Research Article

Some Weighted Estimates for Multilinear Fourier Multiplier Operators

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1. Introduction

During the last several years, considerable attention has been paid to the study of multilinear Fourier multiplier operators. Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of all rapidly decreasing smooth functions on $\mathbb{R}^d$, for some $d \in \mathbb{Z}^+$. The multilinear Fourier multiplier operator $T_\sigma$ associated with a symbol $\sigma$ is defined by

$$T_\sigma(f_1, \ldots, f_m)(x) = \int_{\mathbb{R}^m} e^{i\sum_{k=1}^m (\xi_k \cdot x)} \sigma(\xi_1, \ldots, \xi_m) f_1(\xi_1) \cdots f_m(\xi_m) \, d\xi_1 \cdots d\xi_m$$

for $f_i \in \mathcal{S}(\mathbb{R}^d), i = 1, \ldots, m$.

Coifman and Meyer [1] proved that if $\sigma$ is a bounded function on $\mathbb{R}^m \setminus \{0\}$ that satisfies

$$|\partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_m}^{\alpha_m} \sigma(\xi_1, \ldots, \xi_m)| \leq C_\alpha (|\xi_1| + \cdots + |\xi_m|)^{-|\alpha_1| - \cdots - |\alpha_m|}$$

away from the origin for all sufficiently large multi-indices $\alpha$, then $T_\sigma$ is bounded from the product $L^{p_1}(* \cdots \cdots L^{p_m}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ for all $1 < p_1, \ldots, p_m, p < \infty$ satisfying $1/p_1 + \cdots + 1/p_m = 1/p$. The multiplier theorem of Coifman and Meyer was extended to indices $p < 1$ (and larger than $1/m$) by Grafakos and Torres [2] and Kenig and Stein [3] (when $m = 2$). Exploiting the idea of the proof of the Hörmander multiplier theorem in [4], Tomita [5] gave a Hörmander type theorem for multilinear Fourier multipliers with more weaker smoothness condition assumed on $\sigma$ than (2). Grafakos and Si [6] gave similar results for $p \leq 1$ by using $L^r$-based Sobolev spaces ($1 < r \leq 2$). Grafakos et al. [7] proved the $L^2$-boundedness of $T_\alpha$ with multipliers of limited smoothness.

In order to state other known results, we first introduce some notations. The Laplacian on $\mathbb{R}^d$ is $\Delta g = \sum_{j=1}^d \partial_x^2 g/\partial x_j^2$, that is, the sum of the second partials of $g$ in every variable. We define the operator $(I - \Delta)^{\gamma/2}(g) = \mathcal{F}^{-1}(\omega_\gamma \mathcal{F}(g))$, where $\omega_\gamma(\xi) = (1 + 4\pi^2 |\xi|^2)^{\gamma/2}$ for $\gamma > 0$. Let $L^r_\gamma(\mathbb{R}^d)$ be the $L^r$-based Sobolev space with norm

$$\|f\|_{L^r_\gamma} = \|(I - \Delta)^\gamma f\|_{L^r(\mathbb{R}^d)}$$

where $1 \leq r < \infty$.

Let $\vec{s} = (s_1, \ldots, s_m)$ and let the product type Sobolev space $W^{\vec{s}}_r(\mathbb{R}^m)$ consist of all functions $F$ such that the following norm of $F$ is finite:

$$\|F\|_{W^{\vec{s}}(\mathbb{R}^m)} = \left( \int_{\mathbb{R}^m} \langle \xi_1 \rangle^{2s_1} \cdots \langle \xi_m \rangle^{2s_m} |\hat{F}(\xi)|^2 \, d\xi \right)^{1/2}$$

where $\xi = (\xi_1, \ldots, \xi_m)$ and $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. 

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Let $\psi \in \mathcal{S}(\mathbb{R}^{mn})$ be such that $\text{supp } \psi \subset \{ \xi \in \mathbb{R}^{mn} : 1/2 \leq |\xi| \leq 2 \}$ and $\sum_{j \in \mathbb{Z}} \psi(2^{-j}\xi) = 1$ for $\xi \neq 0$.

Let $\mathcal{S}_1(\mathbb{R}^d)$ be the set of all Schwartz functions $\Psi$ on $\mathbb{R}^d$, whose Fourier transform is supported in an annulus of the form $\{ \xi : c_1 < |\xi| < c_2 \}$, is nonvanishing in a smaller annulus $\{ \xi : c_1' \leq |\xi| \leq c_2' \}$ (for some choice of constants $0 < c_1 < c_1' < c_2 < c_2' < \infty$), and satisfies

$$\sum_{j \in \mathbb{Z}} \Psi(2^{-j}\xi) = \text{constant}, \quad \xi \in \mathbb{R}^d \setminus \{0\}. \quad (5)$$

The weighted estimate for $T_\sigma$ is also an interesting topic in harmonic analysis. And it has attracted many authors in this area. Recently, Fujita and Tomita [8] established some weighted estimates of $T_\sigma$ under the Hörmander condition and classical $A_p$ weights. For other works about the weighted estimates for $T_\sigma$, see [9, 10] and the references therein.

**Theorem A** (see [8]). Let $1 < p_1, p_2, \ldots, p_N < \infty, 1/p_1 + \cdots + 1/p_N = 1/p$, and $Nn/2 < s \leq Nn$. Assume

(i) $\min p_1, \ldots, p_N > Nn/s$ and $\omega \in A_{p_1/Nn, \ldots, p_N/Nn}$;

(ii) $\min p_1, \ldots, p_N > (Nn/\delta)'$ and $1 < p < \infty, \omega^{1-p'} \in A_{p'/Nn}$.

