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Research Article

Perturbation Analysis of the Nonlinear Matrix Equation $X - \sum_{i=1}^{m} A_i^* X^{p_i} A_i = Q$

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Consider the nonlinear matrix equation $X - \sum_{i=1}^{m} A_i^* X^{p_i} A_i = Q$ with $0 < p_i < 1$. Two perturbation bounds and the backward error of an approximate solution to the equation are derived. Explicit expressions of the condition number for the equation are obtained. The theoretical results are illustrated by numerical examples.

1. Introduction

In this paper we consider the Hermitian positive definite solution of the nonlinear matrix equation

$$X - \sum_{i=1}^{m} A_i^* X^{p_i} A_i = Q, \tag{1}$$

where $0 < p_i < 1$ $(i = 1, 2, \ldots, m)$, $A_1, A_2, \ldots, A_m$ are $n \times n$ complex matrices, m is a positive integer, and Q is a positive definite matrix. Here, A_i^* denotes the conjugate transpose of the matrix A_i.

When $m > 1$, (1) is recognized as playing an important role in solving a system of linear equations. For example, in many physical calculations, one must solve the system of linear equation

$$Mx = f, \tag{2}$$

where

$$M = \begin{pmatrix} I & 0 & \cdots & 0 & A_1 \\ 0 & I & \cdots & 0 & A_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & A_m \\ A_1^* & A_2^* & \cdots & A_m^* & -Q \end{pmatrix} \tag{3}$$

arises in a finite difference approximation to an elliptic partial differential equation (for more information, refer to [1]). We can rewrite M as $M = \widetilde{M} + D$, where

$$\widetilde{M} = \begin{pmatrix} X^{-p_1} & 0 & \cdots & 0 & A_1 \\ 0 & X^{-p_2} & \cdots & 0 & A_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & X^{-p_m} & A_m \\ A_1^* & A_2^* & \cdots & A_m^* & -Q \end{pmatrix},$$
$$D = \begin{pmatrix} I - X^{-p_1} & 0 & \cdots & 0 & 0 \\ 0 & I - X^{-p_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I - X^{-p_m} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{4}$$

$\widetilde{M}$ can be factored as

$$\widetilde{M} = \begin{pmatrix} -I & 0 & \cdots & 0 & 0 \\ 0 & -I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -I & 0 \\ -A_1^* X^{p_1} & -A_2^* X^{p_2} & \cdots & -A_m^* X^{p_m} & -I \end{pmatrix}$$

$$\times \begin{pmatrix} -X^{-p_1} & 0 & \cdots & 0 & -A_1 \\ 0 & -X^{-p_2} & \cdots & 0 & -A_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -X^{-p_m} & -A_m \\ 0 & 0 & \cdots & 0 & X \end{pmatrix} \tag{5}$$

if and only if X is a solution of equation $X - \sum_{i=1}^{m} A_i^* X^{p_i} A_i = Q$. When $m = 1$, this type of nonlinear matrix equations arises in ladder networks, dynamic programming, control theory, stochastic filtering, statistics, and so forth [2–7].

For the similar equations $X \pm A^* X^{-p} A = Q$, $X^s \pm A^* X^{-t} A = Q$, $X + \sum_{i=1}^{m} A_i^* X^{-1} A_i = I$, and $X = Q - A^* X^{-1} A + B^* X^{-1} B$, there were many contributions in the literature to the theory, numerical solutions, and perturbation analysis [8–32]. Jia and Gao [33] derived two perturbation estimates for the solution of the equation $X - A^* X^q A = Q$ with $0 < q < 1$. In addition, Duan et al. [34] proved that the equation $X - \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i = Q$ $(0 < |\delta_i| < 1)$ has a unique positive definite solution. They also proposed an iterative method for obtaining the unique positive definite solution. However, to our best knowledge, there has been no perturbation analysis for (1) with $m > 1$ in the known literature.

As a continuation of the previous results, the rest of the paper is organized as follows. In Section 2, some preliminary lemmas are given. In Section 3, two perturbation bounds for the unique solution to (1) are derived. Furthermore, in Section 4, we obtain the backward error of an approximate solution to (1). In Section 5, we also discuss the condition number of the unique solution to (1). Finally, several numerical examples are presented in Section 6.

We denote by $\mathscr{C}^{n\times n}$ the set of $n \times n$ complex matrices, by $\mathscr{H}^{n\times n}$ the set of $n \times n$ Hermitian matrices, by I the identity matrix, by $\mathbf{i}$ the imaginary unit, by $\|\cdot\|$ the spectral norm, by $\|\cdot\|_F$ the Frobenius norm, and by $\lambda_{\max}(M)$ and $\lambda_{\min}(M)$ the maximal and minimal eigenvalues of M, respectively. For $A = (a_1, \ldots, a_n) = (a_{ij}) \in \mathscr{C}^{n\times n}$ and a matrix B, $A \otimes B = (a_{ij}B)$ is a Kronecker product, and $\operatorname{vec} A$ is a vector defined by $\operatorname{vec} A = (a_1^T, \ldots, a_n^T)^T$. For $X, Y \in \mathscr{H}^{n\times n}$, we write $X \geq Y$ ($X > Y$, resp.) if $X - Y$ is Hermitian positive semidefinite (definite, resp.).

2. Preliminaries

Lemma 1 (see [35]). *If $A \geq B > 0$ and $0 \leq \gamma \leq 1$, then $A^\gamma \geq B^\gamma$.*

Lemma 2 (see [33]). *For any Hermitian positive definite matrix X and Hermitian matrix ΔX, one has*

(i) $X^q = (\sin q\pi/\pi) \int_0^\infty X^{1/2}(\lambda I + X)^{-1} X^{1/2} \lambda^{q-1} d\lambda$, $0 < q < 1$;

(ii) $X^q = (\sin q\pi/(1-q)\pi) \int_0^\infty X^{1/2}(\lambda I + X)^{-1} X (\lambda I + X)^{-1} X^{1/2} \lambda^{q-1} d\lambda$, $0 < q < 1$.

In addition, if $X + \Delta X \geq (1/\nu)X > 0$ and $0 < q < 1$, then

(iii) $\|X^{-1/2} A^* ((X + \Delta X)^q - X^q) A X^{-1/2}\| \leq q(\|X^{-1/2} \Delta X X^{-1/2}\| + \nu \|X^{-1/2} \Delta X X^{-1/2}\|^2) \|X^{q/2} A X^{-1/2}\|^2$.

Lemma 3 (see [34]). *The matrix equation $X - \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i = Q$ $(0 < |\delta_i| < 1)$ always has a unique positive definite solution X. The matrix sequence X_k:*

$$X_{s+m+1} = Q + \sum_{i=1}^{m} A_i^* X_{s+i}^{\delta_i} A_i, \quad s = 0, 1, 2, \ldots, \tag{6}$$

converges to the unique positive definite solution X for arbitrary initial positive definite matrices $X_1, X_2, \ldots, X_m$.

