Research Article

Continuity of the Restriction Maps on Smirnov Classes

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We prove the restriction maps define continuous linear operators on the Smirnov classes for some certain domain with analytic boundary.

1. Introduction

As usual, we define the Hardy space $H^2 = H^2(\Delta)$ as the space of all functions $f : z \rightarrow \sum_{n=0}^{\infty} a_n z^n$ for which the norm $(\|f\| = \sum_{n=0}^{\infty} |a_n|^2)^{1/2}$ is finite. Here, Δ is the open unit disc. For a more general simply connected domain D in the sphere or extended plane $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with at least two boundary points, and a conformal mapping φ from D onto Δ (i.e., a Riemann mapping function, abbreviation is RMF), a function g analytic in D is said to belong to the Smirnov class $E^2(D)$ if and only if $g = (f \circ \varphi)\varphi'^{1/2}$ for some $f \in H^2(\Delta)$ where $\varphi'^{1/2}$ is an analytic branch of the square root of φ' . The reader is referred to [1–7] and references therein for the basic properties of these spaces.

Let $C = (C_1, C_2, C_3, \dots, C_N)$ be an N -tuple of closed distinct curves on the sphere $\bar{\mathbb{C}}$ and suppose that, for each i , $1 \leq i \leq N$, C_i is a circle, a line $\cup \{\infty\}$, an ellipse, a parabola $\cup \{\infty\}$, or a branch of a hyperbola $\cup \{\infty\}$. Let D_i be the complementary domain of C_i . Recall that a complementary domain of a closed $F \subseteq \bar{\mathbb{C}}$ is a maximal connected subset of $\bar{\mathbb{C}} - F$, which must be a domain. For $1 \leq i \leq N$, suppose that $\varphi_i : D_i \rightarrow \Delta$ is a conformal equivalence (i.e., RMF) and let $\psi_i : \Delta \rightarrow D_i$ be its inverse. For $1 \leq i \leq N$, let us keep the notations of C_i , D_i , φ_i , ψ_i fixed until the end of the paper.

In this paper we prove the following.

Theorem 1. *Let $1 \leq i, j \leq N$. Suppose that Γ is an open subarc of C_j and suppose also that $\Gamma \subseteq D_i$ if $i \neq j$. Then the restriction $f \rightarrow f|_{\Gamma}$ defines a continuous linear operator mapping $E^2(D_i)$ into $L^2(\Gamma)$.*

For similar work regarding restriction maps, see [8, 9]. Our conjecture is that Theorem 1 is valid if, for each j , $1 \leq j \leq N$, C_j is a σ -rectifiable analytic Jordan curve.

There are some similar results for rectifiable curves in Havin's paper [10]. Also the Cauchy projection operator from L^p to E^p is bounded on all Carleson regular curves; compare the papers of David, starting with [11].

We need the following Theorem to simplify the proof of Theorem 1.

Theorem 2 (Theorem 1 in [12]). *Let D be a complementary domain of $\cup_{i=1}^N C_i$ and suppose that D is simply connected so that D_i is the complementary domain of C_i which contains D . Then*

- (i) ∂D is a σ -rectifiable closed curve and every $f \in E^2(D)$ has a nontangential limit function $\tilde{f} \in L^2(\partial D)$;
- (ii) (Parseval's identity) the map $f \rightarrow \tilde{f} (E^2(D) \rightarrow L^2(\partial D))$ is an isometric isomorphism onto a closed subspace $E^2(\partial D)$ of $L^2(\partial D)$, so

$$\|f\|_{E^2(D)}^2 = \|\tilde{f}\|_{L^2(\partial D)}^2 = \frac{1}{2\pi} \int_{\partial D} |\tilde{f}(z)|^2 |dz|, \quad (1)$$

$$(f \in E^2(D)).$$

If $\Gamma \subseteq C_i$ is an open subarc, then

$$\|\tilde{f}|_{\Gamma}\|_{L^2(\Gamma)}^2 \leq \|\tilde{f}|_{C_i}\|_{L^2(C_i)}^2 = \|f\|_{E^2(D_i)}^2, \quad (2)$$

because Parseval's identity is true for the trivial chain (C_i) of curves. Hence Theorem 1 will be proved if the following theorem can be proved.

Theorem 3. *Let $1 \leq i \neq j \leq N$. Suppose that Γ is an open subarc of C_j and that $\Gamma \subseteq D_i$. Then the restriction $f \rightarrow f|_\Gamma$ defines a continuous linear operator mapping $E^2(D_i)$ into $L^2(\Gamma)$.*

2. Preliminaries for the Proof of Theorem 3

Let us keep the notation of Theorem 3 fixed for the rest of the paper and let us also agree to use l for arc-length measure.

An arc or closed curve γ is called σ -rectifiable if and only if it is a countable union of rectifiable arcs in $\mathbb{C}$, together with (∞) in the case when $\infty \in \gamma$. For instance, a parabola without ∞ is σ -rectifiable arc, and a parabola with ∞ is σ -rectifiable Jordan curve. The following definition will simplify the language.

Definition 4. Let $\gamma \subseteq \mathbb{C}$ be a simple σ -rectifiable arc contained in a simply connected domain $G \subseteq \overline{\mathbb{C}}$. We say that γ has the restriction property in G if and only if the map $g \rightarrow g|_\gamma$ defines a continuous linear operator mapping $E^2(G)$ into $L^2(\gamma)$.

