -simultaneous approximation for a vector subspace K in X . Some other results regarding f -simultaneous approximation in quotient space are presented.

1. Introduction

Let K be a closed subset of a Hausdorff topological vector space X and f a real-valued continuous function on X . For $x \in X$, set $F_K(x) = \inf_{k \in K} f(x - k)$. A point $k_0 \in K$ is called f -best approximation to x in K if $F_K(x) = f(x - k_0)$. The set $P_K^f(x) = \{k \in K : F_K(x) = f(x - k)\}$ denotes the set of all f -best approximations to x in K . Note that this set may be empty. The set K is said to be f -proximal (f -Chebyshev) if for each $x \in X$, $P_K^f(x)$ is nonempty (singleton). The notion of f -best approximation in a vector space X was given by Breckner and Brosowski [1] and in a Hausdorff topological space X by Narang [2, 3]. For a Hausdorff locally convex topological vector space and a continuous sublinear functional f on X , certain results on best approximation relative to the functional f were proved in [1, 4]. By using the existence of elements of f -best approximation, certain results on fixed points were proved by Pai and Veermani in [5]. In addition, for a topological vector space X relative to upper semicontinuous functions, some results on best approximation were proved by Haddadi and Hamzenejad [6]. Moreover, Naidu [7] proved some results on best simultaneous approximation related to f -nearest point and topological vector space X .

Analogous to the problem of simultaneous approximation [8], we introduce the concept of best f -simultaneous approximation as follows.

Definition 1. Let K be a non-empty subset of a Hausdorff topological vector space X , and let f be a real-valued continuous function on X . A point $k_0 \in K$ is called f -best simultaneous approximation in K if there exists $x = (x_1, x_2, \dots, x_n) \in X^n$ such that

$$F_K(x) = \inf \left\{ \sum_{i=1}^n f(x_i - k) : k \in K \right\} = \sum_{i=1}^n f(x_i - k_0). \quad (1)$$

The set of all f -best simultaneous approximations to $x = (x_1, x_2, \dots, x_n) \in X^n$ in K is denoted by

$$P_K^f(x) = \left\{ k \in K : F_K(x) = \sum_{i=1}^n f(x_i - k) \right\}. \quad (2)$$

The set K is called f -simultaneously proximal (f -simultaneously Chebyshev) if for each $x = (x_1, x_2, \dots, x_n) \in X^n$, $P_K^f(x) \neq \emptyset$ (singleton). If $n = 1$, simultaneous f -proximal is precisely f -proximal.

We remark that if $f(x) = \|x\|$, then the concept of f -best approximation is precisely the best approximation.

A set K is said to be inf-compact at a point $x = (x_1, x_2, \dots, x_n) \in X^n$ [5] if each minimizing sequence in K (i.e., $\sum_{i=1}^n f(x_i - k_n) \rightarrow F_K(x)$) has a convergent subsequence in K . The set K is called inf-compact if it is inf-compact at each $x = (x_1, x_2, \dots, x_n) \in X^n$.

It is easy to see that if K is compact or inf-compact, then K is f -simultaneously proximal.

In this paper, we introduce the concept of f -simultaneous approximation and study the existence and uniqueness problem of f -simultaneous approximation of a subspace K of a Hausdorff topological vector space X . Certain results regarding f -simultaneous approximation in quotient spaces are obtained by generalizing some of the results in [9].

Throughout this paper, X is a Hausdorff topological vector space and f is a real-valued continuous function on X .

2. f -Simultaneous Approximation

In this section, we give some characterizations of f -proximal sets in X . We begin with the following definitions.

Definition 2. A function $f : X \rightarrow \mathbb{R}$ is called absolutely homogeneous if $f(\alpha x) = |\alpha|f(x)$, for all $x \in X$ and all $\alpha \in \mathbb{R}$.

Definition 3. A subset K of X is called f -closed if for all sequences $\{k_m\}$ of K and for all $x = (x_1, x_2, \dots, x_n) \in X^n$, such that $\sum_{i=1}^n f(x_i - k_m) \rightarrow 0$, we have $x \in K^n$.