3. Perturbation Bounds

Here the perturbed equation

$$\widetilde{X} - \sum_{i=1}^{m} \widetilde{A}_i^* \widetilde{X}^{p_i} \widetilde{A}_i = \widetilde{Q}. \tag{7}$$

is considered, where $0 < p_i < 1$ and $\widetilde{A}_i$ and $\widetilde{Q}$ are small perturbations of A_i and Q in (1), respectively. We assume that X and $\widetilde{X}$ are the solutions of (1) and (7), respectively. Let $\Delta X = \widetilde{X} - X$, $\Delta Q = \widetilde{Q} - Q$ and $\Delta A_i = \widetilde{A}_i - A_i$.

By Lemma 3, we know that (1) always has a unique positive definite solution X; then in this section two perturbation bounds for the unique positive definite solution of (1) are developed. The relative perturbation bound in Theorem 5 does not depend on any knowledge of the actual solution X of (1). Furthermore, a sharper perturbation bound in Theorem 8 is derived.

To prove the next theorem, we first verify the following lemma.

Lemma 4. *If X is a solution of* (1), *then*

$$X \geq \left(\lambda_{\min}(Q) + \sum_{i=1}^{m} \lambda_{\min}(A_i^* A_i) \lambda_{\min}^{p_i}(Q) \right) I = \beta I. \tag{8}$$

Proof. By Lemma 3, (1) with $0 < p_i < 1$ always has a unique positive definite solution X. Then $X > 0$, and it follows that $X^{p_i} > 0$. Therefore $X \geq Q$. By Lemma 1 and (1), we have $X \geq Q + \sum_{i=1}^{m} A_i^* Q^{p_i} A_i \geq (\lambda_{\min}(Q) + \sum_{i=1}^{m} \lambda_{\min}(A_i^* A_i) \lambda_{\min}^{p_i}(Q)) I = \beta I$. □

The next theorem generalizes [33, Theorem 4] with $m = 1$, $\|\Delta Q\| = 0$ to arbitrary integer $m \geq 1$, $\|\Delta Q\| > 0$.

Theorem 5. *Let $b = \beta + \|\Delta Q\| - \sum_{i=1}^{m} p_i \beta^{p_i} \|A_i\|^2$, $s = \sum_{i=1}^{m} \beta^{p_i} \|\Delta A_i\| (2\|A_i\| + \|\Delta A_i\|)$. If*

$$\begin{gathered} 0 < b < 2(\beta - s), \\ b^2 - 4(\beta - s)(s + \|\Delta Q\|) \geq 0, \end{gathered} \tag{9}$$

then

$$\frac{\|\widetilde{X} - X\|}{\|X\|} \leq \varrho \sum_{i=1}^{m} \|\Delta A_i\| + \omega \|\Delta Q\| \triangleq \xi_1, \tag{10}$$

where

$$\varrho = \frac{2s}{\sum_{i=1}^{m} \|\Delta A_i\| \left(b + \sqrt{b^2 - 4(\beta - s)(s + \|\Delta Q\|)}\right)},$$
$$\omega = \frac{2}{b + \sqrt{b^2 - 4(\beta - s)(s + \|\Delta Q\|)}}. \tag{11}$$

Proof. Let

$$\Omega = \left\{ \Delta X \in \mathscr{H}^{n\times n} : \left\|X^{-1/2}\Delta X X^{-1/2}\right\| \le \varrho \sum_{i=1}^{m} \|\Delta A_i\| + \omega \|\Delta Q\| \right\}. \tag{12}$$

Obviously, Ω is a nonempty bounded convex closed set. Let

$$f(\Delta X) = \sum_{i=1}^{m} \left(\widetilde{A}_i^* (X + \Delta X)^{p_i} \widetilde{A}_i - A_i^* X^{p_i} A_i\right) + \Delta Q, \quad \Delta X \in \Omega. \tag{13}$$

Evidently, $f : \Omega \mapsto \mathscr{H}^{n\times n}$ is continuous. We will prove that $f(\Omega) \subseteq \Omega$.

For every $\Delta X \in \Omega$, it follows that $\|X^{-1/2}\Delta X X^{-1/2}\| \le \varrho \sum_{i=1}^{m} \|\Delta A_i\| + \omega\|\Delta Q\|$. Thus

$$\begin{aligned}
&\left(\varrho \sum_{i=1}^{m} \|\Delta A_i\| + \omega \|\Delta Q\|\right) I \\
&\quad \ge X^{-1/2}\Delta X X^{-1/2} \\
&\quad \ge \left(-\varrho \sum_{i=1}^{m} \|\Delta A_i\| - \omega \|\Delta Q\|\right) I, \\
&\left(1 + \varrho \sum_{i=1}^{m} \|\Delta A_i\| + \omega \|\Delta Q\|\right) X \\
&\quad \ge X + \Delta X \\
&\quad \ge \left(1 - \varrho \sum_{i=1}^{m} \|\Delta A_i\| - \omega \|\Delta Q\|\right) X.
\end{aligned} \tag{14}$$

According to (9), we have

$$\begin{aligned}
\varrho \sum_{i=1}^{m} \|\Delta A_i\| + \omega \|\Delta Q\| &= \frac{2(\|\Delta Q\| + s)}{b + \sqrt{b^2 - 4(\beta - s)(s + \|\Delta Q\|)}} \\
&\le \frac{2(\|\Delta Q\| + s)}{b} \le \frac{b}{2(\beta - s)} < 1.
\end{aligned} \tag{15}$$

Therefore

$$\left(1 - \varrho \sum_{i=1}^{m} \|\Delta A_i\| - \omega \|\Delta Q\|\right) X > 0. \tag{16}$$

From Lemmas 2 and 4, it follows that

$$\begin{aligned}
&\left\|X^{-1/2}\left[\sum_{i=1}^{m} A_i^* \left((X + \Delta X)^{p_i} - X^{p_i}\right) A_i\right] X^{-1/2}\right\| \\
&\quad \le \left(\left\|X^{-1/2}\Delta X X^{-1/2}\right\| + \frac{\left\|X^{-1/2}\Delta X X^{-1/2}\right\|^2}{1 - \varrho \sum_{i=1}^{m} \|\Delta A_i\| - \omega \|\Delta Q\|}\right) \\
&\qquad \times \left(\sum_{i=1}^{m} p_i \left\|X^{p_i/2} A_i X^{-1/2}\right\|^2\right) \\
&\quad \le \left(\left\|X^{-1/2}\Delta X X^{-1/2}\right\| + \frac{\left\|X^{-1/2}\Delta X X^{-1/2}\right\|^2}{1 - \varrho \sum_{i=1}^{m} \|\Delta A_i\| - \omega \|\Delta Q\|}\right) \\
&\qquad \times \left(\sum_{i=1}^{m} \frac{p_i}{\beta^{1-p_i}} \|A_i\|^2\right).
\end{aligned} \tag{17}$$

