Research Article

Characterizing $\xi$-Lie Multiplicative Isomorphisms on Von Neumann Algebras

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Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras without central summands of type $I_1$. Assume that $\xi \in \mathbb{C}$ with $\xi \neq 1$. In this paper, all maps $\Phi : \mathcal{M} \to \mathcal{N}$ satisfying $\Phi(AB - \xi BA) = \Phi(A)\Phi(B) - \xi\Phi(B)\Phi(A)$ are characterized.

1. Introduction

Let $\mathcal{A}$ and $\mathcal{A}'$ be two algebras over a field $\mathbb{F}$. Recall that a map $\Phi : \mathcal{A} \to \mathcal{A}'$ is called a multiplicative map if $\Phi(AB) = \Phi(A)\Phi(B)$ for all $A, B \in \mathcal{A}$; a Lie multiplicative map if $[\Phi(A), \Phi(B)] = \Phi([A, B])$ for all $A, B \in \mathcal{A}$, where $[A, B] = AB - BA$; and a Jordan multiplicative map if $\Phi(AB + BA) = \Phi(A)\Phi(B) + \Phi(B)\Phi(A)$ for all $A, B \in \mathcal{A}$.

The question when a multiplicative map is additive is studied by many mathematicians. As the first result in this line, Matindale [1] proved that every multiplicative bijective map is additive and thus is a ring isomorphism. Recently, Matindale's result has been generalized in several directions, such as multiplicative maps and Jordan multiplicative maps between standard operator algebras or nest algebras (see [2, 3] and the references therein). For Lie multiplicative maps, Bai et al. [4] showed that if $\mathcal{R}$, $\mathcal{R}'$ are prime rings with $\mathcal{R}$ being unital and containing a nontrivial idempotent and if $\Phi : \mathcal{R} \to \mathcal{R}'$ is a Lie multiplicative bijective map, then $\Phi(T + S) = \Phi(T) + \Phi(S) + Z_{TS}$ for all $T, S \in \mathcal{R}$, where $Z_{TS}$ is an element in the center of $\mathcal{R}'$ depending on $T$ and $S$.

Let $\mathcal{A}$ be an algebra over a field $\mathbb{F}$. For a scalar $\xi \in \mathbb{F}$ and for $A, B \in \mathcal{A}$, we say that $A$ commutes with $B$ up to a factor $\xi$ if $AB = \xi BA$. The notion of commutativity up to a factor for pairs of operators is an important concept and has been studied in the context of operator algebras and quantum groups [7, 8]. Motivated by this, a binary operation $[A, B]_\xi = AB - \xi BA$, called $\xi$-Lie product of $A$ and $B$, was introduced in [9]. Moreover, a concept of $\xi$-Lie multiplicative maps was introduced in [10], which unifies the above three kinds of maps. Recall that a map $\Phi : \mathcal{A} \to \mathcal{A}'$ is called a $\xi$-Lie multiplicative map if $\Phi([A, B]_\xi) = [\Phi(A), \Phi(B)]_\xi$ for all $A, B \in \mathcal{A}$. In addition, $\Phi$ is called a $\xi$-Lie multiplicative isomorphism if $\Phi$ is bijective and $\xi$-Lie multiplicative and is called a $\xi$-Lie ring isomorphism if $\Phi$ is bijective, additive, and $\xi$-Lie multiplicative. A linear (resp., conjugate linear) $\xi$-Lie ring isomorphism between two algebras is called a $\xi$-Lie isomorphism (resp., conjugate $\xi$-Lie isomorphism).

Recall that a standard operator algebra on a Banach space $X$ is a subalgebra of the whole operator algebra $\mathcal{B}(X)$ containing the identity operator $I$ and the ideal of all finite rank operators. Qi and Hou in [10] gave a characterization of all $\xi$-Lie multiplicative isomorphisms between standard operator algebras. Let $\mathcal{A}$ and $\mathcal{B}$ be standard operator algebras on infinite dimensional Banach spaces $X$ and $Y$ over the real or complex field $\mathbb{F}$, respectively. Assume that $\Phi : \mathcal{A} \to \mathcal{B}$ is a unital bijection and $\xi$ is a scalar. The main result in [10] states that $\Phi$ is $\xi$-Lie multiplicative if and only if one of the

\[ \forall A, B \in \mathcal{A}, \quad \Phi(AB) = \Phi(A)\Phi(B) + \Phi(B)\Phi(A). \]
following holds: (1) $\xi = 1$, there exists a functional $h : \mathcal{A} \to \mathbb{F}$ with $h([A,B]) = 0$ for all $A,B$, and either there exists an invertible bounded linear or conjugate linear operator $T : X \to Y$ such that $\Phi(A) = TAT^{-1} + h(A)I$ for all $A \in \mathcal{A}$. (2) $\xi = -1$, there exists an invertible bounded linear or conjugate linear operator $T : X^* \to Y$ such that $\Phi(A) = TA^*T^{-1} + h(A)I$ for all $A \in \mathcal{A}$. (3) $\xi \in \mathbb{R} \setminus \{1,-1\}$, there exists an invertible bounded linear operator $T : X \to Y$ such that $\Phi(A) = TAT^{-1}$ for all $A \in \mathcal{A}$ if $\mathbb{F} = \mathbb{C}$; there exists an invertible bounded linear or conjugate linear operator $T : X \to Y$ such that $\Phi(A) = TA^*T^{-1}$ for all $A \in \mathcal{A}$ if $\mathbb{F} = \mathbb{R}$.

A complete characterization of $\xi$-Lie multiplicative isomorphisms on matrix algebras and certain nest algebras was given, respectively, in [10] and [6]. These results reveal the structural properties of the involved operator algebras from some new aspects. However, we have not seen any description on the structure of the $\xi$-Lie multiplicative isomorphisms between nonfactor von Neumann algebras so far. The present paper considers this problem.

The purpose of this paper is to characterize the $\xi$-Lie multiplicative isomorphisms with $\xi \neq 1$ between certain quite general von Neumann algebras. Let $\mathcal{M}$ and $\mathcal{N}$ be two von Neumann algebras without central summands of type $I_1$. Denote by $I_M$ and $I_N$ the unit operators in $\mathcal{M}$ and $\mathcal{N}$, respectively. Assume that $\Phi : \mathcal{M} \to \mathcal{N}$ is a map and $\xi \in \mathbb{C}$ with $\xi \neq 1$. We show that $\Phi$ is a $\xi$-Lie multiplicative isomorphism if and only if one of the following statements is true: (1) $\xi = 0$, $\Phi$ is a ring isomorphism; (2) $\xi = -1$, there exist central projections $P \in \mathcal{M}$ and $Q \in \mathcal{N}$ such that $\Phi = \Phi_1 \oplus \Phi_2$, where $\Phi_1 : P \mathcal{M} \to Q \mathcal{N}$ is a ring isomorphism and $\Phi_2 : (I_M - P) \mathcal{M} \to (I_N - Q) \mathcal{N}$ is a ring anti-isomorphism; (3) $\xi \neq -1$, there exist central projections $P \in \mathcal{M}$ and $Q \in \mathcal{N}$ such that $\Phi = \Phi_1 \oplus \Phi_2$, where $\Phi_1 : P \mathcal{M} \to Q \mathcal{N}$ is a ring isomorphism with $\Phi_1(\mathcal{A}_1) = \xi \Phi_2(\mathcal{A}_1)$ for all $\mathcal{A}_1 \in (I_M - P) \mathcal{M}$. (4) $\xi \neq -1$, there exist central projections $P \in \mathcal{M}$ and $Q \in \mathcal{N}$ such that $\Phi = \Phi_1 \oplus \Phi_2$, where $\Phi_1 : P \mathcal{M} \to Q \mathcal{N}$ is a ring isomorphism with $\Phi_1(\mathcal{A}_1) = \Phi_2(\mathcal{A}_1)$ for all $\mathcal{A}_1 \in (I_M - P) \mathcal{M}$.