Thus, the last sentence of Theorem 3 reads “ Γ has the restriction property in D_i .”

Lemma 5 (Invariance Lemma (Lemma 4 in [9])). *Let $G_1, G_2 \subseteq \overline{\mathbb{C}}$ be simply connected domains and suppose that $\gamma_1 \subseteq G_1 \cap \mathbb{C}$, $\gamma_2 \subseteq G_2 \cap \mathbb{C}$ are simple σ -rectifiable arcs. If $\chi : G_1 \rightarrow G_2$ is a conformal equivalence onto G_2 and $\chi(\gamma_1) = \gamma_2$, then γ_1 has the restriction property in G_1 if and only if γ_2 has the restriction property in G_2 .*

Corollary 6. *Theorem 3 is true; that is, Γ has the restriction property in D_i , if and only if $\varphi_i(\Gamma)$ has the restriction property in Δ , for some RMF $\varphi_i : D_i \rightarrow \Delta$.*

A subarc γ of Γ has the restriction property in D_i if and only if $\varphi_i(\gamma)$ has the restriction property in Δ . Corollary 6 will be used in the following way. Γ will be written as the union of finitely many subarcs and we will show that each of these subarcs has the restriction property in D_i ; it will then follow that Γ itself has the required restriction property. Three different kinds of subarc will be considered.

Definition 7. A subarc $\gamma \subseteq \Gamma$ is said to be of type I if and only if $\bar{\gamma} \subseteq D_i$ (i.e., both of its end-points a, b belong to D_i).

Lemma 8 (Lemma 6 in [9]). *Let γ be a subarc of Γ and suppose that φ_i, θ_i are Riemann mapping functions for D_i .*

- (i) $\varphi_i(\gamma)$ has the restriction property in Δ if and only if $\theta_i(\gamma)$ has the restriction property in Δ ;
- (ii) $\varphi_i(\gamma)$ is rectifiable if and only if $\theta_i(\gamma)$ is rectifiable;
- (iii) if γ is of type I, then $\overline{\varphi_i(\gamma)} \subseteq \Delta$ and $\varphi_i(\gamma)$ is rectifiable;
- (iv) if γ is of type I, it has the restriction property in D_i .

FIGURE 1: Type I, II, and III arcs.

We can now “ignore” subarcs of Γ whose closure (in $\overline{\mathbb{C}}$) is contained in D_i . We will now restrict our attention to subarcs of Γ with a single end-point $a \in \partial D_i$, the other being in D_i . There are two types, depending on whether $a \in \mathbb{C}$ or $a = \infty$.

Definition 9. (i) An open subarc γ of Γ is of type II if and only if it has an end-point $a \in \partial D_i \cap \mathbb{C}$ and $\bar{\gamma} - (a) \subseteq D_i \cap \mathbb{C}$.

(ii) In the case where C_i is unbounded (so that $\infty \in \partial D_i$) an open subarc $\gamma \subseteq \Gamma$ is of type III if and only if ∞ is an end-point of γ and $\bar{\gamma} - (\infty) \subseteq D_i$.

Modulo a finite subset of D_i , Γ is the union of at most three open subarcs, each of which is of type I, II, or III; see Figure 1.

If γ is a type II or type III subarc of Γ then $\varphi_i(\gamma)$ is a simple open analytic arc in Δ with one end-point on the circle $\mathbb{T}$ and the other in Δ . We will show that $\varphi_i(\gamma)$ has the restriction property in Δ using the powerful Carleson theorem (Theorem II below).

Definition 10 (see [1, p.157]). For $0 < h < 1$ and $0 \leq \theta < 2\pi$, let $C_{\theta h} = \{z \in \mathbb{C} : 1-h \leq |z| \leq 1, \theta \leq \arg z \leq \theta+h\}$. A positive regular Borel measure μ on Δ is called a Carleson measure if there exists a positive constant M such that $\mu(C_{\theta h}) \leq Mh$, for every h and every θ .

Theorem II (see [1, p. 157, Theorem 9.3] or see [13, p. 37]). *Let μ be a finite positive regular Borel measure on Δ . In order that there exists a constant $C > 0$ such that*

$$\int_{\Delta} |f(z)|^2 d\mu(z) \leq C \|f\|^2, \quad \forall f \in H^2(\Delta), \quad (3)$$

it is necessary and sufficient that μ be a Carleson measure.

To complete the proof of Theorem 3 it is sufficient to show that arc-length measure on $\varphi_i(\gamma)$ is a Carleson measure whenever γ is of type II or III.

It will be useful to use arc-length to parametrize γ and $\varphi_i(\gamma)$. Recall that a compact arc σ is called *smooth* if there exists some parametrization $g : [a, b] \rightarrow \sigma$ such that $g \in$

$C^1[a, b]$ and $g'(t) \neq 0, \forall t \in [a, b]$. Note that if σ is smooth, then it is rectifiable; that is,

$$l(\sigma) = \int_a^b |g'(t)| dt < \infty. \quad (4)$$

To define the arc-length parametrization of σ put $s = \int_a^t |g'(u)| du$ for $a \leq t \leq b$ so that $0 \leq s \leq \ell(\sigma)$. Then $s'(t) = |g'(t)|$ and $t \rightarrow s(t)$ ($[a, b] \rightarrow [0, \ell]$) is C^1 with strictly positive derivative. Hence also its inverse $s \rightarrow t(s)$ ($[0, \ell] \rightarrow [a, b]$) is C^1 with strictly positive derivative. Recall that the arc-length parametrization of the smooth arc σ is the map $h : [0, \ell] \rightarrow \sigma$ satisfying $h(s) = \{\text{the point on } \sigma \text{ length } s \text{ from the initial point } (g(a))\}$; that is, $h(s) = g(t(s))$ $0 \leq s \leq \ell$.