Therefore

$$\begin{aligned}
&\left\|X^{-1/2} f(\Delta X) X^{-1/2}\right\| \\
&\quad = \left\|X^{-1/2}\left[\sum_{i=1}^{m} \widetilde{A}_i^* (X + \Delta X)^{p_i} \widetilde{A}_i - A_i^* X^{p_i} A_i\right] \times X^{-1/2} + X^{-1/2}\Delta Q X^{-1/2}\right\| \\
&\quad \le \left\|\sum_{i=1}^{m} X^{-1/2} A_i^* \left((X + \Delta X)^{p_i} - X^{p_i}\right) A_i X^{-1/2}\right\| \\
&\qquad + \left\|X^{-1/2}\Delta Q X^{-1/2}\right\| \\
&\qquad + \left\|\sum_{i=1}^{m} X^{-1/2}\left[\Delta A_i^* (X + \Delta X)^{p_i} (A_i + \Delta A_i) + A_i^* (X + \Delta X)^{p_i} \Delta A_i\right] X^{-1/2}\right\| \\
&\quad \le \left(\left\|X^{-1/2}\Delta X X^{-1/2}\right\| + \frac{\left\|X^{-1/2}\Delta X X^{-1/2}\right\|^2}{1 - \varrho \sum_{i=1}^{m} \|\Delta A_i\| - \omega \|\Delta Q\|}\right) \\
&\qquad \times \left(\sum_{i=1}^{m} \frac{p_i}{\beta^{1-p_i}} \|A_i\|^2\right) \\
&\qquad + \sum_{i=1}^{m} \frac{\|\Delta A_i\| (2\|A_i\| + \|\Delta A_i\|)}{\beta^{1-p_i}}
\end{aligned}$$

$$\begin{aligned}
&\times\left(1+\varrho\sum_{i=1}^{m}\|\Delta A_i\|+\omega\|\Delta Q\|\right)+\frac{\|\Delta Q\|}{\beta}\\
&\le\left(\xi_1+\frac{\xi_1^2}{1-\xi_1}\right)\left(\sum_{i=1}^{m}\frac{p_i}{\beta^{1-p_i}}\|A_i\|^2\right)\\
&\quad+\frac{s}{\beta}(1+\xi_1)+\frac{\|\Delta Q\|}{\beta}\\
&=\xi_1.
\end{aligned}\tag{18}$$

That is, $f(\Omega)\subseteq\Omega$. By Brouwer's fixed point theorem, there exists a $\Delta X\in\Omega$ such that $f(\Delta X)=\Delta X$. Moreover, by Lemma 3, we know that X and $\widetilde{X}$ are the unique solutions to (1) and (7), respectively. Then

$$\begin{aligned}
\frac{\|\widetilde{X}-X\|}{\|X\|}&=\frac{\|\Delta X\|}{\|X\|}=\frac{\left\|X^{1/2}\left(X^{-1/2}\Delta XX^{-1/2}\right)X^{1/2}\right\|}{\|X\|}\\
&\le\left\|X^{-1/2}\Delta XX^{-1/2}\right\|\le\varrho\sum_{i=1}^{m}\|\Delta A_i\|+\omega\|\Delta Q\|.
\end{aligned}\tag{19}$$

□

Remark 6. According to

$$\begin{aligned}
&\varrho\sum_{i=1}^{m}\|\Delta A_i\|+\omega\|\Delta Q\|\\
&\quad=\frac{2\left(\sum_{i=1}^{m}\|\Delta A_i\|\left(2\|A_i\|+\|\Delta A_i\|\right)+\|\Delta Q\|\right)}{b+\sqrt{b^2-4(\beta-s)(s+\|\Delta Q\|)}},
\end{aligned}\tag{20}$$

we get $\varrho\sum_{i=1}^{m}\|\Delta A_i\|+\omega\|\Delta Q\|\to 0$ for $\|\Delta Q\|\to 0$ and $\|\Delta A_i\|\to 0$ $(i=1,2,\ldots,m)$. Therefore (1) is well posed.

Next, a sharper perturbation estimate is derived. Subtracting (1) from (7), we have

$$\begin{aligned}
&\Delta X+\sum_{i=1}^{m}\frac{\sin p_i\pi}{\pi}\int_0^\infty\left[(\lambda I+X)^{-1}X^{1/2}A_i\right]^*\\
&\qquad\times\Delta X\left[(\lambda I+X)^{-1}X^{1/2}A_i\right]\lambda^{p_i-1}d\lambda\\
&\quad=E+h(\Delta X),
\end{aligned}\tag{21}$$

where

$$B_i=X^{p_i}A_i,\quad i=1,2,\ldots,m,$$

$$E=\sum_{i=1}^{m}\left(B_i^*\Delta A_i+\Delta A_i^*B_i\right)+\sum_{i=1}^{m}\Delta A_i^*X^{p_i}\Delta A_i+\Delta Q,$$

$$\begin{aligned}
Z_i(\Delta X)&=\frac{\sin p_i\pi}{\pi}\int_0^\infty X^{1/2}(\lambda I+X)^{-1}\Delta X(\lambda I+X+\Delta X)^{-1}\\
&\qquad\times\Delta X(\lambda I+X)^{-1}X^{1/2}\lambda^{p_i-1}d\lambda,\\
V_i(\Delta X)&=\frac{\sin p_i\pi}{\pi}\int_0^\infty X^{1/2}(\lambda I+X)^{-1}\Delta X\\
&\qquad\times(\lambda I+X+\Delta X)^{-1}X^{1/2}\lambda^{p_i-1}d\lambda,\\
h(\Delta X)&=\sum_{i=1}^{m}\left[A_i^*Z_i(\Delta X)A_i-\widetilde{A}_i^*V_i(\Delta X)\Delta A_i\right.\\
&\qquad\left.-\Delta A_i^*V_i(\Delta X)A_i\right].
\end{aligned}\tag{22}$$

Lemma 7. *If $\sum_{i=1}^{m}(\|A_i\|^2/\beta^{1-p_i})<1$, then the linear operator $\mathbf{L}:\mathscr{H}^{n\times n}\to\mathscr{H}^{n\times n}$ defined by*

$$\begin{aligned}
\mathbf{L}W&=W+\sum_{i=1}^{m}\frac{\sin p_i\pi}{\pi}\\
&\quad\times\int_0^\infty\left[(\lambda I+X)^{-1}X^{1/2}A_i\right]^*\\
&\qquad\times W\left[(\lambda I+X)^{-1}X^{1/2}A_i\right]\\
&\qquad\times\lambda^{p_i-1}d\lambda,\quad W\in\mathscr{H}^{n\times n}
\end{aligned}\tag{23}$$

is invertible.