We remark that, from the above result, for bijective maps between von Neumann algebras without central summands of type $I_1$, the $\xi$-Lie multiplicity ($\xi \neq 1$) will imply the additivity; moreover, if $\xi$ is a rational real number and $\xi \notin \{0,1,-1\}$, then by (3), $\xi \Phi_2(\mathcal{A}_2) = (1/\xi) \Phi_2(\mathcal{A}_2)$ for all $\mathcal{A}_1 \in (I_M - P) \mathcal{M}$. Here, $I_M$ denotes the unit in $\mathcal{M}$.

Let $\mathcal{N}$ be a factor von Neumann algebra. Assume that $\mathcal{M}$ is a von Neumann algebra and $\Psi : \mathcal{M} \to \mathcal{N}$ is a ring isomorphism (ring anti-isomorphism). If $\dim \mathcal{N} < \infty$, there exists a field automorphism $\tau : \mathbb{C} \to \mathbb{C}$ such that $\Psi$ is $\tau$-linear; if $\dim \mathcal{N} = \infty$, then $\Psi$ is linear or conjugate linear.

Proof. We only deal in detail with the case that $\Psi$ is a ring isomorphism. The other case can be proved similarly.

Assume that $\Psi$ is a ring isomorphism. It is clear that $\Psi(\mathcal{Z}(\mathcal{M})) = \mathcal{Z}(\mathcal{N})$, which implies that $\Psi(\mathcal{C}I_M) \subseteq \mathcal{C}I_N$ as $\mathcal{N}$ is a factor. Then, since $\Psi$ is a ring isomorphism, it is easy to check that $\Psi$ is unital; that is, $\Psi(I_M) = I_N$.

If $\mathcal{M}$ is not a factor, there exists a nonzero central element $C \in \mathcal{Z}(\mathcal{M})$ such that $C$ is not invertible. Thus, $\Psi(C) \in \mathcal{C}I_N$ is also not invertible, and so $\Psi(C) = 0$, a contradiction. Hence, $\mathcal{M}$ is a factor and $\Psi(\mathcal{C}I_M) = \mathcal{C}I_N$. 

2. Main Result and Corollary

The following is our main result and its proof will be presented in the next section.

Theorem 1. Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras without central summands of type $I_1$. Assume that $\Phi : \mathcal{M} \to \mathcal{N}$ is a map and $\xi \in \mathbb{C}$ with $\xi \neq 1$. Then, $\Phi$ is a $\xi$-Lie multiplicative isomorphism, that is, $\Phi$ is bijective and satisfies $\Phi(AB - BA) = \Phi(A)\Phi(B) - \Phi(B)\Phi(A)$ for all $A,B \in \mathcal{M}$, if and only if one of the following holds:

(1) $\xi = 0$, $\Phi$ is a ring isomorphism.

(2) $\xi = -1$, there exist central projections $P \in \mathcal{M}$ and $Q \in \mathcal{N}$ such that $\Phi = \Phi_1 \oplus \Phi_2$, where $\Phi_1 : P \mathcal{M} \to Q \mathcal{N}$ is a ring isomorphism and $\Phi_2 : (I_M - P) \to (I_N - Q) \mathcal{N}$ is a ring anti-isomorphism.

(3) $\xi \neq 0,-1$, there exist central projections $P \in \mathcal{M}$ and $Q \in \mathcal{N}$ such that $\Phi = \Phi_1 \oplus \Phi_2$, where $\Phi_1 : P \mathcal{M} \to Q \mathcal{N}$ is a ring isomorphism with $\Phi_1(\mathcal{A}_1) = \xi \Phi_2(\mathcal{A}_1)$ for all $\mathcal{A}_1 \in (I_M - P) \mathcal{M}$. Here, $I_M$ denotes the unit in $\mathcal{M}$.

Lemma 2. Let $\mathcal{N}$ be a factor von Neumann algebra. Assume that $\mathcal{M}$ is a von Neumann algebra and $\Psi : \mathcal{M} \to \mathcal{N}$ is a ring isomorphism (ring anti-isomorphism). If $\dim \mathcal{N} < \infty$, then there exists a field automorphism $\tau : \mathbb{C} \to \mathbb{C}$ such that $\Psi$ is $\tau$-linear; if $\dim \mathcal{N} = \infty$, then $\Psi$ is linear or conjugate linear.

Proof. We only deal in detail with the case that $\Psi$ is a ring isomorphism. The other case can be proved similarly.

Assume that $\Psi$ is a ring isomorphism. It is clear that $\Psi(\mathcal{Z}(\mathcal{M})) = \mathcal{Z}(\mathcal{N})$, which implies that $\Psi(\mathcal{C}I_M) \subseteq \mathcal{C}I_N$ as $\mathcal{N}$ is a factor. Then, since $\Psi$ is a ring isomorphism, it is easy to check that $\Psi$ is unital; that is, $\Psi(I_M) = I_N$.

If $\mathcal{M}$ is not a factor, there exists a nonzero central element $C \in \mathcal{Z}(\mathcal{M})$ such that $C$ is not invertible. Thus, $\Psi(C) \in \mathcal{C}I_N$ is also not invertible, and so $\Psi(C) = 0$, a contradiction. Hence, $\mathcal{M}$ is a factor and $\Psi(\mathcal{C}I_M) = \mathcal{C}I_N$. 

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For any \( \lambda \in \mathbb{C} \), let \( \Psi(\lambda I_M) = \tau(\lambda)I_N \). Then,

\[
\Psi(\lambda A) = \Psi(\lambda I_M A) = \tau(\lambda)\Psi(A)
\]

\( \forall \lambda \in \mathbb{C} \) and \( A \in \mathcal{M} \).

We claim that \( \tau : \mathbb{C} \rightarrow \mathbb{C} \) is a field automorphism. In fact, since \( \Psi \) is additive, \( \tau : \mathbb{C} \rightarrow \mathbb{C} \) is additive. Note that

\[
\tau(\lambda y) I_N = \Psi(\lambda y I_M) = \Psi(\lambda I_M) \Psi(y I_M) = \tau(\lambda) \tau(y) I_N.
\]

This implies that \( \tau(\lambda y) = \tau(\lambda) \tau(y) \); that is, \( \tau \) is multiplicative.