Since $h'(s) = g'(t(s))t'(s)$, $h \in C^1[0, \ell]$, with nonzero derivative, necessarily $|h'(s)| = 1$ since

$$h'(s(t)) = g'(t) t'(s) = \frac{g'(t)}{s'(t)} = \frac{g'(t)}{|g'(t)|}. \quad (5)$$

We need the following lemma.

Lemma 12 (Theorem 1 in [14]). *Let $\sigma \subseteq \bar{\Delta}$ be a smooth simple arc with arc-length parametrization $g \in C^1[0, \ell]$. Suppose that $|g(0)| = 1$, $|g(s)| < 1$ for $0 < s \leq \ell$. Then arc-length measure on $\sigma \cap \Delta$ is a Carleson measure; hence $\sigma \cap \Delta$ has the restriction property in Δ .*

3. Type II Subarcs

The following lemma gives the continuity of the restriction map for finite end-points.

Lemma 13. *A type II arc $\gamma \subseteq \Gamma \subseteq D_i$ has the restriction property in D_i .*

Proof. By Lemmas 12 and 5 it is sufficient to show that $\overline{\varphi_i(\gamma)}$ is a smooth arc in $\bar{\Delta}$. Suppose that γ has end-points $a \in \partial D_i \cap \mathbb{C}$ and $b \in D_i \cap \mathbb{C}$, so that $\bar{\gamma} = \gamma \cup (a) \cup (b)$. Clearly $\bar{\gamma}$ is a smooth arc. Because C_i is an open analytic arc, φ_i can be continued analytically into a neighbourhood U of a so as to be conformal in $D_i \cup U$. This means that φ_i is conformal in a neighbourhood of $\bar{\gamma}$ and so $\overline{\varphi_i(\gamma)} = \varphi_i(\bar{\gamma})$ is a smooth arc in $\bar{\Delta}$ with $|\varphi_i(a)| = 1$ and $\varphi_i(\bar{\gamma} - (a)) \subseteq \Delta$. The result now follows from Lemmas 12 and 5. $\square$

We have now made a good deal of progress because of the following.

Lemma 14. *Theorem 3 is true if C_i is a circle or an ellipse.*

Proof. In this case Γ is a finite union of type I and type II arcs only, so the result follows by Lemma 8(iv) and Lemma 13. $\square$

4. Type III Subarcs

The proof of Theorem 3 will be completed by showing that every type III arc in D_i has the restriction property in D_i . We

have an open subarc γ of an open subarc Γ of C_j and $\Gamma \subseteq D_i$. In this case ∞ is an end-point of γ and $\infty \in \partial D_i$, so both C_i and C_j are unbounded. We will use the same strategy we used for type II arcs in Lemma 13; we show that $\sigma = \overline{\varphi_i(\gamma)}$ is a smooth arc in Δ as in Lemma 12, so that $\varphi_i(\gamma)$ has the restriction property in Δ and so γ has the restriction property in D_i . The proof is more complicated because conformality of φ_i at ∞ cannot necessarily be used. Instead we make use of the fact that as $z \rightarrow \infty$ along γ , the unit tangent vector of γ at z tends to a limit. The following two Lemmas help us exploit this fact.

Lemma 15. *Let $g \in C^1[0, \infty)$ with $g'(t) \neq 0$ ($t \geq 0$). Suppose that $c \in \mathbb{C}$ and*

$$\lim_{t \rightarrow \infty} g(t) = c, \quad (6)$$

$$\lim_{t \rightarrow \infty} \frac{g'(t)}{|g'(t)|} = \omega, \quad (|\omega| = 1)$$

exist. Define $\sigma = g([0, \infty)) \cup (c)$. Then

- (i) σ is a compact arc,
- (ii) σ is rectifiable,
- (iii) σ is smooth.

Proof. (i) Define f on $[0, 1]$ by

$$f(t) = \begin{cases} g(\tanh^{-1} t) & 0 \leq t < 1 \\ c & t = 1. \end{cases} \quad (7)$$

Then $f \in C[0, 1]$ is a continuous parametrization of σ .

(ii) To prove that σ is rectifiable, it suffices to show that, for some $T > 0$, $\int_T^\infty |g'(u)| du < \infty$. Let $\varepsilon(t) = \omega - (g'(t)/|g'(t)|)$. So $\varepsilon(t) \rightarrow 0$ as $t \rightarrow \infty$. Choose $T \geq 0$ such that $|\varepsilon(t)| \leq 1/2$ for $t \geq T$. Then, for $t \geq T$,

$$|g'(t)| (1 - \bar{\omega}\varepsilon(t)) = \bar{\omega}g'(t). \quad (8)$$

Hence

$$\int_T^t |g'(u)| (1 - \bar{\omega}\varepsilon(u)) du = \bar{\omega}(g(t) - g(T)), \quad (t > T),$$

$$|\varepsilon| \leq \frac{1}{2} \implies \operatorname{Re}(1 - \bar{\omega}\varepsilon) \geq \frac{1}{2} \implies 2 \operatorname{Re}(1 - \bar{\omega}\varepsilon) \geq 1. \quad (9)$$