If $\sigma \in L^\infty$ satisfies $\sup_{k \in \mathbb{Z}} \|\sigma(2^{k} \cdot)\|_{L^1_1} < \infty$, then $T_\sigma$ is bounded from $L^{p_1}(\omega) \times \cdots \times L^{p_N}(\omega)$ to $L^p(\omega)$.

An improvement of Theorem 1.2 is stated as follows.

**Theorem B** (see [8]). Let $1 < p_1, p_2, \ldots, p_N < \infty, 1/p_1 + \cdots + 1/p_N = 1/p$, and $n/2 < s_j \leq n, j = 1, \ldots, N$. Assume $p_j > n/s_j$ and $\omega_j \in A_{p_j/n}$ for $1 \leq j \leq N$. If $\sigma \in L^\infty$ satisfies $\sup_{k \in \mathbb{Z}} \|\sigma(2^{k} \cdot)\|_{L^1_1} < \infty$, then $T_\sigma$ is bounded from $L^{p_1}(\omega_1) \times \cdots \times L^{p_N}(\omega_N)$ to $L^p(\omega)$, where $\omega = \omega^{p_1/n} \cdots \omega^{p_N/n}$.

The first purpose of this paper is to improve Theorem A by using $L^r$-based Sobolev spaces ($1 < r < 2$). The second purpose is to give a new proof of Theorem B. The following are the main results.

**Theorem 1.** For some $1 < r \leq 2$, suppose that $\sigma \in L^\infty(\mathbb{R}^{mn})$ and $\Psi \in \mathcal{S}_1(\mathbb{R}^{mn})$ satisfy, for some $m/n < r < mn,$

$$\sup_{k \in \mathbb{Z}} \|\sigma(2^k \cdot)\Phi\|_{L^1_1(\mathbb{R}^{mn})} = K < \infty. \quad (6)$$

If $p_1, \ldots, p_m, q$, and the weights $\omega$ satisfy one of the following two conditions:

(i) $\min(p_1, \ldots, p_m) > mn/q$ and $\omega \in A_{\min(p_1/n, \ldots, p_N/n)}$;

(ii) $\min(p_1, \ldots, p_m) > (mn/q)', 1 < p < \infty, \omega^{1-p'} \in A_{p'/n}$,

then there is a number $\delta = \delta(mn, q, r)$ satisfying $0 < \delta \leq r - 1$, such that the $m$-linear operator $T_{\sigma, \Xi}$ associated with the multiplier $\sigma$, is bounded from $L^{p_1}(\omega) \times \cdots \times L^{p_m}(\omega)$ to $L^p(\omega)$, whenever $r - \delta < p_j < \infty$ for all $j = 1, \ldots, m$, and $p$ is given by $1/p = 1/p_1 + \cdots + 1/p_m$.

**Theorem 2.** Let $1 < p_1, \ldots, p_m < 2$ and let $s_1 > n/p_1, \ldots, s_m > n/p_m$ and $s_1 + \cdots + s_m < n/p_1 + \cdots + n/p_m + 1$. If $\sigma$ is $L^\infty(\mathbb{R}^{mn})$ satisfies $\sup_{k \in \mathbb{Z}} \|\sigma(2^k \cdot)\Psi\|_{W^{s_1}_{\infty}, \ldots, W^{s_m}_{\infty}} < \infty$, then $T_{\sigma, \Xi}$ is bounded from $L^{p_1}(w_1^{a_1}) \times \cdots \times L^{p_m}(w_m^{a_m})$ to $L^p(w^a)$, whenever $1 < q_1, \ldots, q_m < \infty$, $1/q_1 + \cdots + 1/q_m = 1/q$, and $(w_1^{a_1}, \ldots, w_m^{a_m}) \in (A_{q_1}, \ldots, A_{q_m})$, with $w = w_1^{a_1} \cdots w_m^{a_m}$.

**2. The Proof of Theorem 1**

In this section we discuss the proof of Theorem 1. We begin with some definitions for maximal operators. Throughout the paper, $M$ denotes the Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy, \quad (7)$$

where $Q$ moves over all cubes containing $x$. For $\delta > 0, M_\delta$ is the maximal function defined by

$$M_\delta f(x) = M\left(\left|\frac{f}{\delta}\right|\right)^{1/\delta}(x) = \left(\sup_{Q \ni x} \frac{1}{|Q|} \int_Q \left|f(y) - f_Q\right|^\delta \, dy\right)^{1/\delta}. \quad (8)$$

In addition, $M^I$ is the sharp maximal function of Fefferman and Stein:

$$M^I f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - c| \, dy = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy, \quad (9)$$

where $f_Q$ denotes the average of $f$ over $Q$ and a variant of $M^I$ is given by

$$M^I f(x) = M\left(\left|\frac{f}{\delta}\right|\right)^{1/\delta}(x). \quad (10)$$

We prepare some lemmas which will be used later.

**Lemma 3** (see [11]). Let $1 < p < \infty$ and $\omega \in A_p$. Then

(i) $\omega^{1-p'} \in A_p'$; (2) there exists $q < p$ such that $\omega \in A_q$.

**Lemma 4** (see [12]). Let $1 < p, q < \infty$, and $\omega \in A_p$. Then there exist positive finite constants $C(p, q)$ such that

$$\left\|\sum_{k \in \mathbb{Z}} |M(f_k)|^q \right\|_{L^p(\omega)}^{1/q} \leq C(p, q) \left\|\sum_{k \in \mathbb{Z}} |f_k|^q \right\|_{L^p(\omega)}^{1/q}, \quad (11)$$

for all sequences $\{f_k\}_{k \in \mathbb{Z}}$ of locally integrable functions on $\mathbb{R}^d$.