In the following, we assume that \( \mathcal{N} \) is infinite dimensional. We will show that \( \tau \) is continuous. As \( \dim \mathcal{N} = \infty \), there exists a sequence \( \{P_n\}_{n=1}^{\infty} \) of projections which are orthogonal to each other. So, we have \( 0 = \Psi(P_n P_m) = \Psi(P_n) \Psi(P_m) \) for \( n \neq m \) and \( 0 \neq \Psi(P_n) = \Psi(P_n^2) = \Psi(P_n)^2 \). If \( \tau \) is not continuous, then \( \tau \) is unbounded on any neighborhood of 0. So \( \tau \) is unbounded on \( \{z : |z| \leq (1/2)\} \) and hence there exists \( \lambda_1 \in (1/2) \) such that

\[
\|\Psi(\lambda_1 N)\| = \|\tau(\lambda_1)\| > \|\Psi(1)\|.
\]

Considering \( \{z : |z| \leq (1/2^2)\} \) gives \( \lambda_2 \) with \( |\lambda_2| < (1/2^2) \) such that

\[
\|\Psi(\lambda_2 N)\| = \|\tau(\lambda_2)\| > \|\Psi(\lambda_1 N)\| > \|\Psi(1)\|.
\]

Generally, for any \( n \), there exists \( \lambda_n \) with \( |\lambda_n| < (1/2^n) \) such that

\[
\|\Psi(\lambda_n N)\| = \|\tau(\lambda_n)\| > \|\Psi(\lambda_{n-1} N)\| > \|\Psi(1)\|.
\]

Let \( A = \sum_{n=1}^{\infty} \lambda_n P_n \); then \( \|A\| < 1 \). This implies that \( A \in \mathcal{M} \) and \( \|\Psi(A)\| < \infty \). However,

\[
\|\Psi(A)\| > \|\Psi(\lambda_n P_n)\| = \|\Psi(\lambda_n)\| \|\Psi(P_n)\| = \|\Psi(\lambda_n)\| > \|\Psi(\lambda_{n-1})\| > \|\Psi(\lambda_{n-2})\| > \cdots > \|\Psi(\lambda_1)\| > \|\Psi(1)\|,
\]

which implies that \( \|\Psi(A)\| > n \) for any \( n \), a contradiction. Hence, \( \tau \) is continuous and by [11, pp. 52–57] is the identity or the conjugation. Therefore, \( \Psi \) is linear or conjugate linear.

**Lemma 3.** Let \( \mathcal{M} \) be any von Neumann algebra and \( \mathcal{N} \) a factor of infinite dimension. Assume that \( -\xi \Psi : \mathcal{M} \rightarrow \mathcal{N} \) is a ring anti-isomorphism and \( \xi \in \mathbb{C} \) with \( \xi \neq 0, \pm 1 \). Then, \( \Psi(\xi I_M) = (1/\xi)I_N \) if and only if \( -\xi \Psi \) is a conjugate algebra anti-isomorphism and \( |\xi| = 1 \).

**Proof.** By Lemma 2, \( \Psi \) is linear or conjugate linear. So \( \Psi(\xi I_M) = \xi I_N \) or \( \Psi(\xi I_M) = \bar{\xi} I_N \). If \( \Psi(\xi I_M) = (1/\xi)I_N \), it follows that \( (1/\xi)I_N = \xi I_N \) or \( (1/\xi)I_N = \bar{\xi} I_N \), which imply that \( \xi^2 = 1 \) or \( \bar{\xi}^2 = 1 \). Since \( \xi \neq 0, \pm 1 \), we see that \( \Psi \) must be conjugate linear and \( |\xi| = 1 \). The converse is obvious.

**Lemma 4.** Let \( \mathcal{M} \) and \( \mathcal{N} \) be von Neumann algebras acting on separable Hilbert spaces and assume that \( \mathcal{N} \) has no central summands of type I\(_n\) for any \( 1 \leq n < \infty \).

\( \Psi : \mathcal{M} \rightarrow \mathcal{N} \) is a ring isomorphism (resp., a ring anti-isomorphism) if and only if \( P \mathcal{M} = \mathcal{M} \) and \( Q \mathcal{N} \) are algebraic isomorphism (resp., an algebraic anti-isomorphism) \( \Psi_1 : P \mathcal{M} \rightarrow Q \mathcal{N} \) and a conjugate algebraic isomorphism (resp., a conjugate algebraic anti-isomorphism) \( \Psi_2 : (I_M - P) \mathcal{M} \rightarrow (I_N - Q) \mathcal{N} \) such that \( \Psi = \Psi_1 \oplus \Psi_2 \).

**Proof.** (1) We consider the case of ring anti-isomorphism; the case of ring isomorphism is treated similarly.

Assume that \( \Psi \) is a ring anti-isomorphism. By [12, pp. 209, 236], there exists a positive measure space \( (X, \Omega, \mu) \) such that

\[
\mathcal{N} = \int_X N d\mu, \mathcal{M} = \int_X M d\mu, \Psi = \int_X \Psi_t d\mu_t,
\]

where every \( \mathcal{N}_t \) is a factor and \( \Psi_t : \mathcal{M}_t = \Psi^{-1}(\mathcal{N}_t) \rightarrow \mathcal{N}_t \) is a ring anti-isomorphism. Since \( \mathcal{N} \) has no central summands of type I\(_n\) for any \( 1 \leq n < \infty \), \( \mathcal{N}_t \) is a factor of infinite dimensional a.e. \( \mu \). By Lemma 2, \( \Psi_t \) is linear or conjugate linear. If \( \Psi_t \) is a conjugate algebraic anti-isomorphism a.e. \( \mu \) for all \( t \), then there exists a measurable subset with nonzero measure such that \( \Psi_t \) is an algebraic anti-isomorphism a.e. \( \mu \) on it. It follows that there exists a proper central projection \( P_t \) such that \( \Psi|_{\mathcal{M}_t} \) is an algebraic anti-isomorphism. Note that \( \Psi(P_t) \) is a central projection. Now it is clear (e.g., using Zorn’s Lemma) that there exist central projections \( P \in \mathcal{M} \) and \( Q \in \mathcal{N} \), an algebraic anti-isomorphism \( \Psi_1 : P \mathcal{M} \rightarrow Q \mathcal{N} \) and a conjugate algebraic anti-isomorphism \( \Psi_2 : (I_M - P) \mathcal{M} \rightarrow (I_N - Q) \mathcal{N} \) such that \( \Psi = \Psi_1 \oplus \Psi_2 \).

Conversely, if \( \Psi \) has the mentioned decomposition, then \( \Psi \) is clearly a ring anti-isomorphism.

(2) The “if” part is clear. To check the “only if” part, assume that \( -\xi \Phi \) is a ring anti-isomorphism and \( \Psi(\xi I_M) = (1/\xi)\Psi(I_M) \). For any \( A \in \mathcal{M} \), writing \( A = \int_X A_t d\mu_t \), we have

\[
\Psi(A) = \int_X \Psi_t(A_t) d\mu_t.
\]

Hence,

\[
\frac{1}{\xi} \int_X I_N d\mu_t = \frac{1}{\xi} I_N = \Psi(\xi I_M) = \int_X \Psi_t(\xi I_M) d\mu_t.
\]

It follows that \( \Psi(\xi I_M) = (1/\xi)I_N \), a.e. \( \mu \). By Lemma 3, we have \( |\xi| = 1 \) and \( -\xi \Psi_t \) is conjugate linear a.e. \( \mu \), and so \( -\xi \Psi \) is a conjugate algebraic anti-isomorphism.

Now, we are in a position to give the following corollary of Theorem 1.

**Corollary 5.** Let \( \mathcal{M} \) and \( \mathcal{N} \) be von Neumann algebras acting on separable Hilbert spaces without central summands of type I\(_1\). Assume further that \( \mathcal{N} \) has no any central summands of type I\(_n\) for \( 1 \leq n < \infty \). Let \( \Phi : \mathcal{M} \rightarrow \mathcal{N} \) be a map and \( \xi \in \mathbb{C} \) with \( \xi \neq 1 \). Then, \( \Phi \) is a \( \xi \)-Lie multiplicative isomorphism if and only if one of the following statements is true.
(1) $\xi \in \mathbb{R} \setminus \{-1\}$, there exist central projections $P \in \mathcal{M}$ and $Q \in \mathcal{N}$, and an algebraic isomorphism $\Phi_1 : P \mathcal{M} \rightarrow Q \mathcal{N}$ and a conjugate algebraic isomorphism $\Phi_2 : (I_M - P) \mathcal{M} \rightarrow (I_N - Q) \mathcal{N}$ such that $\Phi = \Phi_1 \oplus \Phi_2$.