So

$$\begin{aligned} & \int_T^t |g'(u)| du \\ & \leq 2 \int_T^t |g'(u)| \operatorname{Re}(1 - \bar{\omega}\varepsilon(u)) du \\ & = 2 \operatorname{Re}(\bar{\omega}(g(t) - g(T))) \\ & \longrightarrow 2 \operatorname{Re}(\bar{\omega}(c - g(T))) \quad \text{as } t \longrightarrow \infty, \end{aligned} \quad (10)$$

FIGURE 2: The sector S.

and hence

$$\int_T^\infty |g'(u)| du < \infty, \quad (11)$$

which establishes the rectifiability of σ .

(iii) Let $h : [0, \ell] \rightarrow \sigma$ be the arc-length parametrization of σ . Then $h \in C[0, \ell]$, $h(s) = g(t)$ where $\int_0^t |g'(u)| du = s$ and $s'(t) = |g'(t)|$. Therefore the map $t \rightarrow s$ ($[0, \infty) \rightarrow [0, \ell]$) is C^1 with strictly positive derivative. So the inverse map $s \rightarrow t$ ($[0, \ell] \rightarrow [0, \infty)$) is C^1 . Since $t(s(t)) \equiv t$ and $t'(s) = 1/s'(t)$ where $0 \leq t \leq \infty$ and $0 \leq s \leq \ell$, it follows that

$$\lim_{s \rightarrow \ell} h'(s) = \lim_{t \rightarrow \infty} g'(t) t'(s) = \lim_{t \rightarrow \infty} \frac{g'(t)}{s'(t)} = \lim_{t \rightarrow \infty} \frac{g'(t)}{|g'(t)|} = \omega. \quad (12)$$

Hence h' is continuous and so $h \in C^1[0, \ell]$. $\square$

Lemma 16. Let $k \in C^1[0, \infty)$ with $k'(t) \neq 0$ ($t \geq 0$) and suppose that $k(t) \rightarrow \infty$ as $t \rightarrow +\infty$. Then, if $|\omega| = 1$,

$$\frac{k'(t)}{|k'(t)|} \rightarrow \omega \implies \frac{k(t)}{|k(t)|} \rightarrow \omega. \quad (13)$$

Proof. Write $\omega = e^{i\alpha}$. Choose T' such that $t \geq T' \implies \operatorname{Re} e^{-i\alpha}(k'(t)/|k'(t)|) > 0$. Then using $\widehat{\arg}$ to denote the principal value of $\arg$ we see that

$$\theta(t) = \alpha + \widehat{\arg} e^{-i\alpha} \frac{k'(t)}{|k'(t)|} \quad (14)$$

is a branch of $\arg(k'/|k'|)$ and hence also of $\arg k'$ on $[T', \infty)$ which tends to α as $t \rightarrow \infty$. We will find a branch ϑ of $\arg k$ which also tends to α as $t \rightarrow \infty$.

Let $\varepsilon > 0$. Choose T such that $t \geq T \geq T' \implies \alpha - \varepsilon/2 \leq \theta \leq \alpha + \varepsilon/2$. Now $k(t) - k(T) = \int_T^t k'(u) du$ is a limit of Riemann sums $\sum (t_{i+1} - t_i) k'(\xi_i)$.

The sector S (see Figure 2) is closed under addition and multiplication by positive scalars; therefore

$$k(t) - k(T) \in S \quad \text{for } t \geq T. \quad (15)$$

So there is an argument $\mu(t)$ of $k(t) - k(T)$ satisfying

$$\alpha - \frac{\varepsilon}{2} \leq \mu(t) \leq \alpha + \frac{\varepsilon}{2} \quad (t \geq T). \quad (16)$$

Now $k(t)/(k(t) - k(T)) \rightarrow 1$ as $t \rightarrow \infty$. So

$$\exists T_1 \geq T \quad \text{such that } t \geq T_1 \implies -\frac{\varepsilon}{2} < \widehat{\arg} \frac{k(t)}{k(t) - k(T)} < \frac{\varepsilon}{2}. \quad (17)$$

If we define

$$\vartheta(t) = \mu(t) + \widehat{\arg} \frac{k(t)}{k(t) - k(T)} \quad (t \geq T_1), \quad (18)$$

then $\vartheta(t)$ is an argument of $k(t)$ and

$$t \geq T_1 \implies |\vartheta(t) - \alpha| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad (19)$$

Hence also

$$\left| \frac{k(t)}{|k(t)|} - e^{i\alpha} \right| = |e^{i\vartheta(t)} - e^{i\alpha}| < \varepsilon. \quad (20)$$

Consequently,

$$\frac{k(t)}{|k(t)|} \rightarrow \omega = e^{i\alpha}, \quad (21)$$

and our Lemma is proved. $\square$

There are now four cases to prove depending on the geometry of C_i and D_i .