**Lemma 5.** Let $\Delta_k$ be the Littlewood-Paley operator given by

$$\Delta_k(\xi) = \hat{g}(\xi) \hat{\Psi}(2^{-k} \xi), \quad k \in \mathbb{Z}, \quad \Psi$$

is a Schwartz function whose Fourier transform is supported in the annulus $\{ |\xi| : 2^{-b} < |\xi| < 2^b \}$, for some $b \in \mathbb{Z}^+$, and satisfies

$$\sum_{k \in \mathbb{Z}} \hat{\Psi}(2^{-k} \xi) = c_0, \quad \text{for some constant } c_0.$$
and \( \omega \in A^\infty \). Then there is a constant \( c = c(n, p, c_0, \Psi) \), such that for \( L^p(\omega) \) functions \( f \) one has

\[
\|f\|_{L^p(\omega)} \leq c \left( \sum_{k \in \mathbb{Z}} |\Delta_k(f)|^2 \right)^{1/2} \|f\|_{L^p(\omega)}.
\]

(12)

**Proof.** The proof follows from similar steps in Lemma 4 of [6] and combines the method used in Remark 2.6 of [8]. Let \( \Phi \) be a Schwartz function with integral one. Then,

\[
|f(x)| = \lim_{t \to 0} |\Phi_t \ast f(x)| \leq \sup_{t > 0} |\Phi_t \ast f(x)|.
\]

(13)

If \( \omega \in A^\infty \), the weighted Hardy space \( H^p(\omega) \) coincides with the weighted Triebel-Lizorkin space \( \dot{F}^p_2(\omega) \) for \( 0 < p < \infty \). Hence, if \( \omega \) is in \( A^\infty \), we have

\[
\|f\|_{L^p(\omega)} \leq \sup_{t > 0} |\Phi_t \ast f(x)| \leq c \|f\|_{H^p(\omega)} \approx \|f\|_{\dot{F}^p_2(\omega)}
\]

(14)

\[
\leq c \left( \sum_{k \in \mathbb{Z}} |\Delta_k(f)|^2 \right)^{1/2} \|f\|_{L^p(\omega)}.
\]

The proof is complete. \( \square \)

Now we give the proof of Theorem 1.

**Proof.** Since the proof follows from similar steps in Theorem 1 in [6], we just give the different parts. For each \( j = 1, \ldots, m \), we let \( R_j \) be the set of points \( (\xi_1, \ldots, \xi_m) \) in \( \mathbb{R}^m \) such that \( |\xi_j| = \max(|\xi_1|, \ldots, |\xi_m|) \) and we introduce nonnegative smooth functions \( \phi_j \) on \([0, \infty)\) that are supported in \([0, 11/10]^{-m} \) such that

\[
1 = \sum_{j=1}^m \phi_j(\xi_1, \ldots, \xi_m) = 1\quad \text{for all } (\xi_1, \ldots, \xi_m) \neq (0, \ldots, 0).
\]

(15)

for \( (\xi_1, \ldots, \xi_m) \neq 0 \), with the understanding that the variable with the hat is missing. These functions introduce a partition of unity of \( (\mathbb{R}^m)^m \) \( \setminus \{0\} \) subordinate to a conical neighborhood of the region \( R_j \).

Each region \( R_j \) can be written as the union of sets:

\[
R_{j,k} = \{ (\xi_1, \ldots, \xi_m) \in R_j : |\xi_k| \geq |\xi_j| \forall s \neq j \}
\]

(16)

with \( k = 1, \ldots, m \). We need to work with a finer partition of unity, subordinate to each \( R_{j,k} \). To achieve this, for each \( j \), we introduce smooth functions \( \phi_{j,k} \) on \([0, \infty)^{m-2} \) supported in \([0, 11/10]^{m-2} \) such that

\[
1 = \sum_{k \neq \hat{k}}^m \phi_{j,k}(\xi_1/k, \ldots, \xi_{j-1}/k, \xi_j/k, \ldots, \xi_{m-1}/k, \xi_m/k)
\]

(17)

for \( (\xi_1, \ldots, \xi_m) \) in the support of \( \phi_j \) with \( \xi_k \neq 0 \).

We now have obtained the following partition of unity of \( (\mathbb{R}^m)^m \) \( \setminus \{0\} \):

\[
1 = \sum_{j=1}^m \sum_{k \neq \hat{k}}^m \phi_j(\cdot \cdot \cdot) \phi_{j,k}(\cdot \cdot \cdot),
\]

(18)

where the dots indicate the variables of each function.

We now introduce a nonnegative smooth bump \( \psi \) supported in the interval \([10m]^{-1}, 2\] and equal to 1 in the interval \([5m]^{-1}, 12/10\] and we decompose \( \sigma \) into a finite number of multipliers:

\[
\sigma = \sum_{j=1}^m \sum_{k \neq \hat{k}}^m [\sigma \phi_{j,k} + \sigma \Psi_{j,k}],
\]

(19)

where

\[
\Phi_{j,k}(\xi_1, \ldots, \xi_m) = \phi_j(\cdot \cdot \cdot) \phi_{j,k}(\cdot \cdot \cdot) \left( 1 - \psi(\xi_1/\xi_j, \ldots, \xi_m/\xi_j) \right),
\]

\[
\Psi_{j,k}(\xi_1, \ldots, \xi_m) = \phi_j(\cdot \cdot \cdot) \phi_{j,k}(\cdot \cdot \cdot) \psi(\xi_1/\xi_j, \ldots, \xi_m/\xi_j).
\]

(20)

We will prove the required assertion for each piece of this decomposition, that is, for the multipliers \( \sigma \phi_{j,k} \) and \( \sigma \Psi_{j,k} \) for each pair \( (j, k) \) in the previous sum. In view of the symmetry of the decomposition, it suffices to consider the case of a fixed pair \( (j, k) \) in the previous sum. To simplify notation, we fix the pair \( (m, m-1) \); thus, for the rest of the proof we fix \( j = m \) and \( k = m-1 \) and we prove boundedness for the \( m \)-linear operators whose symbols are \( \sigma_1 = \sigma \Phi_{mm-1} \) and \( \sigma_2 = \sigma \Psi_{mm-1} \). These correspond to the \( m \)-linear operators \( T_{\sigma_1} \) and \( T_{\sigma_2} \), respectively.