Definition 4. A subset K of X is called f -compact if for every sequence $\{k_n\}$ in K there exist a subsequence $\{k_{n_k}\}$ of $\{k_n\}$ and $k_0 \in K$ such that $f(k_{n_k} - k_0) \rightarrow 0$.

Definition 5. For $x, y \in X$, where $x = (x_1, x_2, \dots, x_n) \in X^n$ and $y = (y_1, y_2, \dots, y_n) \in X^n$, x is said to be f -orthogonal to y denoted by $x \perp_f y$, if $\sum_{i=1}^n f(x_i) \leq \sum_{i=1}^n f(x_i + \alpha y_i)$ for every scalar $\alpha \in \mathbb{R}$. Also, x is said to be f -orthogonal to a set K if $x \perp_f k$, for all $k \in K$.

Definition 6. We say that K is w -compact if every net $\{k_\alpha\}$ in K has a convergent subnet.

Theorem 7. Let K be a subset of X . Then, one has the following.

- (1) $F_{K+y}(x+Y) = F_K(x)$, for all $x = (x_1, x_2, \dots, x_n)$, where $Y = (y, y, \dots, y) \in X^n$.
- (2) $P_{K+y}^f(x+Y) = P_K^f(x) + y$, for all $x = (x_1, x_2, \dots, x_n)$.
- (3) K is f -simultaneously proximal (f -simultaneously Chebyshev) if and only if $K + y$ is f -simultaneously proximal (f -simultaneously Chebyshev) for every $y \in X$.

Moreover, if f is absolutely homogeneous function, then one has the following.

- (4) $F_{\alpha K}(\alpha x) = |\alpha|F_K(x)$, for all $x = (x_1, x_2, \dots, x_n) \in X^n$ and $\alpha \in \mathbb{R}$.

(5) $P_{\alpha K}^f(\alpha x) = \alpha P_K^f(x)$, for all $x = (x_1, x_2, \dots, x_n) \in X^n$ and $\alpha \in \mathbb{R}$.

(6) K is f -simultaneously proximal (f -simultaneously Chebyshev) if and only if αK is f -simultaneously proximal (f -simultaneously Chebyshev), $\alpha \in \mathbb{R}$.

(7) If f is convex function and K is a convex set, then $P_K^f(x)$ is convex.

Proof. (1) Let $x = (x_1, x_2, \dots, x_n)$ and $Y = (y, y, \dots, y) \in X^n$. Then

$$F_{K+y}(x+Y) = \inf_{k \in K} \sum_{i=1}^n f((x_i + y) - (k + y)) = F_K(x). \quad (3)$$

(2) The equation

$$\begin{aligned} \sum_{i=1}^n f(x_i - k_0) &= \inf_{k \in K} \sum_{i=1}^n f((x_i + y) - (k + y)) \\ &= \inf_{k \in K} \sum_{i=1}^n f(x_i - k) \end{aligned} \quad (4)$$

implies that $k_0 + y \in P_{K+y}^f(x+Y)$ if and only if $k_0 \in P_K^f(x)$. Thus,

$$P_{K+y}^f(x+Y) = P_K^f(x) + y. \quad (5)$$

(3) The proof follows immediately from part (2) above.

(4) Let $x = (x_1, x_2, \dots, x_n) \in X^n$, $\alpha \in \mathbb{R}$. Then,

$$\begin{aligned} F_{\alpha K}(\alpha x) &= \inf_{k \in K} \sum_{i=1}^n f(\alpha x_i - \alpha k) \\ &= |\alpha| \inf_{k \in K} \sum_{i=1}^n f(x_i - k) = |\alpha| F_K(x). \end{aligned} \quad (6)$$

(5) If $\alpha = 0$, then we are done. If $\alpha \neq 0$ and $k_0 \in P_{\alpha K}^f(\alpha x)$, then $k_0 \in \alpha K$ and

$$\sum_{i=1}^n f(\alpha x_i - k_0) = \inf_{k \in K} \sum_{i=1}^n f(\alpha x_i - \alpha k). \quad (7)$$

This implies that

$$\sum_{i=1}^n f\left(x_i - \frac{1}{\alpha} k_0\right) = F_K(x), \quad (8)$$

which implies that $(1/\alpha)k_0 \in P_K^f(x)$.

(6) The proof follows immediately from part (5) above.