4.1. Case 1: D_i Is a Half-Plane. The following lemma will be needed here and in Case 2.

Lemma 17. Let G be the open right half-plane $\operatorname{Re} z > 0$ and let $\theta(z) = (z-1)/(z+1)$ so that θ is a Riemann mapping function for G . Let $k : [0, \infty) \rightarrow G$ be an injective C^1 function such that $k'(t) \neq 0$, for all $t \geq 0$, and $\lim_{t \rightarrow \infty} k(t) = \infty$. Let ρ be the (simple) arc parametrized by k . If $\lim_{t \rightarrow \infty} (k'(t)/|k'(t)|) = \omega$ (with $|\omega| = 1$), then $\sigma = \theta(\rho)$ satisfies the hypothesis of Lemma 12 and, hence, ρ has the restriction property in G .

Proof. Put $g = \theta \circ k$, so that $g \in C^1[0, \infty)$ parametrizes $\theta(\rho)$. Clearly $g(t) \rightarrow 1$ as $t \rightarrow \infty$. Now g satisfies the hypothesis of Lemma 15, for we can show that $g'(t)/|g'(t)| \rightarrow \omega^{-1}$ as $t \rightarrow \infty$. Since $\theta'(z) = 2/(z+1)^2$ it follows that

$$\begin{aligned} \frac{g'(t)}{|g'(t)|} &= \frac{|1+k(t)|^2}{(1+k(t))^2} \frac{k'(t)}{|k'(t)|} \\ &= \frac{k'(t)}{|k'(t)|} \frac{|k(t)|^2}{(k(t))^2} \frac{|1+1/k(t)|^2}{(1+1/k(t))^2} \\ &\rightarrow \omega^{-1}, \end{aligned} \quad (22)$$

using Lemma 16.

So $\sigma = g[0, \infty) \cup (\omega^{-1})$ satisfies Lemma 12; hence $g[0, \infty)$ has the restriction property in Δ . But $g[0, \infty) = \theta(\rho)$ and, therefore, by Lemma 5, ρ has the restriction property in G . $\square$

Now suppose that C_i is a line and D_i is a half-plane. By Invariance Lemma 5 with a linear equivalence $\chi(z) = \alpha z + \beta$ ($\alpha \neq 0$) we can assume that C_i is the imaginary axis and that $D_i = G$, the open right half-plane, as above. If $\gamma \subseteq D_i$ is a type III arc, it is a subarc of a line, parabola, or hyperbola component. Obviously γ has a parametrization k as in Lemma 17. Hence γ has the restriction property in D_i .

4.2. Case 2: D_i Is the Concave Complementary Domain of a Parabola. Any two parabolas are conformally equivalent via a linear equivalence: $\mu(z) = az + b$ ($a, b \in \mathbb{C}, a \neq 0$). So assume that C_i is the parabola

$$y^2 = 4(1 - x) \quad (23)$$

and that D_i is the complementary domain to the “right” of C_i . The function

$$w \longrightarrow (1 + w)^2 \quad (24)$$

maps the open right half-plane G conformally onto D_i and the imaginary axis onto C_i . Its inverse is the function

$$\vartheta(z) = z^{1/2} - 1, \quad (z \in D_i), \quad (25)$$

where $z^{1/2}$ is the principal square-root of z (here and throughout all standard multivalued functions will take their principal values).

Now let $\gamma \subseteq D_i$ be a type III arc. Because G is conformally equivalent to D_i via ϑ it will be sufficient to show that the arc $\vartheta(\gamma) \subseteq G$ has a parametric function k as in Lemma 17. Letting h be the arc-length parametrization of γ , then $h \in C^1[0, \infty)$, $|h'(t)| \equiv 1$ and $h(t) \rightarrow \infty$ as $t \rightarrow \infty$, and h is injective.

Now γ is a subarc of a line, parabola, or hyperbola component. Hence as $z \rightarrow \infty$ along γ the unit tangent vector at z tends to a limit ($|\omega| = 1$). Thus

$$\lim_{t \rightarrow \infty} \frac{h'(t)}{|h'(t)|} = \lim_{t \rightarrow \infty} h'(t) = \omega, \quad (26)$$

and therefore

$$\lim_{t \rightarrow \infty} \frac{h(t)}{|h(t)|} = \omega, \quad (27)$$

by Lemma 16.

Put $k = \vartheta \circ h$. Then k is an injective parametric function for $\vartheta(\gamma)$. Clearly $k \in C^1[0, \infty)$, $k(t) \rightarrow \infty$ as $t \rightarrow \infty$, and

$$k'(t) = \vartheta'(h(t)) h'(t) \neq 0, \quad \forall t \geq 0. \quad (28)$$

Moreover,

$$\frac{k'(t)}{|k'(t)|} = \frac{|h(t)|^{1/2}}{h(t)^{1/2}} \frac{h'(t)}{|h'(t)|} \longrightarrow \omega^{1/2}. \quad (29)$$

So k is as in Lemma 17, which shows that γ has the restriction property in D_i .

Remark 18. The notation $\omega^{1/2}$ is ambiguous when $\omega = -1$ (γ could be part of another parabola). But, because type I arcs can be ignored, we can assume that either γ is contained entirely in the upper half-plane, in which case $(-1)^{1/2} = i$, or else γ is in the lower half-plane and $(-1)^{1/2} = -i$.