Proof. It suffices to show that the following equation:

$$\mathbf{L}W=V\tag{24}$$

has a unique solution for every $V\in\mathscr{H}^{n\times n}$. Define the operator $\mathbf{M}:\mathscr{H}^{n\times n}\to\mathscr{H}^{n\times n}$ by

$$\begin{aligned}
\mathbf{M}Z&=\sum_{i=1}^{m}\frac{\sin p_i\pi}{\pi}\\
&\quad\times\int_0^\infty X^{-1/2}A_i^*X^{1/2}\\
&\qquad\times(\lambda I+X)^{-1}X^{1/2}ZX^{1/2}\\
&\qquad\times(\lambda I+X)^{-1}X^{1/2}A_iX^{-1/2}\\
&\qquad\times\lambda^{p_i-1}d\lambda,\quad Z\in\mathscr{H}^{n\times n}.
\end{aligned}\tag{25}$$

Let $Y=X^{-1/2}WX^{-1/2}$. Thus (21) is equivalent to

$$Y+\mathbf{M}Y=X^{-1/2}VX^{-1/2}.\tag{26}$$

According to Lemma 2, we have

$$\begin{aligned}\|\mathbf{M}Y\| &\le \sum_{i=1}^{m}\left\|\frac{\sin p_i\pi}{\pi}\right.\\ &\qquad\times\int_0^\infty X^{-1/2}A_i^*X^{1/2}\\ &\qquad\times(\lambda I+X)^{-1}X(\lambda I+X)^{-1}\\ &\qquad\left.\times X^{1/2}A_i\,X^{-1/2}\lambda^{p_i-1}d\lambda\right\|\|Y\|\\ &\le \sum_{i=1}^{m}(1-p_i)\left\|X^{p_i/2}A_iX^{-1/2}\right\|^2\|Y\|\\ &\le \sum_{i=1}^{m}\frac{\|A_i\|^2}{\beta^{1-p_i}}\|Y\|<\|Y\|,\end{aligned}\tag{27}$$

which implies that $||\mathbf{M}||<1$ and $I+\mathbf{M}$ is invertible. Therefore, the operator $\mathbf{L}$ is invertible. □

Furthermore, we define operators $\mathbf{P}_i:\mathscr{C}^{n\times n}\to\mathscr{H}^{n\times n}$ by

$$\mathbf{P}_iZ_i=\mathbf{L}^{-1}\left(B_i^*Z_i+Z_i^*B_i\right),\quad Z_i\in\mathscr{C}^{n\times n},\ i=1,2,\ldots,m.\tag{28}$$

Thus, we can rewrite (21) as

$$\begin{aligned}\Delta X &= \mathbf{L}^{-1}\Delta Q\\ &\quad+\sum_{i=1}^{m}\mathbf{P}_i\Delta A_i+\mathbf{L}^{-1}\left(\sum_{i=1}^{m}\Delta A_i^*X^{p_i}\Delta A_i\right)\\ &\quad+\mathbf{L}^{-1}\left(h\left(\Delta X\right)\right).\end{aligned}\tag{29}$$

Define

$$\begin{aligned}\left\|\mathbf{L}^{-1}\right\| &= \max_{\substack{W\in\mathscr{H}^{n\times n}\\ \|W\|=1}}\left\|\mathbf{L}^{-1}W\right\|,\\ \left\|\mathbf{P}_i\right\| &= \max_{\substack{Z\in\mathscr{C}^{n\times n}\\ \|Z\|=1}}\left\|\mathbf{P}_iZ\right\|,\quad i=1,2,\ldots,m.\end{aligned}\tag{30}$$

Now we denote that

$$\begin{gathered}l=\left\|\mathbf{L}^{-1}\right\|^{-1},\qquad \zeta=\left\|X^{-1}\right\|,\\ \xi_i=\left\|X^{p_i}\right\|,\qquad n_i=\left\|\mathbf{P}_i\right\|,\\ \theta=\frac{\zeta^2}{l}\sum_{i=1}^{m}\xi_i\|A_i\|^2,\\ i=1,2,\ldots,m,\\ \epsilon=\frac{1}{l}\|\Delta Q\|+\sum_{i=1}^{m}\left(n_i\|\Delta A_i\|+\frac{\xi_i}{l}\|\Delta A_i\|^2\right),\\ \sigma=\frac{\zeta}{l}\sum_{i=1}^{m}\xi_i\left(2\|A_i\|+\|\Delta A_i\|\right)\|\Delta A_i\|.\end{gathered}\tag{31}$$

Theorem 8. *Suppose that X and $\widetilde{X}$ are the solutions of* (1) *and* (7), *respectively. If $\sum_{i=1}^{m}\|A_i\|^2/\beta^{1-p_i}<1$,*

$$\begin{gathered}\sigma<1,\\ \epsilon<\frac{(1-\sigma)^2}{\zeta+\sigma\zeta+2\theta+2\sqrt{((\zeta+\theta)(\sigma\zeta+\theta))}},\end{gathered}\tag{32}$$

then

$$\left\|\widetilde{X}-X\right\|\le\frac{2\epsilon}{1+\epsilon\zeta-\sigma+\sqrt{(1+\zeta\epsilon-\sigma)^2-4\epsilon(\zeta+\theta)}}\equiv\nu.\tag{33}$$

Proof. Let

$$\begin{aligned}f(\Delta X) &= \mathbf{L}^{-1}\Delta Q+\sum_{i=1}^{m}\mathbf{P}_i\Delta A_i\\ &\quad+\mathbf{L}^{-1}\left(\sum_{i=1}^{m}\Delta A_i^*X^{p_i}\Delta A_i\right)+\mathbf{L}^{-1}\left(h\left(\Delta X\right)\right).\end{aligned}\tag{34}$$

Obviously, $f:\mathscr{H}^{n\times n}\to\mathscr{H}^{n\times n}$ is continuous. The condition (32) ensures that the quadratic equation $(\zeta+\theta)x^2-(1+\zeta\epsilon-\sigma)x+\epsilon=0$ with respect to the variable x has two positive real roots. The smaller one is

$$\nu=\frac{2\epsilon}{1+\epsilon\zeta-\sigma+\sqrt{(1+\zeta\epsilon-\sigma)^2-4\epsilon(\zeta+\theta)}}.\tag{35}$$

Define $\Omega=\{\Delta X\in\mathscr{H}^{n\times n}:\|\Delta X\|\le\nu\}$. Then for any $\Delta X\in\Omega$, by (32), we have

$$\begin{aligned}\left\|X^{-1}\Delta X\right\| &\le \left\|X^{-1}\right\|\|\Delta X\|\le\zeta\nu\\ &\le\zeta\cdot\frac{2\epsilon}{1+\zeta\epsilon-\sigma}=1+\frac{\zeta\epsilon+\sigma-1}{1+\zeta\epsilon-\sigma}\\ &\le 1+\frac{-2(1-\sigma)(\zeta\sigma+\theta)}{(\zeta+\sigma\zeta+2\theta)(1+\zeta\epsilon-\sigma)}<1.\end{aligned}\tag{36}$$