(7) Let $k_1, k_2 \in P_K^f(x)$. Since K is convex, then $\lambda k_1 - (1 - \lambda)k_2 \in K$. We must show that $\lambda k_1 - (1 - \lambda)k_2 \in P_K^f(x)$; that is,

$$\sum_{i=1}^n f(x_i - (\lambda k_1 - (1 - \lambda)k_2)) = \inf_{k \in K} \sum_{i=1}^n f(x_i - k). \quad (9)$$

So,

$$\begin{aligned}
 & \sum_{i=1}^n f(x_i - (\lambda k_1 - (1-\lambda)k_2)) \\
 &= \sum_{i=1}^n f(\lambda(x_i - k_1) + (1-\lambda)(x_i - k_2)) \\
 &= \lambda \sum_{i=1}^n f(x_i - k_1) + (1-\lambda) \sum_{i=1}^n f(x_i - k_2) \quad (10) \\
 &= \lambda F_K(x) + (1-\lambda) F_K(x) \\
 &= F_K(x) = \sum_{i=1}^n f(x_i - k),
 \end{aligned}$$

which implies that $P_K^f(x)$ is convex. $\square$

Example 8. Let $X = \mathbb{R}^2$ and $K = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 4\}$, and let $f(x, y) = x^2 - y^2$. If $z = ((0, 0), (0, 1)) \in X^2$, then one can show that $F_K(z) = f(0, 1/2) = -1/4$.

Theorem 9. Let f be an absolutely homogeneous real-valued function on X and M a vector subspace of X . Then,

- (1) $F_M(\alpha x) = |\alpha| F_M(x)$, for all $x = (x_1, x_2, \dots, x_n) \in X^n$, $\alpha \in \mathbb{R} - \{0\}$;
- (2) $P_M^f(\alpha x) = \alpha P_M^f(x)$, for all $x = (x_1, x_2, \dots, x_n) \in X^n$, $\alpha \in \mathbb{R} - \{0\}$.

Proof. (1) Let $x = (x_1, x_2, \dots, x_n)$. Then,

$$\begin{aligned}
 F_M(\alpha x) &= \inf_{m \in M} \sum_{i=1}^n f(\alpha x_i - m) \\
 &= |\alpha| \inf_{m' \in M} \sum_{i=1}^n f(x_i - m') = |\alpha| F_M(x). \quad (11)
 \end{aligned}$$

(2) Let $m_0 \in P_M^f(\alpha x)$. Then,

$$\sum_{i=1}^n f(\alpha x_i - m_0) = \inf_{m \in M} \sum_{i=1}^n f(\alpha x_i - m) \quad (12)$$

if and only if

$$\sum_{i=1}^n f\left(x_i - \frac{1}{\alpha} m_0\right) = \inf_{m' \in M} \sum_{i=1}^n f(x_i - m') = F_M(x), \quad (13)$$

for all $\alpha \in \mathbb{R} - \{0\}$, which implies that $(1/\alpha)m_0 \in P_M^f(x)$, so, $m_0 \in \alpha P_M^f(x)$. $\square$

Theorem 10. Let f be a positive real-valued function on X such that $x = 0$ if and only if $f(x) = 0$. Then, if K is f -simultaneously proximal, then K is f -closed.

Proof. Since f is a positive function, then $\sum_{i=1}^n f(x_i) \geq 0$ for all $x = (x_1, x_2, \dots, x_n) \in X^n$. Let $\{k_m\}$ be a sequence of K and

$x = (x_1, x_2, \dots, x_n) \in X^n$, such that $\sum_{i=1}^n f(x_i - k_m) \rightarrow 0$. This implies that

$$F_K(x) = \inf_{k \in K} \sum_{i=1}^n f(x_i - k) \leq \sum_{i=1}^n f(x_i - k_m) \rightarrow 0. \quad (14)$$

Since K is f -simultaneously proximal, then there exists $k_0 \in K$ such that

$$F_K(x) = \sum_{i=1}^n f(x_i - k_0) = 0. \quad (15)$$

Hence, for all $i = 1, 2, \dots, n$, $f(x_i - k_0) = 0$. Using the assumption it follows that $x_i - k_0 = 0$, and, hence, $x_i = k_0 \in K$. Consequently, $x \in K^n$ and K is f -closed. $\square$

Theorem 11. Let X be a topological vector space and K a vector subspace of X . Suppose that f is continuous function and K is w -compact; then, K is f -simultaneously proximal.