4.3. Case 3: D_i Is the Convex Complementary Domain of a Parabola. In this case the parabola

$$y^2 = 4\left(\frac{\pi}{4}\right)^2 \left(\left(\frac{\pi}{4}\right)^2 - x\right) \quad (30)$$

will be chosen for C_i , and D_i will be the complementary domain to the “left” of C_i . This choice is made because then we have the relatively simple Riemann mapping function

$$\varphi_i(z) = \tan^2(z^{1/2}), \quad (z \in D_i). \quad (31)$$

This function maps the real interval $(-\infty, (\pi/4)^2)$ in an increasing fashion onto $(-1, 1)$, and so it maps the upper/lower half of D_i onto the upper/lower half of Δ . The formula for φ_i is indeterminate on $(-\infty, 0]$, but these singularities are removable and the formula

$$\varphi_i(x) = -\tanh^2(-x)^{1/2} \quad (32)$$

can be used to define $\varphi_i(x)$, for negative x . This mapping will be examined in detail in a moment, but first we dispose of a trivial case and make some simple observations.

Let $\gamma \subseteq D_i$ be a type III arc. If γ is a real interval $(-\infty, a)$, with $a < (\pi/4)^2$, then $\varphi_i(\gamma)$ is a subinterval of $(-1, 1)$ which obviously has the restriction property in Δ . So this case is trivial and needs no more attention.

The following observations are elementary.

- (i) If γ is part of another line, then it must be parallel to $\mathbb{R}$ and certainly disjoint from $(-\infty, 0]$.
- (ii) If γ is part of another parabola C_j , then C_j must be symmetric about $\mathbb{R}$ and have an equation of the form

$$y^2 = 4a(b - x), \quad (33)$$

where $0 < a \leq (\pi/4)^2$, $b \leq (\pi/4)^2$.

- (iii) If γ is part of a hyperbola, then its asymptote must be parallel to $\mathbb{R}$.
- (iv) In all (nontrivial) cases γ intersects $(-\infty, 0]$ in at most two points. So, because type I arcs can be ignored there is no loss of generality in assuming that $\text{Im } z$ has constant sign on γ and that $\text{Re } z < 0$ on γ .
- (v) Hence, for definiteness, we can assume that γ is contained in the open second quadrant.
- (vi) In all cases y^2/x tends to a limit as $z \rightarrow \infty$ along γ . If γ is part of a line or hyperbola, the limit is 0, and if γ is part of the parabola in (ii) above the limit is $-4a$. For future reference let us note that

$$0 \leq \lim_{x \rightarrow \infty} \frac{y^2}{4|x|} \leq \left(\frac{\pi}{4}\right)^2. \quad (34)$$

- (vii) Because the lim in (34) exists and because type I arcs can be ignored, we can assume that

$$\frac{y^2}{x^2} < 1, \quad \text{on } \gamma. \quad (35)$$

Now let γ be type III arc in D_i as in (v) and (vi). We will show that $\varphi_i(\gamma)$ has the restriction property in Δ . To elucidate $\varphi_i(\gamma)$ it is convenient to work backwards, examining the mapping properties of the square map ($z \rightarrow z^2$), then tan, and then the principal square root.

Lemma 19. *Let Δ^+ be the open semidisc*

$$\Delta^+ = \{z \in \mathbb{C} : |z| < 1, x > 0\}. \quad (36)$$

If σ' is a smooth simple arc in $\overline{\Delta^+}$, if i is an end-point of σ' , and if $\sigma' - \{i\} \subseteq \Delta^+$, then the arc

$$\sigma = \{z^2 : z \in \sigma'\} \quad (37)$$

is a smooth simple arc in $\overline{\Delta}$ satisfying the hypothesis of Lemma 12, so that $\sigma - \{-1\}$ has the restriction property in Δ .

Proof. This is clear: the square map $z \rightarrow z^2$ is conformal in a neighbourhood of σ' . $\square$

Now let S be the open strip

$$S = \left\{z \in \mathbb{C} : 0 < x < \frac{\pi}{4}\right\}. \quad (38)$$

It is well known that $\tan$ maps S conformally onto Δ^+ . The imaginary axis is mapped to the vertical part of $\partial\Delta^+$, and the line $\pi/4 + i\mathbb{R}$ is mapped to the semicircular part of $\partial\Delta^+$. Moreover, if z tends to infinity in S in such a way that $y \rightarrow +\infty$, then $\tan z \rightarrow i$.

Lemma 20. *Let $k \in C^1[0, \infty)$ be injective and satisfy $k'(t) \neq 0$, for $t \geq 0$. Suppose also that*

- (i) $k(t) \in S$ for all $t \geq 0$,
- (ii) $\operatorname{Im} k(t) \rightarrow +\infty$ as $t \rightarrow +\infty$,
- (iii) $\lim_{t \rightarrow \infty} \operatorname{Re} k(t) = x_0$ exists ($0 \leq x_0 \leq \pi/4$),
- (iv) $\lim_{t \rightarrow \infty} (k'(t)/|k'(t)|) = i$.

If γ' is the arc parametrized by k , then $\sigma' = (\tan \gamma') \cup \{i\}$ satisfies the hypothesis of Lemma 19, so that $\tan^2 \gamma'$ has the restriction property in Δ .