We first prove Theorem 1 under assumption (i). Since \( 1 \leq mn/\gamma < \min(r_1, p_1, \ldots, p_m) \), we can take \( \rho \) such that \( 1 \leq mn/\gamma < \rho < \min(r_1, p_1, \ldots, p_m) \) and \( \omega \in A_{\min(r_1, p_1, \ldots, p_m)/\rho} \).

We first consider \( T_{\sigma_1}(f_1, \ldots, f_m) \), where \( f_j \) are fixed Schwartz functions. We fix a Schwartz radial function \( \eta \) whose Fourier transform is supported in the annulus \( 1 - (1/25) \leq |\xi| \leq 2 \) and satisfies

\[
\sum_{j \in \mathbb{Z}} (2^{-j} \xi^2) = 1, \quad \xi \in \mathbb{R}^m \setminus \{0\}.
\]

(21)

Associated with \( \eta \) we define the Littlewood-Paley operator \( \Delta_j(f) = f \ast \eta_{2^{-j}} \), where \( \eta_j(x) = (2^{-j} \gamma) \eta(2^{-j} x) \) for \( t > 0 \). We also define an operator \( S_j \) by setting

\[
S_j(g) = g \ast \eta_{2^{-j}}.
\]

(22)

where \( \eta \) is a smooth function whose Fourier transform is equal to 1 on the ball \( |z| < 3/5m \) and vanishes outside
the double of this ball. As in [6, page 143], by using Lemma 5 we get
\[
\|T_{\sigma_1}(f_1, \ldots, f_m)\|_{L^p(\omega)} \\
\leq C \left\| \sqrt{\sum_j |T_{\sigma_1}(S_j(f_1), \ldots, S_j(f_{m-1}), \Delta_j(f_m))|^2} \right\|_{L^p(\omega)}^{1/2}.
\]
(23)

We will use the following estimate for \(T_{\sigma_1}\) (see [6, page 145]):
\[
|T_{\sigma_1}(S_j(f_1), \ldots, S_j(f_{m-1}), \Delta_j(f_m))| \\
\leq CK \prod_{i=1}^{m-1} \left( M \left( M(f_i)^p \right)^{1/p} \right)^{1/p} \left( M \left( |\Delta_j(f_m)|^p \right)^{1/p} \right)^{1/p}.
\]
(24)

We now square the previous expression, sum over \(j \in \mathbb{Z}\), and take square roots. Since \(r - \delta = \rho\), the hypothesis \(p_j > r - \delta\) implies \(p_j > \rho\), and thus each term \(M(M(f_j)^p)^{1/p}\) is bounded on \(L^p(\omega)\). We obtain
\[
|T_{\sigma_1}(f_1, \ldots, f_{m-1}, f_m)| \\
\leq CK \| \sqrt{\sum_j |T_{\sigma_1}(S_j(f_1), \ldots, S_j(f_{m-1}), \Delta_j(f_m))|^2} \|_{L^p(\omega)}^{1/2} \\
\leq C'K \left\{ \sqrt{\sum_j M(|\Delta_j(f_m)|^p)^{2/p}} \right\}^{1/2} \\
\times \prod_{j=1}^{m-1} \left( M \left( M(f_j)^p \right)^{1/p} \right)^{1/p} \\
\leq C''K \left\{ \sqrt{\sum_j M(|\Delta_j(f_m)|^p)^{2/p}} \right\}^{1/2} \\
\times \prod_{j=1}^{m-1} \| f_j \|_{L^p(\omega)}^{1/p} \\
\leq C''K \prod_{j=1}^{m} \| f_j \|_{L^p(\omega)}.
\]
(25)

where the last step holds due to Lemma 4 with \(q = 2/p\) and the weighted Littlewood-Paley theorem.

Next we deal with \(T_{\sigma_2}\). Following [6, page 146], we write
\[
T_{\sigma_2}(f_1, \ldots, f_{m-1}, f_m) \\
= \sum_{j \in \mathbb{Z}} T_{\sigma_2}(S'_j(f_1), \ldots, S'_j(f_{m-2}), \Delta'_j(f_{m-1}), \Delta'_j(f_m)).
\]

We use the weighted Littlewood-Paley theorem to obtain
\[
\|T_{\sigma_2}(f_1, \ldots, f_{m-1}, f_m)\|_{L^p(\omega)} \\
\leq CK \prod_{i=1}^{m-2} \left( M \left( M(f_i)^p \right)^{1/p} \right)^{1/p} \left( M \left( |\Delta_j(f_{m-1})|^p \right)^{1/p} \right)^{1/p} \\
\times \left( M \left( |\Delta'_j(f_m)|^p \right)^{1/p} \right)^{1/p},
\]
(26)

for some other Littlewood-Paley operator \(\Delta'_j\), which is given on the Fourier transform by multiplication with a bump \(\Theta(2^{-j}z)\), where \(\Theta\) is equal to one on the annulus \(|x| \in \mathbb{R}^n : (24/25) \cdot (1/10m) \leq |x| \leq 4\) and vanishes on a larger annulus. Also, \(S'_j\) is given by convolution with \(\xi_j^{1/\rho}\), where \(\xi_j^{1/\rho}\) is a smooth function whose Fourier transform is equal to 1 on the ball \(|z| < (22/10)\) and vanishes outside the double of this ball.