Proof. Let $x = (x_1, x_2, \dots, x_n) \in X^n$. Since

$$F_K(x) = \inf_{k \in K} \sum_{i=1}^n f(x_i - k), \quad \text{where } k \in K, \quad (16)$$

then, for any constant α , there exists $\{k_\alpha\}$ such that

$$\sum_{i=1}^n f(x_i - k_\alpha) \leq \sum_{i=1}^n f(x_i - k) + \frac{1}{\alpha}. \quad (17)$$

But K is w -compact; then, there exists a subnet $\{k_{\alpha_\beta}\}$ such that $k_{\alpha_\beta} \rightarrow k_0$. Thus,

$$x_i - k_{\alpha_\beta} \rightarrow x_i - k_0, \quad \forall i = 1, 2, \dots, n. \quad (18)$$

Since f is continuous, then

$$\sum_{i=1}^n f(x_i - k_{\alpha_\beta}) \leq \sum_{i=1}^n f(x_i - k) + \frac{1}{\alpha}. \quad (19)$$

Also,

$$\begin{aligned}
 \sum_{i=1}^n f(x_i - k_0) &= \liminf \sum_{i=1}^n f(x_i - k_{\alpha_\beta}) \\
 &\leq \sum_{i=1}^n f(x_i - k). \quad (20)
 \end{aligned}$$

Hence, $k_0 \in P_K^f(x)$. $\square$

For a subset K of X , let us define $\widehat{K}_F$ to be such that

$$\widehat{K}_F = \left\{ x = (x_1, x_2, \dots, x_n) \in X^n : F_K(x) = \sum_{i=1}^n f(x_i) \right\}. \quad (21)$$

Example 12. Consider $X = (\mathbb{R}^2)^2$ and $K = \{((x_1, y_1), (x_2, y_2)) : x_i = y_i, \text{ for all } i = 1, 2\}$. Let $f(x, y) = x^2 + y^2$; then, one can see that

$$\widehat{K}_F = \{((x_1, -x_1), (x_2, -x_2))\}. \quad (22)$$

Using the previous definition of $\widehat{K}_F$, we prove the following theorem characterizing f -simultaneously proximal subspaces.

Theorem 13. *Let K be a vector subspace of X . Then, K is f -simultaneously proximal in X if and only if $X^n = D_K + \widehat{K}_F$, where $D_K = \{(k, k, \dots, k) : k \in K\}$.*

Proof. Suppose that $X^n = D_K + \widehat{K}_F$. Then, for $x = (x_1, x_2, \dots, x_n) \in X^n$, there exists $k_1 = (k_0, k_0, \dots, k_0) \in D_K$ and $y = (y_1, y_2, \dots, y_n) \in \widehat{K}_F$ such that $x = y + k_1$. Hence, $x - k_1 = y \in \widehat{K}_F$, and

$$F_K(y) = F_K(x - k_1) = \sum_{i=1}^n f(x_i - k_0), \quad (23)$$

and so

$$\begin{aligned} \sum_{i=1}^n f(x_i - k_0) &= \inf_{k \in K} \sum_{i=1}^n f(x_i - k_0 - k) \\ &= \inf_{k' \in K} \sum_{i=1}^n f(x_i - k') = F_K(x). \end{aligned} \quad (24)$$

So, K is f -simultaneously proximal.

Conversely, suppose that K is f -simultaneously proximal and $x = (x_1, x_2, \dots, x_n) \in X^n$. Then, there exists $k_0 \in K$ such that

$$\begin{aligned} \sum_{i=1}^n f(x_i - k_0) &= \inf_{k \in K} \sum_{i=1}^n f(x_i - k) \\ &= \inf_{k \in K} \sum_{i=1}^n f(x_i - (k' + k_0)), \end{aligned} \quad (25)$$

where $k = k' + k_0$. If $k_1 = (k_0, k_0, \dots, k_0) \in D_K$, then

$$\sum_{i=1}^n f(x_i - k_0) = F_K(x - k_1), \quad (26)$$

which implies that $x - k_1 = k_2 \in \widehat{K}_F$ and $X^n = D_K + \widehat{K}_F$. $\square$