Proof. Let $g = \tan \circ k$, so that g parametrizes γ' and $\tan \gamma' = g[0, \infty)$. Now $g \in C^1[0, \infty)$, $g'(t) \neq 0$, for all $t \geq 0$, and $g(t) \rightarrow i$ as $t \rightarrow +\infty$. Lemma 15 will be used to show that $\sigma' = g[0, \infty) \cup \{i\}$ satisfies the hypothesis of Lemma 19. For all $t \geq 0$,

$$\frac{g'(t)}{|g'(t)|} = \frac{|\cos k(t)|^2}{(\cos k(t))^2} \frac{k'(t)}{|k'(t)|}. \quad (39)$$

Let $k(t) = x(t) + iy(t)$. Since $x(t) \rightarrow x_0$ and $y(t) \rightarrow +\infty$, as $t \rightarrow +\infty$, and because $\cos x, \cosh y > 0$ on γ ,

$$\begin{aligned} \frac{|\cos k(t)|^2}{\cos^2 k(t)} &= \left(\frac{|\cos x(t) \cosh y(t) - i \sin x(t) \sinh y(t)|}{\cos x(t) \cosh y(t) - i \sin x(t) \sinh y(t)} \right)^2 \\ &= \frac{|1 - i \tan x(t) \tanh y(t)|^2}{(1 - i \tan x(t) \tanh y(t))^2} \\ &\rightarrow \frac{|1 - i \tan x_0|^2}{(1 - i \tan x_0)^2}. \end{aligned} \quad (40)$$

So $\lim_{t \rightarrow \infty} (g'(t)/|g'(t)|)$ exists. $\square$

The function

$$\vartheta(z) = z^{1/2} \quad (41)$$

maps $D_i - (-\infty, 0]$ conformally onto the vertical strip S as above. The limiting values of ϑ from above and below a point x on $(-\infty, 0]$ are at $\pm i(-x)^{1/2}$, respectively. Now $\tan$ maps S conformally onto Δ^+ and $\tan \pm i(-x)^{1/2} = \pm i \tanh(-x)^{1/2}$. Finally the square function maps Δ^+ conformally onto $\Delta - ((-1, 0])$, and it maps both of $\pm i \tanh(-x)^{1/2}$ and $-\tanh^2(-x)^{1/2}$. Thus the cut made by ϑ is repaired by the square function (by Schwarz's Reflection Principle): φ_i is continuous at all points of $(-\infty, 0]$ and therefore analytic on D_i . Because $\varphi_i(z) \in (-1, 0]$ if and only if $z \in (-\infty, 0]$ the injectivity of φ_i on D_i is clear.

Let $\gamma \subseteq D_i$ be a type III arc. Assume that $y > 0$ and $x < 0$ when $z = x + iy \in \gamma$. Let $\gamma' = \vartheta(\gamma)$ so that $\gamma' \subseteq S$. We show that γ' is as in Lemma 20 so that $\tan^2 \gamma'$ has the restriction property in Δ and, hence, γ has the restriction property in D_i .

Let $z = x + iy$ be an arbitrary point of γ and write

$$z^{1/2} = u + iv, \quad (42)$$

for the corresponding point $\vartheta(z) \in \gamma'$; then

$$x + iy = u^2 - v^2 + 2iuv. \quad (43)$$

Eliminating v , and remembering that $x < 0$, we see that

$$\begin{aligned} u^2 &= \frac{1}{2} \left(x + (x^2 + y^2)^{1/2} \right) \\ &= \frac{|x|}{2} \left(\left(1 + \frac{y^2}{x^2} \right)^{1/2} - 1 \right). \end{aligned} \quad (44)$$

Since $y^2/x^2 < 1$ (observation (vii)), the binomial series implies that

$$\begin{aligned} u^2 &= \frac{y^2}{4|x|} - \frac{1}{16} \frac{y^4}{|x|^3} + \cdots \\ &\sim \frac{y^2}{4|x|}, \end{aligned} \quad (45)$$

as z tends to ∞ along γ . It follows from (34) that

$$\lim_{t \rightarrow \infty} u^2 = a \text{ exists, } 0 \leq a \leq \left(\frac{\pi}{4}\right)^2. \quad (46)$$

Now let h be the arc-length parametrization of γ and write $h(t) = x(t) + iy(t)$. Let $k = \vartheta \circ h = h^{1/2}$ so that k parametrizes γ' . Write $k(t) = u(t) + iv(t)$. (i), (ii), (iii), and (iv) of Lemma 20 can now be verified.

Obviously $k(t) \in S$, for all $t \geq 0$, so (i) is true. As $t \rightarrow \infty$, $|k(t)| = |h(t)|^{1/2} \rightarrow \infty$, but since $0 \leq u(t) \leq \pi/4$ we must have $v(t) \rightarrow +\infty$, so that (ii) is true. Item (iii) follows from (46). Now $h(t) \rightarrow \infty$ as $t \rightarrow \infty$, $|h'(t)| \equiv 1$, and $h'(t) \rightarrow -1$ as $t \rightarrow \infty$. So, by Lemma 16,

$$\lim_{t \rightarrow \infty} \frac{k'(t)}{|k'(t)|} = \frac{|h(t)|^{1/2} h'(t)}{h(t)^{1/2} |h'(t)|} \rightarrow -i(-1) = i. \quad (47)$$

So (iv) is true and we have now completed the proof.

4.4. Case 4: C_i Is a Hyperbola Component. We can deal simultaneously with the convex and concave complementary domains of a hyperbola component as follows. Let $-\pi/2 < \alpha < \pi/2$ and let $C_i = \sin(\alpha + i\mathbb{R})$. If $\alpha < 0$, C_i is the arc

$$C_i = \left\{ z = x + iy \in \mathbb{C} : x < 0, \frac{x^2}{\sin^2 \alpha} - \frac{y^2}{\cos^2 \alpha} = 1 \right\}, \quad (48)$$

and if $\alpha > 0$, C_i is the arc

$$C_i = \left\{ z = x + iy \in \mathbb{C} : x > 0, \frac{x^2}{\sin^2 \alpha} - \frac{y^2}{\cos^2 \alpha} = 1 \right\}. \quad (49)$$

Let D_i be the complementary domain to the “left” of C_i ; then D_i is convex when $\alpha < 0$ and concave when $\alpha > 0$. Linear equivalence will be used as before to reduce the general case to this one.