Summing over \(j\) and taking \(L^p(\omega)\) norms yield
\[
\|T_{\sigma_2}(f_1, \ldots, f_{m-1}, f_m)\|_{L^p(\omega)} \\
\leq CK \prod_{i=1}^{m-2} \left( M \left( M(f_i)^p \right)^{1/p} \right)^{1/p} \sum_{j \in \mathbb{Z}} \left( M \left( |\Delta_j(f_{m-1})|^p \right)^{1/p} \right)^{1/p} \\
\times \left( M \left( |\Delta'_j(f_m)|^p \right)^{1/p} \right)^{1/p},
\]
(27)

where the last step holds due to the Cauchy-Schwarz inequality and we omitted the prime from the term with \(i = m - 1\) for the matter of simplicity. Applying Hölder’s inequality and using that \(\rho < 2\) and Lemma 4 we obtain the conclusion that the expression above is bounded by
\[
C'K \left\| f_1 \right\|_{L^p(\omega)} \cdots \left\| f_m \right\|_{L^p(\omega)}.
\]
(28)

We next prove Theorem 1 under assumption (ii). It was proven in [6, page 136] that condition (6) is invariant under the adjoints; that is, it is also valid for the symbols of the dual operators \(\sigma^{m*}(\xi_1, \ldots, \xi_m) = \sigma(\xi_1, \ldots, \xi_{m-1}, -\xi_1 + \cdots + \xi_m)\). To prove the required assertion, by duality, it is enough to prove that \(T_{\sigma_1}^{m*}\) and \(T_{\sigma_2}^{m*}\) are bounded from \(L^p(\omega) \times \cdots \times L^p(\omega)\) to \(L^{p_0}(\omega_0^{1-p_0})\). We may assume that \(p_m = \min(p_1, \ldots, p_m)\). Since \(p_m < (mm/\gamma)\), we see \(1/p, 1/p_k < 1/p + 1/p_2 + \cdots + 1/p_m = 1/p_m < \gamma/(mm)\). Hence, \(mn/\gamma < \min(p, p, p, \ldots, p_m)\). Since \(p_m < p_m\) and \(\omega^{1-p_0} \in A_{p_0/\gamma(mn)} < A_{p_0}\), we deduce that \(\omega \in A_{p_0}\) and then \(\omega^{1-p_0} \in A_{p_0} \omega^{1-p_0} = A_{p_0} \omega^{1-p_0} \cdots \omega^{1-p_0} \omega^{1-p_0} \cdots \omega^{1-p_0}\). Since \(p_m < p_k, 1/p = 1/p_1 + \cdots + 1/p_k + \cdots + 1/p_m\),
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Then $p < p_k/2$; then $\omega \in A_{p \cdot} \subset A_{p_{k/2} \cdot} \subset A_{p_{k} \cdot}$. We have

$$\|T_{\sigma^m}(f_1, \ldots, f_m, m)\|_{L^p(|\omega|^{-1/1})} \leq CK \left\{ \sum_{j} \left| \sigma_j(f_1), \ldots, \sigma_j(f_{m-1}) \right| \right\}^{1/22} \left\{ L^p(|\omega|^{-1/1}) \right\}$$

$$\leq C' K \left\{ \sum_{j} \left| M \left( \left| \Delta_j(f_m) \right|^p \right) \right|^{1/2} \right\}^{1/22} \left\{ L^p(|\omega|^{-1/1}) \right\}$$

$$\times \prod_{i=1}^{m-1} \left\{ \left| M \left( \left| \Delta_j(f_m) \right|^p \right) \right|^{1/2} \right\}^{1/2} \left\{ L^p(|\omega|^{-1/1}) \right\} (29)$$

This concludes the proof of Theorem 1.

3. The Proof of Theorem 2

We begin with some lemmas which will be used in the proof of Theorem 2.

Lemma 6 (see [11]). Let $0 < p$ and $\Delta < \infty$ and let $\omega$ be a weight in $A_{\infty}$. Then, there exists $C > 0$ (depending on the $A_{\infty}$ constant of $\omega$) such that

$$\int_{R^n} \left( \sum_{j} \left| M \left( \left| \Delta_j(f) \right|^p \right) \right| \right)^{p/2} \left\{ L^{p/2}(|\omega|^{-1/1}) \right\} \leq C K \left\{ \sum_{i} \left| f_i \right| \right\} \left\{ L^{p/2}(|\omega|^{-1/1}) \right\} \left\{ f_m \right\} \left\{ L^{p/2}(|\omega|^{-1/1}) \right\}$$

(30)

for all function $f$ for which the left-hand side is finite.

Lemma 7 (see [13]). Let $0 < p_1, p_2, p \leq \infty$, and $1/p_1 + 1/p_2 = 1/p$. Let $\sigma$ be a multiplier satisfying $\sup_{\xi \in Z} \sigma(2^k\xi) \psi \in \sigma(2^k\xi)|\sigma(2^k\xi)| \leq C \left\{ L^{p/2}(|\omega|^{-1/1}) \right\}$ for all $\omega$.

Then $T_{\sigma}$ is bounded from $L^p(|\omega|^{-1/1})$ to $L^p(|\omega|^{-1/1})$.

Remark 8. It should be pointed out that Lemma 7 can be extended to the case $m \geq 3$.

Lemma 9 (see [8]). Let $r > 0$, $q_1, \ldots, q_m \in [2, \infty)$, and $s_1, \ldots, s_m \geq 0$. Then there exists a constant $C > 0$ such that

$$\left( \int_{R^n} \cdots \left( \sum_{j} \left| \Delta_j(f) \right|^p \right) \right)^{q_1/2} 
\times \left( \sum_{j} \left| \Delta_j(f) \right|^p \right)^{q_2/2} \cdots \left( \sum_{j} \left| \Delta_j(f) \right|^p \right)^{q_m/2} \leq C \left\{ L^{p/2}(|\omega|^{-1/1}) \right\}$$

(32)

for all $F \in W^{s/1/q_i, \ldots, s/m/q_i}(|\omega|^{-1/1})$ with supp $F \in \{ x_1^2 + \cdots + x_m^2 \leq r \}$.