It follows that $I-X^{-1}\Delta X$ is nonsingular and

$$\left\|I-X^{-1}\Delta X\right\|\le\frac{1}{1-\left\|X^{-1}\Delta X\right\|}\le\frac{1}{1-\zeta\|\Delta X\|}.\tag{37}$$

Using (22) and Lemma 2, we have

$$\begin{aligned}\|Z_i(\Delta X)\| &\le (1-p_i)\|\Delta X\|^2\|X^{-1}\|^2 \\ &\quad\times\left\|\left(I+X^{-1}\Delta X\right)^{-1}\right\|\|X^{p_i}\| \\ &\le \xi_i\zeta^2\frac{\|\Delta X\|^2}{1-\zeta\|\Delta X\|},\end{aligned} \tag{38}$$

$$\begin{aligned}\|V_i(\Delta X)\| &\le \|X^{p_i}\|\|\Delta X\|\|X^{-1}\|\left\|\left(I+X^{-1}\Delta X\right)^{-1}\right\| \\ &\le \xi_i\zeta\frac{\|\Delta X\|}{1-\zeta\|\Delta X\|},\end{aligned}$$

$$\begin{aligned}\|h(\Delta X)\| &\le \sum_{i=1}^{m}\Big(\|A_i\|^2\|Z_i(\Delta X)\| \\ &\quad+(2\|A_i\|+\|\Delta A_i\|) \\ &\quad\times\|\Delta A_i\|\|V_i(\Delta X)\|\Big) \\ &\le \sum_{i=1}^{m}\Bigg(\xi_i\zeta^2\|A_i\|^2\frac{\|\Delta X\|^2}{1-\zeta\|\Delta X\|} \\ &\quad+(2\|A_i\|+\|\Delta A_i\|) \\ &\quad\times\|\Delta A_i\|\xi_i\zeta\frac{\|\Delta X\|}{1-\zeta\|\Delta X\|}\Bigg).\end{aligned} \tag{39}$$

Noting (31) and (34), it follows that

$$\begin{aligned}\|f(\Delta X)\| &\le \frac{1}{l}\|\Delta Q\|+\sum_{i=1}^{m}\left(n_i\|\Delta A_i\|+\frac{\zeta_i}{l}\|\Delta A_i\|^2\right) \\ &\quad+\frac{1}{l}\|h(\Delta X)\| \\ &\le \epsilon+\frac{\sigma\|\Delta X\|}{1-\zeta\|\Delta X\|}+\frac{\theta\|\Delta X\|^2}{1-\zeta\|\Delta X\|} \\ &\le \epsilon+\frac{\sigma\nu}{1-\zeta\nu}+\frac{\theta\nu^2}{1-\zeta\nu}=\nu,\end{aligned} \tag{40}$$

for $\Delta X\in\Omega$. That is, $f(\Omega)\subseteq\Omega$. According to Schauder fixed point theorem, there exists $\Delta X_*\in\Omega$ such that $f(\Delta X_*)=X_*$. It follows that $X+\Delta X_*$ is a Hermitian solution of (7). By Lemma 3, we know that the solution of (7) is unique. Then $\|\Delta X_*\|=\|\widetilde{X}-X\|\le\nu$. □

Remark 9. From Theorem 8, we get the first order perturbation bound for the solution as follows:

$$\begin{aligned}\left\|\widetilde{X}-X\right\| &\le \frac{1}{l}\|\Delta Q\|+\sum_{i=1}^{m}n_i\|\Delta A_i\| \\ &\quad+O\left(\|(\Delta A_1,\Delta A_2,\ldots,\Delta A_m,\Delta Q)\|_F^2\right),\end{aligned} \tag{41}$$

as $(\Delta A_1,\Delta A_2,\ldots,\Delta A_m,\Delta Q)\to 0$.

Combining this with (29) gives

$$\begin{aligned}\Delta X &= \mathbf{L}^{-1}\Delta Q+\mathbf{L}^{-1}\sum_{i=1}^{m}\left(B_i^*\Delta A_i+\Delta A_i^*B_i\right) \\ &\quad+O\left(\|(\Delta A_1,\Delta A_2,\ldots,\Delta A_m,\Delta Q)\|_F^2\right).\end{aligned} \tag{42}$$

as $(\Delta A_1,\Delta A_2,\ldots,\Delta A_m,\Delta Q)\to 0$.

4. Backward Error

In this section, a backward error of an approximate solution for the unique solution to (1) is obtained.

Theorem 10. *Let $\widetilde{X}>0$ be an approximation to the solution X of (1). If $\Sigma=\sum_{i=1}^{m}p_i\|\widetilde{X}^{p_i/2}A_i\widetilde{X}^{-1/2}\|^2<1$ and the residual $R(\widetilde{X})\equiv Q+\sum_{i=1}^{m}A_i^*\widetilde{X}^{p_i}A_i-\widetilde{X}$ satisfies*

$$\begin{aligned}&\left\|R\left(\widetilde{X}\right)\right\|<\frac{\theta_1}{2\left\|\widetilde{X}^{-1}\right\|}\min\left\{1,\frac{\theta_1}{2}\right\}, \\ &\text{where } \theta_1\equiv 1+\left\|\widetilde{X}^{-1}\right\|\left\|R\left(\widetilde{X}\right)\right\|-\Sigma>0,\end{aligned} \tag{43}$$

then

$$\begin{aligned}&\left\|\widetilde{X}-X\right\|\le\mu\left\|R\left(\widetilde{X}\right)\right\|, \\ &\text{where } \mu=\frac{2\left\|\widetilde{X}\right\|\left\|\widetilde{X}^{-1}\right\|}{\theta_1+\sqrt{\theta_1^2-4\left\|\widetilde{X}^{-1}\right\|\left\|R\left(\widetilde{X}\right)\right\|}}.\end{aligned} \tag{44}$$

Proof. Let

$$\Psi=\left\{\Delta X\in\mathscr{H}^{n\times n}:\left\|\widetilde{X}^{-1/2}\Delta X\widetilde{X}^{-1/2}\right\|\le\theta_2\left\|R\left(\widetilde{X}\right)\right\|\right\}, \tag{45}$$

where $\theta_2=\mu/\|\widetilde{X}\|$. Obviously, Ψ is a nonempty bounded convex closed set. Let

$$g(\Delta X)=\sum_{i=1}^{m}A_i^*\left[\left(\widetilde{X}+\Delta X\right)^{p_i}-\widetilde{X}^{p_i}\right]A_i+R\left(\widetilde{X}\right). \tag{46}$$

Evidently $g:\Psi\mapsto\mathscr{H}^{n\times n}$ is continuous. We will prove that $g(\Psi)\subseteq\Psi$. For every $\Delta X\in\Psi$, we have

$$\left\|\widetilde{X}^{-1/2}\Delta X\widetilde{X}^{-1/2}\right\|\le\theta_2\left\|R\left(\widetilde{X}\right)\right\|. \tag{47}$$