Proposition 14. *Let X be a topological vector space and Kf -simultaneous proximal subset of X . Then,*

- (1) $k_0 \in P_K^f(x)$ if and only if $x - k_0 \in \widehat{K}_F$;
- (2) if f is symmetric (i.e., $f(-x) = f(x)$ for all $x \in X$), then $x \in \widehat{K}_F$ if and only if $-x \in \widehat{K}_F$;
- (3) if $x \perp_F K$, then $x \in \widehat{K}_F$, where $x = (x_1, x_2, \dots, x_n)$;
- (4) if $x \in \widehat{K}_F$ and $\alpha K = K$, then $x \perp_F K$, where $x = (x_1, x_2, \dots, x_n)$.

Proof. (1) Let $k_0 \in P_K^f(x)$ if and only if $\sum_{i=1}^n f(x_i - k_0) = \inf\{\sum_{i=1}^n f(x_i - k) : k \in K\}$.

Thus,

$$\begin{aligned} \sum_{i=1}^n f(x_i - k_0) &= \inf \left\{ \sum_{i=1}^n f(x_i - k_0 + k_0 - k) : k \in K \right\} \\ &= \inf \left\{ \sum_{i=1}^n f(x_i - k_0 - k') : k' \in K \right\}, \end{aligned} \quad (27)$$

which implies that $x - k_0 \in \widehat{K}_F$.

(2) Let $x = (x_1, x_2, \dots, x_n) \in \widehat{K}_F$. Since f is symmetric, then

$$\begin{aligned} \sum_{i=1}^n f(-x_i) &= \sum_{i=1}^n f(x_i) \\ &= \inf \left\{ \sum_{i=1}^n f(x_i - k) : k \in K \right\} \\ &= \inf \left\{ \sum_{i=1}^n f(-(-x_i + k)) : -k \in K \right\} \\ &= \inf \left\{ \sum_{i=1}^n f(-x_i + k) : -k \in K \right\}. \end{aligned} \quad (28)$$

Hence, $\sum_{i=1}^n f(-x_i) = \inf\{\sum_{i=1}^n f(-x_i + k) : -k \in K\}$, which implies that

$$-x = (-x_1, -x_2, \dots, -x_n) \in \widehat{K}_F. \quad (29)$$

(3) Let $x = (x_1, x_2, \dots, x_n)$. Since $x \perp_F K$, then

$$\begin{aligned} \sum_{i=1}^n f(x_i) &\leq \sum_{i=1}^n f(x_i + \alpha k) \quad \forall \alpha \in \mathbb{R}, k \in K. \\ &= \sum_{i=1}^n f(x_i - (-\alpha k)) \quad \forall \alpha \in \mathbb{R}, k \in K. \\ &= \sum_{i=1}^n f(x_i - k'), \quad k' \in K. \end{aligned} \quad (30)$$

So,

$$\sum_{i=1}^n f(x_i) \leq \sum_{i=1}^n f(x_i - k'), \quad k' \in K. \quad (31)$$

Hence, $x = (x_1, x_2, \dots, x_n) \in \widehat{K}_F$.

(4) Let $x \in \widehat{K}_F$ and $\alpha K = K$. Then,

$$\begin{aligned} \sum_{i=1}^n f(x_i) &= \inf_{k \in K} \sum_{i=1}^n f(x_i - k) \\ &= \inf_{\alpha k \in K} \sum_{i=1}^n f(x_i - \alpha k), \quad \text{since } \alpha K = K, \\ &= \inf \sum_{i=1}^n f(x_i + (-\alpha k)), \quad \forall k \in K. \end{aligned} \quad (32)$$

Thus,

$$\sum_{i=1}^n f(x_i) \leq \sum_{i=1}^n f(x_i + \alpha' k), \quad \forall \alpha' \in \mathbb{R}, \quad \forall k \in K. \quad (33)$$

Hence, $x \perp_F K$. $\square$

Theorem 15. Let K be a vector subspace of X . If $\pi(\widehat{K_F}) = X^n/D_K$, then K is f -simultaneously proximal, where π is the canonical map $x \rightarrow x + D_K$.