(2) $\xi = -1$, there exist central projections $P_1, \ldots, P_4 \in \mathcal{M}$ and $Q_1, \ldots, Q_4 \in \mathcal{N}$ with $P_1 + \cdots + P_4 = I_M$ and $Q_1 + \cdots + Q_4 = I_N$ such that $\Phi = \Phi_1 \oplus \cdots \oplus \Phi_4$, where $\Phi_1 = \Phi_{P_1,\mathcal{M}} : P_1 \mathcal{M} \rightarrow Q_1 \mathcal{N}$ is an algebraic isomorphism, $\Phi_2 = \Phi_{P_2,\mathcal{M}} : P_2 \mathcal{M} \rightarrow Q_2 \mathcal{N}$ is a conjugate algebraic isomorphism, $\Phi_3 = \Phi_{P_3,\mathcal{M}} : (I_M - P_1 - P_2) \mathcal{M} \rightarrow (I_N - Q_1 - Q_2) \mathcal{N}$ is an algebraic anti-isomorphism, and $\Phi_4 = \Phi_{P_4,\mathcal{M}} : P_4 \mathcal{M} \rightarrow Q_4 \mathcal{N}$ is a conjugate algebraic anti-isomorphism.

(3) $\xi \in \mathbb{C} \setminus \mathbb{R}$ with $|\xi| \neq 1$, $\Phi$ is a conjugate algebraic isomorphism.

(4) $\xi \in \mathbb{C} \setminus \mathbb{R}$ with $|\xi| = 1$, there exist central projections $P \in \mathcal{M}$ and $Q \in \mathcal{N}$ such that $\Phi = \Phi_{P,\mathcal{M}} \oplus \Phi_{Q,\mathcal{N}}$, where $\Phi_{P,\mathcal{M}} : P \mathcal{M} \rightarrow Q \mathcal{N}$ is an algebraic isomorphism and $-\xi \Phi_{Q,\mathcal{N}} : (I_M - P) \mathcal{M} \rightarrow (I_N - Q) \mathcal{N}$ is a conjugate algebraic anti-isomorphism.

Proof. We only need to check the "only if" part. Assume that $\Phi$ is a $\xi$-Lie multiplicative isomorphism. By Theorem 1 and Lemma 4(1), if $\xi = -1$, (2) is true; if $\xi = 0$, (1) holds; if $\xi \neq 0$, then there exists central projections $P \in \mathcal{M}$ and $Q \in \mathcal{N}$ such that $\Phi = \Phi_{P,\mathcal{M}} \oplus \Phi_{Q,\mathcal{N}}$, where $\Phi_{P,\mathcal{M}} : P \mathcal{M} \rightarrow Q \mathcal{N}$ is a ring isomorphism with $\Psi_1(\xi A_1) = \xi \Psi_1(A_1)$ for all $A_1 \in P \mathcal{M}$ and $-\xi \Psi_2 : (I_M - P) \mathcal{M} \rightarrow (I_N - Q) \mathcal{N}$ is a ring anti-isomorphism with $\Psi_2(\xi A_2) = (1/\xi) \Psi_2(A_2)$ for all $A_2 \in (I_M - P) \mathcal{M}$.

For $\Psi_1$, by Lemma 4(1), there exist central projections $P_1 \in \mathcal{P} \mathcal{M}$ and $Q_1 \in \mathcal{Q} \mathcal{N}$ such that $\Psi_1 = \Phi_{P_1,\mathcal{M}} \oplus \Phi_{Q_1,\mathcal{N}}$, where $\Phi_{P_1,\mathcal{M}} : P_1 \mathcal{M} \rightarrow Q_1 \mathcal{N}$ is linear and $\Phi_{Q_1,\mathcal{N}} : (P_1 \mathcal{M})^2 \rightarrow (Q_1 \mathcal{N})^2$ is conjugate linear. Note that $\Psi_1(\xi A_1) = \xi \Psi_1(A_1)$ for all $A_1 \in P \mathcal{M}$. This implies that $\Phi_2 = 0$ if $\xi \notin \mathbb{R}$.

For $\Psi_2$, by Lemma 4(2), $\Psi_2 = 0$ if $|\xi| 
eq 1$ and $-\xi \Psi_2$ is a conjugate algebraic anti-isomorphism if $|\xi| = 1$. Thus, if $\xi \notin \mathbb{R} \setminus (0, -1)$, $\Phi$ is a ring isomorphism and has the form (1); if $\xi \in \mathbb{C} \setminus \mathbb{R}$ with $|\xi| = 1$, then $\Psi_1 = \Phi_{P,\mathcal{M}}$ is an algebraic isomorphism and $\Psi_2 = 0$, and consequently, $\Phi$ is an algebraic isomorphism, which implies the form (3); if $\xi \in \mathbb{C} \setminus \mathbb{R}$ with $|\xi| = 1$, then $\Psi_1 = \Phi_{P,\mathcal{M}}$ is an algebraic isomorphism and $-\xi \Psi_2$ is a conjugate algebraic anti-isomorphism, which implies (4).

\section*{3. Proof of the Main Result}

In this section, we present a proof of the main result Theorem 1. Before doing this, we need some notions. Let $\mathcal{M}$ be any von Neumann algebra and $A \in \mathcal{M}$. Recall that the central carrier of $A$, denoted by $\mathcal{A}$, is the intersection of all central projections $P$ such that $PA = 0$. If $A$ is self-adjoint, then the core of $A$, denoted by $\mathcal{A}$, is $\sup\{S \in \mathcal{S}(\mathcal{M}) : S = S^*, S \leq A\}$. Particularly, if $A = P$ is a projection, it is clear that $P$ is the largest central projection $\leq P$. A projection $P$ is core-free if $P = 0$. It is easy to see that $P = 0$ if and only if $I_M - P = I_M$ [13].

The following two lemmas are needed.

\textbf{Lemma 6} (see [13]). Let $\mathcal{M}$ be a von Neumann algebra without central summands of type $I_1$. Then, each nonzero central projection $C \in \mathcal{M}$ is the carrier of a core-free projection $P \in \mathcal{M}$ with $P = I_M$.

\textbf{Lemma 7} (see [14]). Let $\mathcal{M}$ be a von Neumann algebra without central summands of type $I_1$ or $I_2$. Then, the ideal $\mathcal{Z}$ of $\mathcal{M}$ generated algebraically by $\{[A^2, C]B, [A, C] - [A, C][B, A^2, C] : A, B, C \in \mathcal{M}\}$ is equal to $\mathcal{M}$.

\textbf{Proof of Theorem 1.} For $\mathcal{M}$, by Lemma 6, we can find a nonzero central core-free projection $P_0 \in \mathcal{M}$ with central carrier $I_M$. By the definitions of core and central carrier, $I_M - P_0$ is core-free and $I_M - P_0 = I_M$. For the convenience, denote $\mathcal{M}_{ij} = P_i \mathcal{M} P_j$, $i, j \in \{1, 2\}$, where $P_1 = P_0$ and $P_2 = I_M - P_0$. Then, $\mathcal{M} = \mathcal{M}_{11} + \mathcal{M}_{12} + \mathcal{M}_{21} + \mathcal{M}_{22}$. In the sequel, when we write $S_{ij}$, we always indicate $S_{ij} \in \mathcal{M}_{ij}$.