Next, we give a pointwise control of $M_{\sigma}^\ast T_{\sigma}(\tilde{f})$ which becomes very useful in the proof of Theorem 2.

Lemma 10. Let $1 < p_1, \ldots, p_m < 2$. Assume that $\sigma \in L^{\infty}(|\omega|^{-1/1})$ which satisfies $\sup_{\xi \in Z} \sigma(2^k\xi) \psi \in \sigma(2^k\xi)|\sigma(2^k\xi)| \leq C \left\{ L^{p/2}(|\omega|^{-1/1}) \right\}$ for $s_1 > n/p_1, \ldots, s_m > n/p_m$, and $s_1 + \cdots + s_m < n/p_1 + \cdots + n/p_m + 1$. For any $0 < \delta \leq 1/m$, one has $M_{\sigma}^\ast T_{\sigma}(\tilde{f})(x) \leq C \prod_{i=1}^{m} M_{p_i} \left( \left| \Delta_j(f) \right|^p \right)^{x} \delta^{x/m}$.

Proof. For simplicity, we only prove for the case $m = 2$, since there is no essential difference for the general case. Fix an $x \in R^n$ and a cube $Q$ with side length $l$, such that $x \in Q$. 
Let $f_i = f_i^0 + f_i^\infty$, where $f_i^0 = f_i \chi_{Q^*}$ and $f_i^\infty = f_i \chi_{(Q^*)^c}$ for $i = 1, 2$ and $Q^* = 4 \sqrt{\pi} Q$. Since $0 < \delta < 1/2$, we have

$$
\left( \frac{1}{|Q|} \int_Q |T_\sigma (f_1, f_2) (z)|^\delta - |C|^\delta \, dz \right)^{1/\delta} \\
\leq C \left( \frac{1}{|Q|} \int_Q |T_\sigma (f_1, f_2) (z) - C|^\delta \, dz \right)^{1/\delta} \\
\leq C \left( \frac{1}{|Q|} \int_Q |T_\sigma (f_1^0, f_2^0) (z) + T_\sigma (f_1^\infty, f_2^\infty) (z) + T_\sigma (f_1^0, f_2^\infty) (z) + T_\sigma (f_1^\infty, f_2^0) (z) - C|^\delta \, dz \right)^{1/\delta} \\
= U_1 + U_2.
$$

(33)

We first consider $U_1$. By Kolmogorov’s inequality, Hölder’s inequality, and Lemma 7, we have

$$
\left( \frac{1}{|Q|} \int_Q |T_\sigma (f_1^0, f_2^0) (z)|^\delta \, dz \right)^{1/\delta} \\
\leq C \|T_\sigma (f_1^0, f_2^0) \|_{L^p(Q; dx/|Q|)}
$$

where $1/p = 1/p_1 + 1/p_2$ with $p > \delta$ and $1 < p_1, p_2 < \infty$. Next we deal with $U_2$. We choose $C = \sum_{j=1}^3 C_j$, where

$$
C_1 = T_\sigma (f_1^\infty, f_2^\infty) (x), \\
C_2 = T_\sigma (f_1^0, f_2^\infty) (x), \\
C_3 = T_\sigma (f_1^\infty, f_2^0) (x).
$$

(35)

We may split $U_2$ as $U_2 \leq U_{21} + U_{22} + U_{23}$, where

$$
U_{21} = \left( \frac{1}{|Q|} \int_Q |T_\sigma (f_1^\infty, f_2^\infty) (z) - T_\sigma (f_1^0, f_2^\infty) (x)|^\delta \, dz \right)^{1/\delta}, \\
U_{22} = \left( \frac{1}{|Q|} \int_Q |T_\sigma (f_1^0, f_2^\infty) (z) + T_\sigma (f_1^\infty, f_2^0) (z) - T_\sigma (f_1^0, f_2^\infty) (x)|^\delta \, dz \right)^{1/\delta}, \\
U_{23} = \left( \frac{1}{|Q|} \int_Q |T_\sigma (f_1^\infty, f_2^0) (z) - T_\sigma (f_1^\infty, f_2^\infty) (x)|^\delta \, dz \right)^{1/\delta}.
$$

(36)

Now we estimate $U_{21}$ first. We decompose $\sigma$ as

$$
\sigma = \sum_{j \in Z} \sigma (\cdot) \psi (\cdot/2^j) \equiv \sum_{j \in Z} \sigma_j.
$$

(37)

Let $\sigma_j = \sigma (\cdot) \psi (\cdot/2^j)$, where $\psi \in \mathcal{S}(\mathbb{R}^n)$ with supp $\psi \subset \{ x \in \mathbb{R}^n : 1/2 \leq |x| \leq 2 \}$ and $\sum_{j \in Z} \psi (2^{-j} x) = 1, \chi \neq 0$. Thus, we have

$$
|T_\sigma (f_1^\infty, f_2^\infty) (x) - T_\sigma (f_1^0, f_2^\infty) (x)| \\
\leq C \sum_{j \in Z} \left| T_{\sigma_j} (f_1^\infty, f_2^\infty) (z) - T_{\sigma_j} (f_1^0, f_2^\infty) (x) \right| \\
\leq C \sum_{j \in Z} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \int_{2^{j+1} Q^* \setminus 2^j Q^*} \int_{2^{j+1} Q^* \setminus 2^j Q^*} \left| \sigma_j^y (z - y_1, z - y_2) - \sigma_j^y (x - y_1, x - y_2) \right| |f_1 (y_1) f_2 (y_2)| \, dy_1 \, dy_2 \\
\leq C \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \int_{2^{j+1} Q^* \setminus 2^j Q^*} \int_{2^{j+1} Q^* \setminus 2^j Q^*} \left| \sigma_j^y (z - y_1, z - y_2) - \sigma_j^y (x - y_1, x - y_2) \right| |f_1 (y_1) f_2 (y_2)| \, dy_1 \, dy_2 \\
\leq C \prod_{j=1}^{\infty} \prod_{p_j=1} \left( \frac{1}{|Q|} \int_{Q_j} |f_j (x)|^p \, dx \right)^{1/p_j}.
$$