Proof. Let $\pi(\widehat{K_F}) = X^n/D_K$ and $x = (x_1, x_2, \dots, x_n) \in X^n$. Then, $x + D_K = y + D_K$ for some $y \in \widehat{K_F}$. Hence, $x - y = k_0$ for some $k_0 \in D_K$. Thus, $x = y + k_0 \in \widehat{K_F} + D_K$. Therefore, $\widehat{K_F} + D_K = X^n$. By Theorem 15, K is f -simultaneously proximal. $\square$

3. f -Simultaneous Approximation in Quotient Space

Definition 16. Let K and M be two vector subspaces of X such that M is closed and $M \subset K$. Suppose that f is a positive real-valued function defined on X . Then, a function $\tilde{f} : (X/M)^n \rightarrow \mathbb{R}$ can be defined as follows:

$$\begin{aligned} \tilde{f}(x_1 + M, x_2 + M, \dots, x_n + M) \\ = \inf \left\{ \sum_{i=1}^n f(x_i + y) : y \in M \right\} \end{aligned} \quad (34)$$

for each $(x_1, x_2, \dots, x_n) \in X^n$.

Theorem 17. Let K and M be two vector subspaces of X such that $M \subset K$. If k_0 is a point of f -best simultaneous approximation to $(x_1, x_2, \dots, x_n)$ in K , then $k_0 + M$ is an $\tilde{f}$ -best simultaneous approximation to $(x_1, x_2, \dots, x_n) + M$ in K/M .

Proof. Suppose that $k_0 + M$ is not $\tilde{f}$ -best simultaneous approximation to $(x_1 + M, x_2 + M, \dots, x_n + M)$ in K/M . Then,

$$\tilde{f}((x_i - k_0 + M)_{i=1}^n) \not\leq \tilde{f}((x_i - k + M)_{i=1}^n) \quad (35)$$

for at least $k \in K$, say $k_1 \in K$, such that

$$\tilde{f}((x_i - k_1 + M)_{i=1}^n) < \tilde{f}((x_i - k_0 + M)_{i=1}^n). \quad (36)$$

Since

$$\begin{aligned} \tilde{f}((x_i - k_0 + M)_{i=1}^n) &= \inf \left\{ \sum_{i=1}^n f(x_i - k_0 + y) : y \in M \right\} \\ &\leq \sum_{i=1}^n f(x_i - k_0), \end{aligned} \quad (37)$$

we have

$$\tilde{f}((x_i - k_1 + M)_{i=1}^n) < \sum_{i=1}^n f(x_i - k_0). \quad (38)$$

Thus, for some $m_0 \in M$, we have

$$\sum_{i=1}^n f(x_i - k_1 + m_0) < \sum_{i=1}^n f(x_i - k_0), \quad (39)$$

so,

$$\sum_{i=1}^n f(x_i - (k_1 - m_0)) < \sum_{i=1}^n f(x_i - k_0). \quad (40)$$

Since $M \subset K$ implies that $k_1 - m_0 \in K$, therefore, k_0 is not f -best simultaneous approximation to $(x_1, x_2, \dots, x_n)$ in K , which is a contradiction. $\square$

Corollary 18. Let K and M be two vector subspaces of X such that $M \subset K$. Then, if K is f -simultaneously proximal in X , then K/M is $\tilde{f}$ -simultaneously proximal in X/M .

Proof. If K is f -simultaneously proximal in X , then there exists at least $k_0 \in K$ such that k_0 is f -best simultaneous approximation to $(x_1, x_2, \dots, x_n)$ in K . Thus by Theorem 11, $k_0 + M$ is an $\tilde{f}$ -best simultaneous approximation to $(x_1, x_2, \dots, x_n) + M$ in K/M , so, K/M is $\tilde{f}$ -simultaneously proximal in X/M . $\square$

Theorem 19. Let K and M be two vector subspaces of X such that $M \subset K$. If M is f -simultaneously proximal in X and K/M is $\tilde{f}$ -simultaneously proximal in X/M , then K is f -simultaneously proximal in X .