If the statement (1) holds, it is clear that $\Phi$ is a multiplicative isomorphism. If the statement (2) holds, it is easy to check that $\Phi$ is a Jordan multiplicative isomorphism. If the statement (3) holds, then for any $A_1, B_1 \in \mathcal{M}$, we have

$$\Phi_1 (A_1 B_1 - \xi B_1 A_1) = \Phi_1 (A_1) \Phi_1 (B_1) - \Phi_1 (\xi B_1) \Phi_1 (A_1) = \Phi_1 (A_1) \Phi_1 (B_1) - \xi \Phi_1 (B_1) \Phi_1 (A_1),$$

and for any $A_2, B_2 \in (I_M - P) \mathcal{M}$, we have

$$-\xi \Phi_2 (A_2 B_2 - \xi B_2 A_2) = \xi \Phi_2 (B_2) (\xi \Phi_2 (A_2)) - \xi \Phi_2 (A_2) \Phi_2 (B_2) = \xi \Phi_2 (B_2) \Phi_2 (A_2) - \xi \Phi_2 (A_2) \Phi_2 (B_2) = \xi \Phi_2 (B_2) \Phi_2 (A_2) - \xi \Phi_2 (B_2) \Phi_2 (A_2).$$

which implies that $\Phi_2 (A_2 B_2 - \xi B_2 A_2) = \Phi_2 (A_2) \Phi_2 (B_2) - \xi \Phi_2 (B_2) \Phi_2 (A_2)$. Hence, for any $A, B \in \mathcal{M}$, one obtains that $\Phi(AB - \xi BA) = \Phi(A) \Phi(B) - \xi \Phi(B) \Phi(A)$. This completes the proof of "if" part.

We will prove the "only if" part by checking a series of claims.

\textbf{Claim 1.} $\Phi$ is additive. We will complete the proof of Claim 1 by nine steps.

\textbf{Step 1.} $\Phi(0) = 0$. Since $\Phi$ is surjective, there exists an element $A \in \mathcal{M}$ such that $\Phi(A) = 0$. So $\Phi(0) = \Phi(0A - \xi A0) = \Phi(0) \Phi(A) - \xi \Phi(A) \Phi(0) = 0$.

In the sequel, we will use a so-called standard argument: suppose that $S, A, B \in \mathcal{M}$ are such that $\Phi(S) = \Phi(A) + \Phi(B)$. Multiplying this equation by $\Phi(T)$ from the left and the right,
respectively, we get $\Phi(T)\Phi(S) = \Phi(T)\Phi(A) + \Phi(T)\Phi(B)$ and $\Phi(S)\Phi(T) = \Phi(A)\Phi(T) + \Phi(B)\Phi(T)$. Then,

$$
\Phi(T)\Phi(S) - \xi\Phi(S)\Phi(T) = \Phi(T)\Phi(A) - \xi\Phi(A)\Phi(T)
+ \Phi(T)\Phi(B) - \xi\Phi(B)\Phi(T),
$$

$$
\Phi(S)\Phi(T) - \xi\Phi(T)\Phi(S) = \Phi(A)\Phi(T) - \xi\Phi(T)\Phi(A)
+ \Phi(B)\Phi(T) - \xi\Phi(T)\Phi(B).
$$

(11)

It follows that

$$
\Phi(TS - \xi ST) = \Phi(TA - \xi AT) + \Phi(TB - \xi BT),
$$

$$
\Phi(ST - \xi TS) = \Phi(AT - \xi TA) + \Phi(BT - \xi TB).
$$

(12)

Moreover, if we have $\Phi(TA - \xi AT) + \Phi(TB - \xi BT) = \Phi(TA - \xi AT + TB - \xi BT)$ and $\Phi(AT - \xi TA) + \Phi(BT - \xi TB) = \Phi(AT - \xi TA + BT - \xi TB)$, by the injectivity of $\Phi$, one gets

$$
TS - \xi ST = TA - \xi AT + TB - \xi BT,
$$

$$
ST - \xi TS = AT - \xi TA + BT - \xi TB.
$$

(13)

Step 2. For any $A_{i\overline{i}} \in \mathcal{M}_{ij}$ and $A_{\overline{i}j} \in \mathcal{M}_{ij}$, we have $\Phi(A_{i\overline{i}} + A_{\overline{i}j}) = \Phi(A_{i\overline{i}}) + \Phi(A_{\overline{i}j})$, $1 \leq i \neq j \leq 2$.

By the surjectivity of $\Phi$, there is an element $S = S_{i1} + S_{i2} + S_{21} + S_{22} \in \mathcal{M}$ such that

$$
\Phi(S) = \Phi(A_{i\overline{i}}) + \Phi(A_{\overline{i}j}).
$$

(14)

For any $T_{ij} \in \mathcal{M}_{ij}$, applying the standard argument to (14), we obtain

$$
\Phi(ST_{ij} - \xi T_{ij}S) = \Phi(A_{i\overline{i}}T_{ij} - \xi T_{ij}A_{i\overline{i}})
+ \Phi(A_{\overline{i}j}T_{ij} - \xi T_{ij}A_{\overline{i}j}) = \Phi(A_{i\overline{i}}T_{ij} - \xi T_{ij}A_{i\overline{i}}),
$$

$$
\Phi(T_{ij}S - \xi ST_{ij}) = \Phi(-\xi A_{\overline{i}j}T_{ij}).
$$

(15)

(16)

By (15) and the injectivity of $\Phi$, one gets $ST_{ij} - \xi T_{ij}S = S_{i1}T_{ij} + S_{i2}T_{ij} - \xi T_{ij}S = S_{j1}T_{ij}$ for all $T_{ij} \in \mathcal{M}_{ij}$, which implies that $S_{i1}T_{ij} = S_{j1}T_{ij}$ and $S_{i2}T_{ij} = S_{j2}T_{ij}$ for all $T_{ij} \in \mathcal{M}_{ij}$. Particularly, taking $T_{ij} = P_{ij}$, one gets $S_{ij} = A_{ij}$ and, as $\xi \neq 1$, $S_{ij} = 0$. This, combining (16) and the injectivity of $\Phi$, yields $T_{ij}S_{ij} = 0$, and so $S_{ij} = 0$.

For any $T_{ij} \in \mathcal{M}_{ij}$, applying the standard argument to (14) again, we get

$$
\Phi(ST_{ij} - \xi T_{ij}S) = \Phi(A_{i\overline{i}}T_{ij} - \xi T_{ij}A_{i\overline{i}})
+ \Phi(A_{\overline{i}j}T_{ij} - \xi T_{ij}A_{\overline{i}j}) = \Phi(A_{i\overline{i}}T_{ij} - \xi T_{ij}A_{i\overline{i}}).
$$

(17)

It follows from the injectivity of $\Phi$ that

$$
ST_{ij} - \xi T_{ij}S = A_{i\overline{i}}T_{ij}
$$

(18)

for every $T_{ij} \in \mathcal{M}_{ij}$. Note that $S_{ij} = S_{ij} = 0$ and $S_{ij} = A_{ij}$. The above equation reduces to $S_{ij}T_{ij} = A_{ij}T_{ij}$; that is, $S_{ij}TP_{ij} = A_{i\overline{i}}TP_{ij}$ for all $T \in \mathcal{M}$. Note that $P_{ij} = I_{M}$. It follows from the definition of the central carrier that span $\{TP_{ij}x : T \in \mathcal{M}, x \in H\}$ is dense in $H$. Hence, $S_{ij} = A_{ij}$. Consequently, $S_{ij} = A_{ij} + A_{ij}$.