(38)
Applying Hölder's inequality we have

\[ I_{k_1, k_2} = \sum_{j \in \mathbb{Z}} \left( \int_{2^{k_1+1}Q \setminus 2^{k_2}Q} |\sigma_j^y (z - y_1, z - y_2) - \sigma_j^y (x - y_1, x - y_2)| \left| f_1 (y_1) f_2 (y_2) \right| dy_1 dy_2 \right) \]

\[ \leq C \sum_{j \in \mathbb{Z}} \left( \int_{2^{k_1+1}Q \setminus 2^{k_2}Q} \left( \int_{2^{k_1+1}Q \setminus 2^{k_2}Q} |\sigma_j^y (z - y_1, z - y_2) - \sigma_j^y (x - y_1, x - y_2)|^{p_1/2} dy_1 \right)^{1/p_1} \right) \]

\[ \times \left( \int_{2^{k_1+1}Q} |f_1 (y_1)|^{p_1} dy_1 \right)^{1/p_1} \left( \int_{2^{k_1+1}Q} |f_2 (y_2)|^{p_2} dy_2 \right)^{1/p_2} \]

\[ \leq C \sum_{j \in \mathbb{Z}} I_{k_1, k_2-j} \left( \int_{2^{k_1+1}Q} |f_1 (y_1)|^{p_1} dy_1 \right)^{1/p_1} \left( \int_{2^{k_1+1}Q} |f_2 (y_2)|^{p_2} dy_2 \right)^{1/p_2} \]

Let \( h = z - x \) and \( \overline{Q} = x - Q^* \). Then we have

\[ I_{k_1, k_2-j} = \left( \int_{2^{k_1+1}Q \setminus 2^{k_2}Q} \left( \int_{2^{k_1+1}Q \setminus 2^{k_2}Q} |\sigma_j^y (z - y_1, z - y_2) - \sigma_j^y (x - y_1, x - y_2)|^{p_1/2} dy_1 \right)^{1/p_1} \right) \]

\[ \times \left( \int_{2^{k_1+1}Q} |f_1 (y_1)|^{p_1} dy_1 \right)^{1/p_1} \left( \int_{2^{k_1+1}Q} |f_2 (y_2)|^{p_2} dy_2 \right)^{1/p_2} \]

\[ \leq C \left( \int_{2^{k_1+1}Q \setminus 2^{k_2}Q} \left( \int_{2^{k_1+1}Q \setminus 2^{k_2}Q} |\sigma_j^y (h + y_1, h + y_2) - \sigma_j^y (y_1, y_2)|^{p_1/2} dy_1 \right)^{1/p_1} \right) \]

\[ \times \left( \int_{2^{k_1+1}Q} |f_1 (y_1)|^{p_1} dy_1 \right)^{1/p_1} \left( \int_{2^{k_1+1}Q} |f_2 (y_2)|^{p_2} dy_2 \right)^{1/p_2} \]

\[ \leq C \left( 2^{k_1} \right)^{-7s} \left( 2^{k_2} \right)^{-7s} \left( \int_{2^{k_1+1}Q \setminus 2^{k_2}Q} \left( \int_{2^{k_1+1}Q \setminus 2^{k_2}Q} |\sigma_j^y (y_1, y_2)|^{p_1} \right) \right) \]

\[ \times \left( 1 + |y_1|^2 \right)^{s_1 p_1/2} \left( 1 + |y_2|^2 \right)^{s_2 p_2/2} \]

\[ \leq C \left( 2^{k_1} \right)^{-7s} \left( 2^{k_2} \right)^{-7s} \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \left| \sigma_j^y (2^{-j}y_1, 2^{-j}y_2) \right|^{p_1} \right) \right) \]

\[ \times \left( 1 + |2^{-j}y_1|^2 \right)^{s_1 p_1/2} \left( 1 + |2^{-j}y_2|^2 \right)^{s_2 p_2/2} \]
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\[ \leq C(2^k)^{-\gamma_1}(2^k)^{-\gamma_2}2^{-j(n(1/p_1)+(1/p_2))} \]

\[ \times \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |2^{-j}a_j^\gamma(y_1, 2^{-j}y_2)|^{p'_2} (1 + |y_1|^{2^{s_2}p'_2/2} dy_1) \right)^{p'_2/p_2} \right) \]

\[ \leq C(2^k)^{-\gamma_1}(2^k)^{-\gamma_2}2^{-j(n(1/p_1)+(1/p_2))} \left\| \sigma(2^j \psi) \right\|_{W^{n,2}}, \]

(40)

where the last inequality holds due to Lemma 9. Suppose that \(2^{-R} \leq I < 2^{-R+1}\). Since \(n/p_1 + n/p_2 - s_1 - s_2 < 0\), we have

\[ \sum_{j \in \mathbb{R}} I_{k_1,k_2,j} \leq \sup_j \left\| \sigma(2^j \psi) \right\|_{W^{n,2}} \]