Hence

$$\widetilde{X}^{-1/2}\Delta X\widetilde{X}^{-1/2}\ge-\theta_2\left\|R\left(\widetilde{X}\right)\right\|I. \tag{48}$$

That is,

$$\widetilde{X}+\Delta X\ge\left(1-\theta_2\left\|R\left(\widetilde{X}\right)\right\|\right)\widetilde{X}. \tag{49}$$

Using (43), one sees that

$$\begin{aligned}\theta_2\left\|R\left(\widetilde{X}\right)\right\| &= \frac{2\left\|\widetilde{X}^{-1}\right\|\left\|R\left(\widetilde{X}\right)\right\|}{\theta_1+\sqrt{\theta_1^2-4\left\|\widetilde{X}^{-1}\right\|\left\|R\left(\widetilde{X}\right)\right\|}} \\ &< \frac{2\left\|\widetilde{X}^{-1}\right\|\left\|R\left(\widetilde{X}\right)\right\|}{\theta_1}<1.\end{aligned} \tag{50}$$

Therefore $(1-\theta_2\|R(\widetilde{X})\|)\widetilde{X}>0$.

According to (17), we obtain

$$\begin{aligned}
&\left\|\widetilde{X}^{-1/2} g(\Delta X)\widetilde{X}^{-1/2}\right\| \\
&\quad\le \left(\left\|\widetilde{X}^{-1/2}\Delta X\widetilde{X}^{-1/2}\right\| + \frac{\left\|\widetilde{X}^{-1/2}\Delta X\widetilde{X}^{-1/2}\right\|^2}{1-\theta_2\left\|R\left(\widetilde{X}\right)\right\|}\right)\Sigma \\
&\qquad + \left\|\widetilde{X}^{-1/2}R\left(\widetilde{X}\right)\widetilde{X}^{-1/2}\right\| \\
&\quad\le \left(\theta_2\left\|R\left(\widetilde{X}\right)\right\| + \frac{\left(\theta_2\left\|R\left(\widetilde{X}\right)\right\|\right)^2}{1-\theta_2\left\|R\left(\widetilde{X}\right)\right\|}\right)\Sigma + \left\|\widetilde{X}^{-1}\right\|\left\|R\left(\widetilde{X}\right)\right\| \\
&\quad= \theta_2\left\|R\left(\widetilde{X}\right)\right\|,
\end{aligned} \tag{51}$$

for $\Delta X \in \Psi$. That is, $g(\Psi) \subseteq \Psi$. By Brouwer's fixed point theorem, there exists a $\Delta X \in \Psi$ such that $g(\Delta X) = \Delta X$. Hence $\widetilde{X} + \Delta X$ is a solution of (1). Moreover, by Lemma 3, we know that the solution X of (1) is unique. Then

$$\begin{aligned}
\left\|\widetilde{X} - X\right\| = \|\Delta X\| &\le \left\|\widetilde{X}\right\|\left\|\widetilde{X}^{-1/2}\Delta X\widetilde{X}^{-1/2}\right\| \\
&= \theta_2\left\|\widetilde{X}\right\|\left\|R\left(\widetilde{X}\right)\right\| = \mu\left\|R\left(\widetilde{X}\right)\right\|.
\end{aligned} \tag{52}$$

□

5. Condition Number

In this section, we apply the theory of condition number developed by Rice [36] to study condition numbers of the unique solution to (1).

5.1. The Complex Case. Suppose that X and $\widetilde{X}$ are the solutions of the matrix equations (1) and (7), respectively. Let $\Delta A_i = \widetilde{A}_i - A_i$, $\Delta Q = \widetilde{Q} - Q$ and $\Delta X = \widetilde{X} - X$. Using Theorem 8 and Remark 9, we have

$$\begin{aligned}
\Delta X = \widetilde{X} - X &= \mathbf{L}^{-1}\Delta Q \\
&\quad + \mathbf{L}^{-1}\sum_{i=1}^{m}\left(B_i^*\Delta A_i + \Delta A_i^* B_i\right) \\
&\quad + O\left(\left\|\left(\Delta A_1, \Delta A_2, \ldots, \Delta A_m, \Delta Q\right)\right\|_F^2\right),
\end{aligned} \tag{53}$$

as $(\Delta A_1, \Delta A_2, \ldots, \Delta A_m, \Delta Q) \to 0$.

By the theory of condition number developed by Rice [36], we define the condition number of the Hermitian positive definite solution X to (1) by

$$c(X) = \lim_{\delta\to 0}\sup_{\left\|\left(\Delta A_1/\eta_1, \Delta A_2/\eta_2, \ldots, \Delta A_m/\eta_m, \Delta Q/\rho\right)\right\|_F \le \delta} \frac{\|\Delta X\|_F}{\xi\delta}, \tag{54}$$

where ξ, ρ, and η_i, $i = 1, 2, \ldots, m$, are positive parameters. Taking $\xi = \eta_i = \rho = 1$ in (54) gives the absolute condition number $c_{\text{abs}}(X)$, and taking $\xi = \|X\|_F$, $\eta_i = \|A_i\|_F$, and $\rho = \|Q\|_F$ in (54) gives the relative condition number $c_{\text{rel}}(X)$.

Substituting (53) into (54), we get

$$\begin{aligned}
c(X) &= \frac{1}{\xi}\max_{\substack{\left(\Delta A_1/\eta_1, \Delta A_2/\eta_2, \ldots, \Delta A_m/\eta_m, \Delta Q/\rho\right)\neq 0 \\ \Delta A_i\in\mathscr{C}^{n\times n}, \Delta Q\in\mathscr{H}^{n\times n}}} \frac{\left\|\mathbf{L}^{-1}\left(\Delta Q + \sum_{i=1}^{m}\left(B_i^*\Delta A_i + \Delta A_i^* B_i\right)\right)\right\|_F}{\left\|\left(\Delta A_1/\eta_1, \Delta A_2/\eta_2, \ldots, \Delta A_m/\eta_m, \Delta Q/\rho\right)\right\|_F} \\
&= \frac{1}{\xi}\max_{\substack{\left(E_1, E_2, \ldots, E_m, H\right)\neq 0 \\ E_i\in\mathscr{C}^{n\times n}, H\in\mathscr{H}^{n\times n}}} \frac{\left\|\mathbf{L}^{-1}\left(\rho H + \sum_{i=1}^{m}\eta_i\left(B_i^* E_i + E_i^* B_i\right)\right)\right\|_F}{\left\|\left(E_1, E_2, \ldots, E_m, H\right)\right\|_F}.
\end{aligned} \tag{55}$$