Similarly, one can check that the following Step 3 holds.

Step 3. For any $A_{i\overline{i}} \in \mathcal{M}_{ij}$ and $A_{\overline{i}j} \in \mathcal{M}_{ij}$, we have $\Phi(A_{i\overline{i}} + A_{\overline{i}j}) = \Phi(A_{i\overline{i}}) + \Phi(A_{\overline{i}j})$, $1 \leq i \neq j \leq 2$.

Step 4. $\Phi$ is additive on $\mathcal{M}_{ij}$, $1 \leq i \neq j \leq 2$.

For any $A_{12}, B_{12} \in \mathcal{M}_{12}$, since

$$
A_{12} + B_{12} = (P_{1} + B_{12})(A_{12} + P_{2})
= (P_{1} + B_{12})(A_{12} + P_{2}) - \xi(A_{12} + P_{2})(P_{1} + B_{12}),
$$

(19)

by Steps 2-3, one can obtain

$$
\Phi(A_{12} + B_{12}) = \Phi(P_{1} + B_{12})\Phi(A_{12} + P_{2})
- \xi\Phi(A_{12} + P_{2})\Phi(P_{1} + B_{12})
= (\Phi(P_{1}) + \Phi(B_{12}))\Phi(A_{12} + P_{2})
- \xi(\Phi(A_{12}) + \Phi(P_{2}))\Phi(P_{1} + B_{12})
\times (\Phi(P_{1}) + \Phi(B_{12}))
= \Phi(P_{1}A_{12} - \xi A_{12}P_{1}) + \Phi(P_{1}P_{2} - \xi P_{1}P_{2})
+ \Phi(B_{12}A_{12} - \xi A_{12}B_{12})
+ \Phi(B_{12}P_{2} - \xi P_{2}B_{12})
= \Phi(A_{12}) + \Phi(B_{12}).
$$

(20)

For any $A_{21}, B_{21} \in \mathcal{M}_{21}$, note that

$$
A_{21} + B_{21} = (A_{21} + P_{2})(P_{1} + B_{21})
= (A_{21} + P_{2})(P_{1} + B_{21}) - \xi(P_{1} + B_{21})(A_{21} + P_{2}).
$$

(21)

By a similar computation as above, one can show $\Phi(A_{21} + B_{21}) = \Phi(A_{21}) + \Phi(B_{21})$.

Step 5. $\Phi$ is additive on $\mathcal{M}_{ij}$, $i = 1, 2$.

Take any $A_{i\overline{i}}, B_{\overline{i}j} \in \mathcal{M}_{ij}$ and choose $S = S_{i1} + S_{i2} + S_{21} + S_{22} \in \mathcal{M}$ such that

$$
\Phi(S) = \Phi(A_{i\overline{i}}) + \Phi(B_{\overline{i}j}).
$$

(22)

Let $j \neq i$. For $P_{ij} \in \mathcal{M}_{ij}$, applying the standard argument to (22) and the injectivity of $\Phi$, we get $0 = P_{ij}S - \xi P_{ij}S = S_{ij} - \xi S_{ij} - S_{ij} - \xi S_{ij}$ and $0 = P_{ij}S - \xi P_{ij}S = S_{ij} - S_{ij} - \xi S_{ij} - \xi S_{ij}$. Note that $\xi \neq 1$, these two equations imply that $S_{ij} - S_{ij} = S_{ij} = 0$. For any $T_{ij} \in \mathcal{M}_{ij}$, applying the standard argument to (22), and by Step 4, one obtains

$$
\Phi(ST_{ij} - \xi T_{ij}S) = \Phi(A_{i\overline{i}}T_{ij}) + \Phi(B_{\overline{i}j}T_{ij})
= \Phi(A_{i\overline{i}}T_{ij} + B_{\overline{i}j}T_{ij}).
$$

(23)
Note that $\Phi$ is injective and $S_{ij} = S_{ji} = S_{jj} = 0$. The above equation implies $S_{ij}T_{ij} = (A_{ij} + B_{ij})T_{ij}$ for all $T_{ij} \in M_{ij}$; that is, $S_{ij}T_{P_j} = (A_{ij} + B_{ij})T_{P_j}$ for all $T \in M$. It follows from $TP_j = I_M$ that $S_{ij} = A_{ij} + B_{ij}$.

**Step 6.** Consider $\Phi(A_{12} + A_{21}) = \Phi(A_{12}) + \Phi(A_{21})$.

Choose $S = S_{11} + S_{12} + S_{21} + S_{22} \in M$ such that

$$\Phi(S) = \Phi(A_{12}) + \Phi(A_{21}).$$

(24)

For any $T_{12} \in M_{12}$, applying the standard argument to (24), and the injectivity of $\Phi$, we have $ST_{12} - \xi T_{12}S = A_{21}T_{12} - \xi T_{12}A_{21}$ for all $T_{12} \in M_{12}$. Multiplying this equation by $P_2$ from both sides, we get $S_{21}T_{12} = A_{21}T_{12}$ for all $T_{12} \in M_{12}$, which implies that $S_{21} = A_{21}$.

Similarly, for any $T_{21} \in M_{21}$, applying the standard argument to (24), one can prove that $S_{12} = A_{12}$.

For $P_1$ and $P_2$, applying the standard argument to (24), respectively, we have

$$\Phi(P_2S - \xi SP_2) = \Phi(-\xi A_{12}) + \Phi(A_{21}),$$

(25)

$$\Phi(S_1 - \xi P_1S) = \Phi(-\xi A_{12}) + \Phi(A_{21}).$$

(26)

Therefore, $\Phi(P_2S - \xi SP_2) = \Phi(S_1 - \xi P_1S)$, which implies that $P_2S - \xi SP_2 = S_1 - \xi P_1S$. A simple computation reveals that $S_{11} = S_{22} = 0$. Consequently, $S = A_{12} + A_{21}$. 

**Step 7.** Consider $\Phi(A_{11} + A_{12} + A_{21}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21})$.

Let $S = S_{11} + S_{12} + S_{21} + S_{22} \in M$ be such that $\Phi(S) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21})$. Then, by Steps 2-3, we have

$$\Phi(S) = \Phi(A_{11} + A_{12}) + \Phi(A_{21}),$$

(27)

For any $T_{12} \in M_{12}$, applying the standard argument to (27) and the injectivity of $\Phi$, we get

$$ST_{12} - \xi T_{12}S = A_{11}T_{12} + A_{21}T_{12} - \xi T_{12}A_{21}.$$ 

(28)

Multiplying by $P_2$ from the left in (28), one obtains $S_{21}T_{12} = A_{21}T_{12}$ for each $T_{12} \in M_{12}$, and so $S_{21} = A_{21}$. Multiplying by $P_1$ and $P_2$ from the left and the right, respectively, in (28), one gets

$$\xi T_{12}S_{22} = (S_{11} - A_{11})T_{12} \quad \forall T_{12} \in M_{12}.$$ 

(29)

Similarly, for any $T_{21} \in M_{21}$, applying the standard argument to (26), one can get $S_{12} = A_{12}$.