On the other hand

\[ I_{k_1,k_2,j} \leq \left( \int_{2^{k_1}+1}^{2^{k_2}+1} \left( \int_{2^{k_1}+1}^{2^{k_2}+1} \right) \left\| \sigma_1^\gamma(y_1 + h, y_2 + h) - \sigma_1^\gamma(y_1, y_2) \right\|^{p_1'} dy_1 \right)^{1/p_1'} \]

\[ \leq C \left( \int_{2^{k_1}+1}^{2^{k_2}+1} \left( \int_{2^{k_1}+1}^{2^{k_2}+1} \right) \left\| \tilde{h} \cdot \nabla \left( \sigma_1^\gamma(y_1 + \theta h, y_2 + \theta h) \right) \right\|^{p_1'} dy_1 \right)^{1/p_1'} \]

(42)

\[ \leq C \left( \int_{2^{k_1}+1}^{2^{k_2}+1} \left( \int_{2^{k_1}+1}^{2^{k_2}+1} \right) \left\| \tilde{h} \cdot \nabla \left( \sigma_1^\gamma(y_1 + \theta h, y_2 + \theta h) \right) \right\|^{p_1'} dy_1 \right)^{1/p_1'} \]

where \(\tilde{h} = (h, h) \in \mathbb{R}^{2n}\). Since \(\tilde{h} \cdot \nabla(\sigma_1^\gamma)(y_1, y_2) = \sum_{r=1}^{2n} h_r \partial_r(\sigma_1^\gamma)(y_1, y_2)\), we have

\[ I_{k_1,k_2,j} \leq \sum_{r=1}^{2n} \left( \int_{2^{k_1}+1}^{2^{k_2}+1} \left( \int_{2^{k_1}+1}^{2^{k_2}+1} \right) \left\| \partial_r(\sigma_1^\gamma)(y_1, y_2) \right\|^{p_1'} dy_1 \right)^{1/p_1'} \]

\[ \leq C \sum_{r=1}^{2n} (2^k)^{-\gamma_1}(2^k)^{-\gamma_2} \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \left\| \partial_r(\sigma_1^\gamma)(2^{-j}y_1, 2^{-j}y_2) \right\|^{p_1'} (1 + |2^{-j}y_2|^{2^{s_2}p'_2/2} 2^{-jn} dy_1) \right)^{p'_2/p_2} \right) \]

\[ \times \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \left\| \partial_r(\sigma_1^\gamma)(2^{-j}y_1, 2^{-j}y_2) \right\|^{p_1'} (1 + |2^{-j}y_2|^{2^{s_2}p'_2/2} 2^{-jn} dy_1) \right)^{p'_2/p_2} \right)^{1/p'_2} \]
\[
\sum_{j \in R} \left| T_\sigma (f_1^{co}, f_2^{co}) (z) - T_\sigma (f_1^0, f_2^{co}) (x) \right| \\
\leq C \sum_{j \in Z, k_j = 0} \int_{2^{k_j} \cdot Q} \left| f_2 (y_2) \right| \left| f_1 (y_1) \right| dy_1 dy_2 \\
\leq C \sup_j \left\| \sigma (2^j \cdot \psi) \right\|_{W^{m,n}} 2^{k_j} 2^{-n/\rho} L^{m \cdot n/\rho} L^{m \cdot n/\rho}. \tag{45}
\]

Combining the above arguments we have

\[
|T_\sigma (f_1^{co}, f_2^{co}) (z) - T_\sigma (f_1^0, f_2^{co}) (x)| \\
\leq \sum_{k_j = 0} \left\| T_\sigma (f_1^{co}, f_2^{co}) (z) - T_\sigma (f_1^0, f_2^{co}) (x) \right\|_{L^1} \\
\leq \sum_{k_j = 0} \left( \int_{2^{k_j} \cdot Q} \left| f_1^0 (y_1) \right| \left| f_2 (y_2) \right| dy_1 dy_2 \right)^{1/p_1} \\
\times \left( \int_{2^{k_j} \cdot Q} \left| f_2 (y_2) \right|^{p_2} dy_2 \right)^{1/p_2}. \tag{46}
\]

Then by similar arguments as the above mentioned we get that

\[
\left| T_\sigma (f_1^0, f_2^{co}) (z) - T_\sigma (f_1^0, f_2^{co}) (x) \right| \\
\leq \sum_{k_j = 0} \left( \int_{Q^*} \left| f_1 (y_1) \right|^{p_1} dy_1 \right)^{1/p_1} \\
\times \left( \int_{2^{k_j} \cdot Q^*} \left| f_1 (y_1) \right|^{p_2} dy_2 \right)^{1/p_2}. \tag{47}
\]

The proof of Lemma 10 is complete."
Now we are ready to give the proof of Theorem 2.

**Proof.** By Lemma 3, we can choose $1 < p_1 < q_1$ and $1 < p_2 < q_2$ such that $\omega_1^{p_1} \in A_\beta^{q_1}$ and $\omega_2^{p_2} \in A_\beta^{q_2}$. Then by the Hölder inequality, Lemma 10, and the weighted boundedness of $M$, we deduce that

\[
\left\| T_\sigma (f_1, f_2) \right\|_{L^q(\omega)} \leq \left\| M_{\delta} T_\sigma (f_1, f_2) \right\|_{L^q(\omega)} \\
\leq C \left\| M_{\delta} f_1 M_{\delta} f_2 \right\|_{L^q(\omega)} \\
\leq C \left\| \frac{M_{\delta} f_1}{M_{\delta} f_2} \right\|_{L^q(\omega)} \left\| M_{\delta} f_1 \right\|_{L^q(\omega)} \\
\leq C \left\| f_1 \right\|_{L^{q_1}(\omega_1^{p_1})} \left\| f_2 \right\|_{L^{q_2}(\omega_2^{p_2})}. \tag{48}
\]

The proof of Theorem 2 is complete.

**Conflict of Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.

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**References**