The function $\sin^{-1}$ maps the double cut plane $\mathbb{C} - \{(-\infty, -1] \cup [1, \infty)\}$ conformally onto the vertical strip $|x| < \pi/2$, mapping the upper/lower parts of the first domain onto the upper/lower parts of the second. The upper and lower limits of $\sin^{-1}$ at a point $-x \in (-\infty, -1]$ are $-\pi/2 \pm i \cosh^{-1} x$. The arc $C_i = \sin(\alpha + i\mathbb{R})$ is mapped to the line $\operatorname{Re} z = \alpha$. Therefore $\sin^{-1}$ maps $D_i - (-\infty, -1]$ conformally onto the strip

$$D_\alpha = \left\{ z = x + iy \in \mathbb{C} : -\frac{\pi}{2} < x < \alpha \right\}. \quad (50)$$

If

$$\lambda(z) = \frac{\pi z + (\pi/2)}{4\alpha + (\pi/2)}, \quad (51)$$

then λ maps D_α conformally onto the strip

$$S = \left\{ z = x + iy \in \mathbb{C} : 0 < x < \frac{\pi}{4} \right\}. \quad (52)$$

Therefore

$$\varphi_i(z) = \tan^2 \lambda(\sin^{-1} z) \quad (53)$$

is a Riemann mapping function for D_i . Now let γ be a type III arc in D_i . As in Case 3 the case $\gamma \subseteq \mathbb{R}$ is trivial, so we can assume that γ lies entirely in the upper half-plane. It will be sufficient for us to show that $\lambda(\sin^{-1} \gamma)$ has a parametric function k as in Lemma 20.

Let $z = x + iy$ be arbitrary point of γ and write $\sin^{-1} z = u + iv$ for the corresponding point of $\sin^{-1} \gamma$. Clearly, by (50),

$$u + iv \in D_\alpha. \quad (54)$$

Now

$$z = x + iy = \sin(u + iv) = \sin u \cosh v + i \cos u \sinh v, \quad (55)$$

so that

$$|z|^2 = \sin^2 u \cosh^2 v + \cos^2 u \sinh^2 v = \sin^2 u + \sinh^2 v. \quad (56)$$

As $z \rightarrow \infty$ along γ , $|z|^2 \rightarrow +\infty$ and $\sin^2 u$ remains bounded; therefore

$$v \rightarrow +\infty \text{ as } z \rightarrow \infty \text{ along } \gamma. \quad (57)$$

It now follows from (56) and (57) that

$$\sin u = \frac{x}{|z|} \left(\tanh^2 v + \frac{\sin^2 u}{\cosh^2 v} \right)^{1/2} \sim \frac{x}{|z|} \text{ as } z \rightarrow \infty. \quad (58)$$

Let h be the arc-length parametrization of γ . As $z \rightarrow \infty$ along γ its unit tangent vector has a limit $e^{i\theta}$, say. The asymptotes of C_i are the rays $\arg z = \pm(\pi/2 - \alpha)$. Therefore

$$\lim_{t \rightarrow \infty} \frac{h'(t)}{|h'(t)|} = \lim_{t \rightarrow \infty} h'(t) = e^{i\theta}, \text{ where } \frac{\pi}{2} - \alpha \leq \theta \leq \pi. \quad (59)$$

So, by (57) and Lemma 16,

$$\lim_{t \rightarrow \infty} \frac{h(t)}{|h(t)|} = e^{i\theta}. \quad (60)$$

Now $g = \sin^{-1} \circ h$ is a parametric function for $\sin^{-1} \gamma$. By (54) it follows that

- (i) $g(t) \in D_\alpha$ ($t \geq 0$), and (57) shows that
- (ii) $\operatorname{Im} g(t) \rightarrow +\infty$ as $t \rightarrow \infty$.

Equation (60) shows that

- (iii) $\lim_{t \rightarrow \infty} \operatorname{Re} g(t) = \sin^{-1} \cos \theta = (\pi/2) - \theta$ and we notice that $-\pi/2 \leq (\pi/2) - \theta \leq \alpha$, by (59).

Finally observe that

$$\frac{g'(t)}{|g'(t)|} = \frac{|1 - h(t)^2|^{1/2} h'(t)}{(1 - h(t)^2)^{1/2} |h'(t)|}. \quad (61)$$

Now in the upper half-plane $(1 - w^2)^{1/2} \sim -iw$, as $w \rightarrow \infty$. So, as $t \rightarrow \infty$,

$$\frac{g'(t)}{|g'(t)|} \sim \frac{|h(t)|}{-ih(t)} \frac{h'(t)}{|h'(t)|}, \quad (62)$$

and therefore

$$(iv) \lim_{t \rightarrow \infty} (g'(t)/|g'(t)|) = i.$$

It follows easily that $k = \lambda \circ g$ satisfies the hypothesis of Lemma 20, and therefore $\varphi_i(\gamma)$ has the restriction property in Δ .

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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