For $P_3$, applying the standard argument to (26), by Step 6, we have $\Phi(S_2 - \xi P_3S) = \Phi(A_{12}) + \Phi(-\xi A_{21}) = \Phi(A_{21} - \xi A_{21})$, which implies that $S_2 - \xi P_3S = A_{12} - \xi A_{21}$. As $S_{12} = A_{12}$ and $S_{21} = A_{21}$, a direct computation leads to $S_{22} = 0$. This fact and (29) yield $(S_{11} - A_{11})T_{12} = 0$ for all $T_{12} \in M_{12}$, and so $S_{11} = A_{11}$. Consequently, $S = A_{11} + A_{12} + A_{21}$, as desired.

**Step 8.** Consider $\Phi(A_{11} + A_{12} + A_{21} + A_{22}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22})$.

Let $S = S_{11} + S_{12} + S_{21} + S_{22} \in M$ be such that

$$\Phi(S) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22}).$$

(30)

For $P_1$, applying the standard argument to (30), by Step 7, we have

$$\Phi(P_1S - \xi SP_1) = \Phi(P_1)\Phi(S) - \xi \Phi(S) \Phi(P_1)$$

$$= \Phi((1 - \xi)A_{11} + A_{12} - \xi A_{21})$$

and $\Phi(SP_1 - \xi P_1S) = \Phi((1 - \xi)A_{11} - A_{21} + A_{22})$. It follows that $P_1S - \xi SP_1 = (1 - \xi)A_{11} + A_{12} - \xi A_{21}$ and $SP_1 - \xi P_1S = (1 - \xi)A_{11} - \xi A_{12} + A_{21}$. By a simple computation, one obtains $S_{11} = A_{11}, S_{12} = A_{12},$ and $S_{21} = A_{21}$.

For any $T_{12} \in M_{12}$, applying the standard argument to (30), one gets

$$\Phi(T_{12}S - \xi ST_{12}) = \Phi(-\xi A_{11}T_{12})$$

$$+ \Phi(T_{12}A_{21} - \xi A_{21}T_{12}) + \Phi(T_{12}A_{22}).$$

(32)

Furthermore, for $P_1$, applying the standard argument to the above equation, by Steps 2 and 4, one gets

$$\Phi(P_1(T_{12}S - \xi ST_{12}) - \xi (T_{12}S - \xi ST_{12})P_1)$$

$$= \Phi(-\xi A_{11}T_{12}) + \Phi(T_{12}A_{21} - \xi T_{12}A_{21}) + \Phi(T_{12}A_{22})$$

$$= \Phi(-\xi A_{11}T_{12} + T_{12}A_{21} - \xi T_{12}A_{21} + T_{12}A_{22}).$$

(33)

Thus, we have

$$T_{12}S_{21} + T_{12}S_{22} - \xi S_{11}T_{12} - \xi T_{12}S_{21}$$

$$= -\xi A_{11}T_{12} + T_{12}A_{21} - \xi T_{12}A_{21} + T_{12}A_{22}.$$ 

(34)

Note that $S_{11} = A_{11}, S_{12} = A_{12},$ and $S_{21} = A_{21}$. It follows that $T_{12}S_{22} = T_{12}A_{22}$ for all $T_{12} \in M_{12}$, and hence $S_{22} = A_{22}$. Consequently, $S = A_{11} + A_{12} + A_{21} + A_{22}$.

**Step 9.** $\Phi$ is additive, and so Claim 1 is true.

For any $A, B \in M$, write $A = A_{11} + A_{12} + A_{21} + A_{22}$ and $B = B_{11} + B_{12} + B_{21} + B_{22}$. By Steps 2–8, we have

$$\Phi(A + B) = \Phi((A_{11} + B_{11}) + (A_{12} + B_{12})$$

$$+ (A_{21} + B_{21}) + (A_{22} + B_{22}))$$

$$= \Phi(A_{11} + B_{11}) + \Phi(A_{12} + B_{12})$$

$$+ \Phi(A_{21} + B_{21}) + \Phi(A_{22} + B_{22})$$

$$= \Phi(A_{11}) + \Phi(B_{11}) + \Phi(A_{12}) + \Phi(B_{12})$$

$$+ \Phi(A_{21}) + \Phi(B_{21}) + \Phi(A_{22}) + \Phi(B_{22})$$

$$= \Phi(A_{11} + A_{12} + A_{21} + A_{22})$$

$$+ \Phi(B_{11} + B_{12} + B_{21} + B_{22})$$

$$= \Phi(A) + \Phi(B).$$

(35)
Claim 2. The statements (1)-(2) hold in the theorem. By Claim 1, \( \Phi \) is additive. So, in the case of \( \xi = 0 \), the statement (1) is true; in the case of \( \xi = -1 \), by [14] (also see [15]), it is also easy to see that the statement (2) is true.

In the sequel, we always assume that \( \xi \neq 0, \pm 1 \).

Claim 3. Consider \( \Phi(I_M) \in \mathcal{X}(\mathcal{N}) \).

For any \( A, B \in \mathcal{M} \), we have
\[
\Phi(A) - \xi \Phi(B) \Phi(A) = (1 - \xi) \Phi(1_M) \Phi(A),
\]
(36)

Note that, by Claim 1, \( \Phi \) is additive. Thus, the above two equations imply that
\[
\Phi((1 + \xi)(AB - BA)) = (1 + \xi)(\Phi(A) \Phi(B) - \Phi(B) \Phi(A)), \quad \forall A, B \in \mathcal{M}.
\]
(37)

As \( \Phi(0) = 0 \) and \( \xi \neq 0, -1 \), the above equation ensures that \( AB = BA \) if and only if \( \Phi(A) \Phi(B) = \Phi(B) \Phi(A) \). So \( \Phi(A) \Phi(I_M) = \Phi(I_M) \Phi(A) \) holds for all \( A \in \mathcal{M} \). It follows from the surjectivity of \( \Phi \) that \( \Phi(I_M) \in \mathcal{X}(\mathcal{N}) \).

Claim 4. \( \Phi(I_M) \) is invertible.

For any \( A \in \mathcal{M} \), by Claim 3, we have
\[
\Phi((1 - \xi)A) = \Phi(1_M) \Phi(I_M) - \xi \Phi(I_M) \Phi(A) = (1 - \xi) \Phi(I_M) \Phi(A) - (1 - \xi) \Phi(1_M) \Phi(I_M).
\]
(38)

Taking \( A = A_0 = \frac{1}{1 - \xi} \Phi^{-1}(1 - \xi)I_N \) in the above equation, one gets \( (1 - \xi)I_N = (1 - \xi)I_N \Phi(1_M)(A_0) = (1 - \xi)\Phi(A_0) \Phi(I_M) \). It follows from the fact \( \xi \neq 1 \) that \( \Phi(I_M) \Phi(A_0) = \Phi(A_0) \Phi(I_M) = I_N \). So \( \Phi(I_M) \) is invertible and \( \Phi(A_0) \) is its inverse. The claim holds.