Let L be the matrix representation of the linear operator $\mathbf{L}$. Then it is easy to see that

$$\begin{aligned}
L = I + \sum_{i=1}^{m}\frac{\sin p_i\pi}{\pi}\int_0^\infty &\left[(\lambda I + X)^{-1}X^{1/2}A_i\right]^T \\
&\otimes\left[(\lambda I + X)^{-1}X^{1/2}A_i\right]^* \\
&\times \lambda^{p_i - 1}d\lambda.
\end{aligned} \tag{56}$$

Let

$$\begin{aligned}
&L^{-1} = S + \mathbf{i}\Sigma, \\
&L^{-1}\left(I\otimes B_i^*\right) = L^{-1}\left(I\otimes\left(X^{p_i}A_i\right)^*\right) = U_{i1} + \mathbf{i}\Omega_{i1}, \\
&L^{-1}\left(B_i^T\otimes I\right)\Pi = L^{-1}\left(\left(X^{p_i}A_i\right)^T\otimes I\right)\Pi = U_{i2} + \mathbf{i}\Omega_{i2}, \\
&S_c = \begin{bmatrix} S & -\Sigma \\ \Sigma & S \end{bmatrix}, \\
&U_i = \begin{bmatrix} U_{i1} + U_{i2} & \Omega_{i2} - \Omega_{i1} \\ \Omega_{i1} + \Omega_{i2} & U_{i1} - U_{i2} \end{bmatrix}, \quad i = 1, 2, \ldots, m,
\end{aligned} \tag{57}$$

$\operatorname{vec} H = x + \mathbf{i}y$, $\operatorname{vec} E_i = a_i + \mathbf{i}b_i$, $g = (x^T, y^T, a_1^T, b_1^T, \ldots, a_m^T, b_m^T)^T$, $M = (E_1, E_2, \ldots, E_m, H)$, where $x, y, a_i, b_i \in \mathscr{R}^{n^2}$, $S, \Sigma, U_{i1}, U_{i2}, \Omega_{i1}, \Omega_{i2} \in \mathscr{R}^{n^2\times n^2}$, $i = 1, 2, \ldots, m$, and Π is the vec-permutation matrix, such that

$$\operatorname{vec} A^T = \Pi \operatorname{vec} A. \tag{58}$$

Then we obtain that

$$\begin{aligned}
c(X) &= \frac{1}{\xi}\max_{M\neq 0}\frac{\left\|\mathbf{L}^{-1}\left(\rho H + \sum_{i=1}^m \eta_i\left(B_i^* E_i + E_i^* B_i\right)\right)\right\|_F}{\left\|\left(E_1, E_2, \ldots, E_m, H\right)\right\|_F} \\
&= \frac{1}{\xi}\max_{M\neq 0}\left(\left\|\rho L^{-1}\operatorname{vec} H\right.\right. \\
&\qquad + \sum_{i=1}^m \eta_i L^{-1}\left(\left(I\otimes B_i^*\right)\operatorname{vec} E_i\right. \\
&\qquad \left.\left.\left. + \left(B_i^T\otimes I\right)\operatorname{vec} E_i^*\right)\right\|\right) \\
&\qquad \times\left(\left\|\operatorname{vec}\left(E_1, E_2, \ldots, E_m, H\right)\right\|\right)^{-1} \\
&= \frac{1}{\xi}\max_{M\neq 0}\left(\left\|\rho\left(S + \mathbf{i}\Sigma\right)\left(x + \mathbf{i}y\right)\right.\right. \\
&\qquad + \sum_{i=1}^m \eta_i\left[\left(U_{i1} + \mathbf{i}\Omega_{i1}\right)\left(a_i + \mathbf{i}b_i\right)\right. \\
&\qquad \left.\left.\left. + \left(U_{i2} + \mathbf{i}\Omega_{i2}\right)\left(a_i - \mathbf{i}b_i\right)\right]\right\|\right) \\
&\qquad \times\left(\left\|\operatorname{vec}\left(E_1, E_2, \ldots, E_m, H\right)\right\|\right)^{-1} \\
&= \frac{1}{\xi}\max_{g\neq 0}\frac{\left\|\left(\rho S_c, \eta_1 U_1, \eta_2 U_2, \ldots, \eta_m U_m\right) g\right\|}{\|g\|} \\
&= \frac{1}{\xi}\left\|\left(\rho S_c, \eta_1 U_1, \eta_2 U_2, \ldots, \eta_m U_m\right)\right\|, \\
&\qquad E_i \in \mathscr{C}^{n\times n}, \quad H \in \mathscr{H}^{n\times n}.
\end{aligned} \tag{59}$$

Then we have the following theorem.

Theorem 11. *If $\sum_{i=1}^m \|A_i\|^2/\beta^{1-p_i} < 1$, then the condition number $c(X)$ defined by* (54) *has the explicit expression*

$$c(X) = \frac{1}{\xi}\left\|\left(\rho S_c, \eta_1 U_1, \eta_2 U_2, \ldots, \eta_m U_m\right)\right\|, \tag{60}$$

where the matrices S_c and U_i are defined as in (57).

Remark 12. From (60) we have the relative condition number

$$c_{\text{rel}}(X) = \frac{\left\|\left(\|Q\|_F S_c, \|A_1\|_F U_1, \|A_2\|_F U_2, \ldots, \|A_m\|_F U_m\right)\right\|}{\|X\|_F}. \tag{61}$$

5.2. The Real Case. In this subsection we consider the real case. That is, all the coefficient matrices A_i, Q of (1) are real. In such a case the corresponding solution X is also real. Completely similar arguments as Theorem 11 give the following theorem.

Theorem 13. *Let A_i, Q be real and let $c(X)$ be the condition number defined by* (54). *If $\sum_{i=1}^m \|A_i\|^2/\beta^{1-p_i} < 1$, then $c(X)$ has the explicit expression*

$$c(X) = \frac{1}{\xi}\left\|\left(\rho S_r, \eta_1 U_1, \eta_2 U_2, \ldots, \eta_m U_m\right)\right\|, \tag{62}$$

where

$$\begin{aligned}
S_r &= \left(I + \sum_{i=1}^m \frac{\sin p_i\pi}{\pi}\int_0^\infty\left[(\lambda I + X)^{-1}X^{1/2}A_i\right]^T\right. \\
&\qquad \left.\otimes\left[(\lambda I + X)^{-1}X^{1/2}A_i\right]^T\lambda^{p_i-1}d\lambda\right)^{-1} \\
U_i &= S_r\left[I\otimes\left(A_i^T X^{p_i}\right) + \left(\left(A_i^T X^{p_i}\right)\otimes I\right)\Pi\right], \\
&\qquad i = 1, 2, \ldots, m.
\end{aligned} \tag{63}$$

Remark 14. In the real case the relative condition number is given by

$$c_{\text{rel}}(X) = \frac{\left\|\left(\|Q\|_F S_r, \|A_1\|_F U_1, \|A_2\|_F U_2, \ldots, \|A_m\|_F U_m\right)\right\|}{\|X\|_F}. \tag{64}$$

6. Numerical Examples

To illustrate the results of the previous sections, in this section three simple examples are given, which were carried out using MATLAB 7.1. For the stopping criterion we take $\varepsilon_{k+1}(X) = \|X_k - \sum_{i=1}^m A_i^* X_k^{p_i} A_i - Q\| < 1.0e-10$.