Proof. Since K/M is $\tilde{f}$ -simultaneously proximal in X/M , then there exists $k_0 \in K$ such that $k_0 + M$ is $\tilde{f}$ -best simultaneous approximation to $(x_1, x_2, \dots, x_n) + M$ from K/M , so,

$$\begin{aligned} \tilde{f}((x_i - k_0 + M)_{i=1}^n) &\leq \tilde{f}((x_i - k + M)_{i=1}^n), \quad \forall k \in K, \\ &\Downarrow \end{aligned}$$

$$\begin{aligned} \tilde{f}((x_i - k_0 + M)_{i=1}^n) &= \inf_{m \in M} \sum_{i=1}^n f(x_i - k_0 + m) \\ &\leq \inf_{m \in M} \sum_{i=1}^n f(x_i - k + m), \end{aligned} \quad (41)$$

for all $k \in K$. Note that

$$\begin{aligned} \inf_{m \in M} \sum_{i=1}^n f(x_i - k_0 + m) \\ = F_M(x_1 - k_0, x_2 - k_0, \dots, x_n - k_0) \\ \leq F_M(x_1 - k, x_2 - k, \dots, x_n - k). \end{aligned} \quad (42)$$

Since M is f -simultaneously proximal in X , then there exists $m_0 \in M$ such that

$$\begin{aligned} F_M(x_1 - k_0, x_2 - k_0, \dots, x_n - k_0) \\ = \sum_{i=1}^n f(x_i - k_0 - m_0) \\ \leq \sum_{i=1}^n f(x_i - k - m), \end{aligned} \quad (43)$$

for all $m \in M$ and $k \in K$. So,

$$\sum_{i=1}^n f(x_i - (k_0 + m_0)) \leq \sum_{i=1}^n f(x_i - (k + m)), \quad (44)$$

for all $m \in M$ and $k \in K$. Hence,

$$\begin{aligned} \sum_{i=1}^n f(x_i - (k_0 + m_0)) \\ = \inf \left\{ \sum_{i=1}^n f(x_i - (k + m)) : m \in M, k \in K \right\}. \end{aligned} \quad (45)$$

So, $k_0 + m_0$ is an f -best simultaneous approximation to $(x_1, x_2, \dots, x_n)$ from K and K is f -simultaneously proximal in X . $\square$

Theorem 20. Let K and M be two vector subspaces of X such that $M \subset K$. If M is f -simultaneously proximal in X and K is f -simultaneously Chebyshev in X , then K/M is $\tilde{f}$ -simultaneously Chebyshev in X/M .

Proof. Suppose not, then there exists $(x_1, x_2, \dots, x_n) + M \in X/M$, and $k_1 + M, k_2 + M \in P_{K/M}^{\tilde{f}}((x_1, x_2, \dots, x_n) + M)$ such that $k_1 + M \neq k_2 + M$. Thus, $k_1 - k_2 \notin M$. Since M is f -simultaneously proximal in X , then

$$\begin{aligned} P_M^f(x_1 - k_1, x_2 - k_1, \dots, x_n - k_1) &\neq \phi, \\ P_M^f(x_1 - k_2, x_2 - k_2, \dots, x_n - k_2) &\neq \phi. \end{aligned} \quad (46)$$

Let $m_1 \in P_M^f(x_1 - k_1, x_2 - k_1, \dots, x_n - k_1)$ and $m_2 \in P_M^f(x_1 - k_2, x_2 - k_2, \dots, x_n - k_2)$. By Theorem 13, $k_1 + m_1$ and $k_2 + m_2$ are f -best simultaneous approximation to $(x_1, x_2, \dots, x_n)$ from K . Since K is f -simultaneously Chebyshev in X , then $k_1 + m_1 = k_2 + m_2$, and, hence, $k_1 - k_2 = m_1 - m_2 \in M$, which is a contradiction. $\square$

Theorem 21. Let K and M be two vector subspaces of a topological vector space X . If M is f -simultaneously Chebyshev in X , then the following assertions are equivalent:

- (i) K/M is $\tilde{f}$ -simultaneously Chebyshev in X/M ;
- (ii) $K + M$ is simultaneously Chebyshev in X .