Note that \( \Phi(I_M)^{-1} \in \mathcal{X}(\mathcal{N}) \) as \( \Phi(I_M) \in \mathcal{X}(\mathcal{N}) \). For any \( A \in \mathcal{M} \), let \( \Psi(A) = \Phi(I_M)^{-1} \Phi(A) \). Since
\[
\Phi((1 - \xi)A^2) = \Phi(A^2) - \xi \Phi(I_M) \Phi(A^2) = (1 - \xi) \Phi(I_M) \Phi(A^2),
\]
(39)

we get \( \Phi(A)^2 = \Phi(I_M) \Phi(A^2) \). So
\[
\Psi(A)^2 = \Phi(I_M)^{-1} \Phi(A) \Phi(I_M)^{-1} \Phi(A) = \Phi(I_M)^{-1} \Phi(I_M)^{-1} \Phi(A^2) = \Phi(I_M)^{-1} \Phi(A^2) \quad \text{(40)}
\]
\[
\Psi(A) = \Phi(I_M)^{-1} \Phi(A).
\]
It follows that \( \Psi : \mathcal{M} \to \mathcal{N} \) is a Jordan ring isomorphism and \( \Phi(A) = \Phi(I_M)^{-1} \Phi(A) \). Thus, by [14], there exist central projections \( P \in \mathcal{M} \) and \( Q \in \mathcal{N} \) such that \( \Psi_{|P,\mathcal{M}} : P,\mathcal{M} \to Q,\mathcal{N} \) is a ring isomorphism and \( \Psi_{|I_N \setminus P,\mathcal{M}} : (I_N - P,\mathcal{M}) \to (I_N - Q,\mathcal{N}) \) is a ring anti-isomorphism.

For the convenience, write \( \Phi_{|P,\mathcal{M}} = \Phi_1, \Phi_{|I_N \setminus P,\mathcal{M}} = \Phi_2, \Psi_1, \text{ and } \Psi_{|I_N \setminus P,\mathcal{M}} = \Psi_2 \). Then, we may write \( \Phi = \Phi_1 \Phi_2 \).

Note that \( \Phi(I_M) = \Phi(P) + \Phi(I_M - P) \in \mathcal{X}(\mathcal{N}), \Phi(P) = \Phi(I_M)\Psi(P) \in Q,\mathcal{N}, \text{ and } \Phi(I_M - P) = \Phi(I_M)\Psi(I_M - P) \in (I_N - Q,\mathcal{N}). \)

It is easy to check that \( \Phi(P) \in \mathcal{X}(\mathcal{N}), \Phi(I_M - P) \in (I_N - Q,\mathcal{N}), \Phi_1(P)^{-1} \in Q,\mathcal{N}, \text{ and } \Phi_2(I_M - P)^{-1} \in (I_N - Q,\mathcal{N}). \)

Hence,
\[
\Psi_1(A)(\xi A) = \Phi_1(I_M)^{-1} \Phi_1(1_M) = \Phi_1(P)^{-1} \Phi_1(A),
\]
(41)

\[
\Psi_2(A)(\xi A) = \Phi_2(I_M)^{-1} \Phi_2(1_M) = \Phi_2(I_M - P)^{-1} \Phi_2(A),
\]
(42)

These imply that \( \Psi_1(A), \Psi_2(B) = \Phi_1(P)[\Psi_1(A), \Psi_2(B)] \) holds for all \( A, B \in P,\mathcal{M} \). It follows from the surjectivity of \( \Phi_1 \) that \( (Q - \Phi_1(P))[T, S] = 0 \) holds for all \( T, S \in Q,\mathcal{N} \). Furthermore, it is easily checked that \( (Q - \Phi_1(P))[T, S]X[T, S] = \)
\[ [T,S]X[T^2,S] = 0 \] holds for every \( T,S,X \in Q.N. \) Note that \( Q.N \) is a von Neumann algebra without central summands of type \( I_1. \) By Lemma 7, one obtains \( \Phi_1(P) = Q. \) Hence, \( \Phi_1(A_1) = \Phi_1(P)\Psi_1(A_1) = Q\Psi_1(A_1) = \Psi_1(A_1) \) for all \( A_1 \in P.M; \) that is, \( \Phi_1 = \Phi_1|_{P,M} : P.M \to Q.N \) is a ring isomorphism.

Also note that, for any \( A_1 \in P.M, \) we have

\[
\Phi_1(A_1) - \Phi_1(\xi A_1) = \Phi_1(A_1P - \xi PA_1) = \Phi_1(A_1P) - \xi \Phi_1(\xi A_1)\] (43)

This leads to \( \Phi_1(\xi A_1) = \xi \Phi_1(A_1) \) which completes the proof of Claim 5.

Claim 6. \( -\xi \Phi_2 : (I_M - P).M \to (I_N - Q).N \) is a ring anti-isomorphism and \( \Phi_2(\xi A_2) = (1/\xi)\Phi_2(\xi A_2) \) for all \( A_2 \in (I_M - P).M. \)

For every \( A_2, B_2 \in (I_M - P).M, \) we have

\[
\Psi_2(A_2B_2 - \xi B_2A_2) = \Psi_2(B_2)\Psi_2(A_2) - \Psi_2(\xi B_2) \times \Psi_2(A_2) + (1 - \xi) \times \Phi_2((I_M - P)\Psi_2(B_2)) = \Psi_2(B_2)\Psi_2(A_2) - \Psi_2(\xi B_2) \times \Psi_2(A_2) + (1 - \xi) \times \Phi_2((I_M - P)\Psi_2(B_2))
\]

\[ \Phi_2(A_2B_2 - \xi B_2A_2) = \Phi_2((I_M - P)^{-1} \Phi_2(A_2B_2 - \xi B_2A_2)) = \Phi_2((I_M - P)^{-1} \Phi_2(A_2B_2 - \xi B_2A_2)) = \Phi_2((I_M - P)^{-1} \Phi_2(A_2B_2 - \xi B_2A_2)) = \Phi_2((I_M - P)^{-1} \Phi_2(A_2B_2 - \xi B_2A_2)). \] (44)

Then, \( [\Psi_2(B_2),\Psi_2(A_2)] = -\xi \Phi_2((I_M - P)[\Psi_2(B_2),\Psi_2(A_2)] \) and so \( ([I_N - Q] + \xi \Phi_2((I_M - P)[T,S])X[T^2,S] = 0 \) holds for all \( T,S,X \in (I_N - Q).N. \) It is easily checked that \( ([I_N - Q] + \xi \Phi_2((I_M - P)[T,S])X[T^2,S] = 0 \) holds for each \( T,S,X \in (I_N - Q).N. \) It follows from Lemma 7 that \( \Phi_2((I_M - P)^{-1} \Phi_2(A_2) - \xi \Phi_2((I_M - P)^{-1} \Phi_2(A_2)). \) Hence, \( -\xi \Phi_2((I_M - P)^{-1} \Phi_2(A_2) - \xi \Phi_2((I_M - P)^{-1} \Phi_2(A_2)). \) and \( -\xi \Phi_2((I_M - Q)\Psi_2(A_2) = \Psi_2(A_2). \) and \( -\xi \Phi_2((I_M - Q)\Psi_2(A_2) = \Psi_2(A_2). \)

Since

\[
\Phi_2(A_2) - \Phi_2(\xi A_2) = \Phi_2((1 - \xi) A_2)
\]

\[ = (1 - \xi) \Phi_2((I_M - P)\Phi_2(A_2)) \] (45)

\[ = \Phi_2(A_2) - 1/\xi \Phi_2(A_2), \]

we get \( \Phi_2(\xi A_2) = (1/\xi)\Phi_2(A_2) \) for all \( A_2 \in (I_M - P).M. \) Hence, Claim 6 is true. Combining Claims 3–6, one sees that the statement (3) in Theorem 1 holds.

The proof of the theorem is therefore completed. \( \square \)

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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