Example 15. We consider the matrix equation

$$X - A_1^* X^{1/2} A_1 - A_2^* X^{1/3} A_2 = I, \tag{65}$$

with

$$\begin{aligned}
A_1 &= \frac{(1/3) + 2\times 10^{-2}}{\|A\|}A, \\
A_2 &= \frac{(1/6) + 3\times 10^{-2}}{\|A\|}A, \\
A &= \begin{pmatrix} 0 & 0.95 \\ 0 & 1 \end{pmatrix}.
\end{aligned} \tag{66}$$

Table 1: Assumptions check for Example 15 with different values of j.

j	4	5	6	7
ass_1	1.0749	1.0753	1.0753	1.0753
ass_2	0.9248	0.9247	0.9247	0.9247
ass_3	0.8543	0.8550	0.8550	0.8550
ass_4	0.9999	1.0000	1.0000	1.0000
ass_5	0.7645	0.7648	0.7648	0.7648

Suppose that the coefficient matrices A_1 and A_2 are perturbed to $\widetilde{A}_i = A_i + \Delta A_i$, $i = 1, 2$, where

$$\Delta A_1 = \frac{10^{-j}}{\left\|C^T + C\right\|}\left(C^T + C\right),$$
$$\Delta A_2 = \frac{3 \times 10^{-j-1}}{\left\|C^T + C\right\|}\left(C^T + C\right), \tag{67}$$

and C is a random matrix generated by MATLAB function **randn**.

We now consider the corresponding perturbation bounds for the solution X in Theorems 5 and 8.

The assumptions in Theorem 5 are

$$\text{ass}_1 = 2(\beta - s) - b > 0,$$
$$\text{ass}_2 = \beta + \|\Delta Q\| - \sum_{i=1}^{m} p_i \beta^{p_i} \|A_i\|^2 > 0, \tag{68}$$
$$\text{ass}_3 = b^2 - 4(\beta - s)(s + \|\Delta Q\|) \geq 0.$$

The assumptions in Theorem 8 are

$$\text{ass}_4 = 1 - \sigma > 0,$$
$$\text{ass}_5 = \frac{(1-\sigma)^2}{\zeta + \sigma\zeta + 2\theta + 2\sqrt{((\zeta+\theta)(\sigma\zeta+\theta))}} - \epsilon > 0. \tag{69}$$

By computation, we list them in Table 1.

The results listed in Table 1 show that the assumptions of Theorems 5 and 8 are satisfied.

By Theorems 5 and 8, we can compute the relative perturbation bounds ξ_1 and $\xi_2 = \nu/\|X\|$, respectively. These results averaged as the geometric mean of 10 randomly perturbed runs. Some results are listed in Table 2.

The results listed in Table 2 show that the perturbation bound ξ_2 given by Theorem 8 is fairly sharp, while the bound ξ_1 given by Theorem 5 which does not depend on the exact solution is conservative.

Example 16. We consider the matrix equation

$$X - A_1^* X^{0.5} A_1 - A_2^* X^{0.25} A_2 = I, \tag{70}$$

with

$$A_1 = \frac{(1/3) + 2 \times 10^{-2}}{\|A\|} A,$$
$$A_2 = \frac{(1/6) + 3 \times 10^{-2}}{\|A\|} A,$$
$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}. \tag{71}$$

Choose $\widetilde{X}_1 = A$, $\widetilde{X}_2 = 2A$. Let the approximate solution $\widetilde{X}_k$ of X be given with the iterative method (6), where k is the iteration number.

The residual $R(\widetilde{X}_k) \equiv I + A_1^* \widetilde{X}_k^{0.5} A_1 + A_2^* \widetilde{X}_k^{0.25} A_2 - \widetilde{X}_k$ satisfies the conditions in Theorem 10.

By Theorem 10, we can compute the backward error bound for $\widetilde{X}_k$ as follows:

$$\left\|\widetilde{X}_k - X\right\| \leq \mu \left\|R\left(\widetilde{X}_k\right)\right\|, \tag{72}$$

where

$$\mu = \frac{2\left\|\widetilde{X}_k\right\|\left\|\widetilde{X}_k^{-1}\right\|}{\theta_1 + \sqrt{\theta_1^2 - 4\left\|\widetilde{X}_k^{-1}\right\|\left\|R\left(\widetilde{X}_k\right)\right\|}},$$
$$\theta_1 \equiv 1 + \left\|\widetilde{X}_k^{-1}\right\|\left\|R\left(\widetilde{X}_k\right)\right\| - \left(0.5\left\|\widetilde{X}_k^{1/4} A_1 \widetilde{X}_k^{-1/2}\right\|^2 + 0.25\left\|\widetilde{X}_k^{1/8} A_2 \widetilde{X}_k^{-1/2}\right\|^2\right). \tag{73}$$

Some results are listed in Table 3.

The results listed in Table 3 show that the error bound given by Theorem 10 is fairly sharp.

Example 17. We study the matrix equation

$$X - A_1^* X^{1/2} A_1 - A_2^* X^{1/3} A_2 = Q, \tag{74}$$

with

$$A_1 = \begin{pmatrix} 0 & 0.55 + 10^{-k} \\ 0 & 0 \end{pmatrix},$$
$$A_2 = \frac{1}{2} A_1, \tag{75}$$
$$Q = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

By Remark 14, we can compute the relative condition number $c_{\text{rel}}(X)$. Some results are listed in Table 4.

Table 4 shows that the unique positive definite solution X is well conditioned.

Table 2: Perturbation bounds for Example 15 with different values of j.

j	4	5	6	7
$\left\|\widetilde{X}-X\right\| / \|X\|$	2.5627×10^{-5}	3.8447×10^{-6}	5.1681×10^{-7}	2.1776×10^{-8}
ξ_1	2.1885×10^{-4}	1.9891×10^{-5}	2.4026×10^{-6}	1.8251×10^{-7}
ξ_2	8.0828×10^{-5}	7.4741×10^{-6}	8.8011×10^{-7}	6.9496×10^{-8}

Table 3: Backward error bound for Example 16 with different values of k.

k	8	10	12	14
$\left\|\widetilde{X}_k-X\right\|$	6.1091×10^{-4}	4.0865×10^{-5}	2.6837×10^{-6}	1.7372×10^{-7}
$\mu\left\|R(\widetilde{X}_k)\right\|$	7.1435×10^{-4}	4.7784×10^{-5}	3.1381×10^{-6}	2.0318×10^{-7}

Table 4: Relative condition number for Example 17 with different values of k.

k	1	3	5	7	9
$c_{\mathrm{rel}}(X)$	1.0717	1.0228	1.0225	1.0225	1.0225

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