Proof. (i $\Rightarrow$ ii) By hypothesis, $(K + M)/M = K/M$ is $\tilde{f}$ -simultaneous Chebyshev. Assume that $K + M$ is not

f -simultaneous Chebyshev in X . Then, there exists $x = (x_1, \dots, x_n) \in X^n$ which has two distinct f -best simultaneous approximations, say ℓ_0 and $\ell_1 \in K + M$. Thus, we have ℓ_0 and $\ell_1 \in P_{K+M}^f(x)$. Since $M \subseteq K + M$, we have that $\ell_0 + M$ and $\ell_1 + M \in P_{(K+M)/M}^f(x + M) = P_{K/M}^f(x + M)$. By hypothesis, K/M is $\tilde{f}$ -simultaneous Chebyshev, and so $\ell_0 + M = \ell_1 + M$. Then, there exists $m_0 \in M \setminus \{0\}$ such that $\ell_1 = \ell_0 + m_0$. Thus, we conclude that

$$\begin{aligned} \sum_{i=1}^n f((x_i - \ell_0) - m_0) \\ = \sum_{i=1}^n f(x_i - \ell_1) \\ = \inf_{m \in M} \left\{ \sum_{i=1}^n f(x_i - (\ell_0 + m)) \right\} \\ \leq \left\{ \sum_{i=1}^n f((x_i - \ell_0) - m) \right\}, \quad \forall m \in M \\ = F_M(x - \ell_0). \end{aligned} \quad (47)$$

So, m_0 and 0 are f -best simultaneous approximations to $x - \ell_0$ from M . Hence, M is not f -simultaneously Chebyshev. This is a contradiction.

(ii $\Rightarrow$ i) Assume that (i) does not hold. Then, there exists $x + M \in K/M$ which has two distinct $\tilde{f}$ -best simultaneous approximations, say $k + M$ and $k' + M \in K/M$; thus, $k - k' \notin M$. Since M is f -simultaneously proximal, so there exist f -best simultaneous approximations m and m' to $x - k$ and $x - k'$ from M , respectively. Therefore, we have $m \in P_M^f(x - k)$ and $m' \in P_M^f(x - k')$. Since $M \subseteq K + M$, $k + M$ and $k' + M \in P_{K/M}^f(x + M) = P_{(K+M)/M}^f(x + M)$, so $k + m$ and $k' + m' \in P_{K+M}^f(x)$. But $K + M$ is f -simultaneously Chebyshev. Thus we get $k + m = k' + m'$, and therefore $k - k' \in M$. This is a contradiction. $\square$

Definition 22. A subset K of X is called f -quasisimultaneously Chebyshev if $P_K^f(x)$ is non-empty and f -compact set in X , for all $x = (x_1, x_2, \dots, x_n) \in X^n$.

Theorem 23. Let f be a positive function, M an f -simultaneously proximal vector subspace of X , and K f -quasisimultaneously Chebyshev of X such that $M \subset K$. Then, K/M is $\tilde{f}$ -quasi-simultaneously Chebyshev in X^n/M .

Proof. Since K is f -simultaneously proximal in X , then by Corollary 12, K/M is $\tilde{f}$ -simultaneously proximal in X/M . Let $x = (x_1, x_2, \dots, x_n) \in X^n$ and $(k_n + M)$ a sequence in $P_{K/M}^{\tilde{f}}(x + M)$. For every n , there exists $m_n \in M$ such that $k_n + m_n = k'_n \in P_K^f(x)$. But since M is a vector subspace, we have

$$k'_n + M = k_n + m_n + M = k_n + M. \quad (48)$$

Since K is f -quasi-simultaneously Chebyshev of X , the sequence $\{k_n\}$ has a subsequence $\{k_{n_i}\}$ which is f -convergent to $k_0 \in P_K^f(x)$, meaning that

$$f(k_{n_i} - k_0) \longrightarrow 0. \quad (49)$$

But

$$\tilde{f}(k_{n_i} - k_0 + M) \leq f(k_{n_i} - k_0) \longrightarrow 0. \quad (50)$$

Hence,

$$\tilde{f}(k_{n_i} - k_0 + M) \longrightarrow 0. \quad (51)$$

Consequently, $P_{K/M}^{\tilde{f}}(x + M)$ is $\tilde{f}$ -compact and K/M is $\tilde{f}$ -quasi-simultaneously Chebyshev. This completes the proof. $\square$

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