Research Article

Weighted Composition Operator from Mixed Norm Space to Bloch-Type Space on the Unit Ball

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We discuss the boundedness and compactness of the weighted composition operator from mixed norm space to Bloch-type space on the unit ball of $\mathbb{C}^n$.

1. Introduction

Let $H(B_n)$ be the class of all holomorphic functions on $B_n$ and $S(B_n)$ the collection of all the holomorphic self-mappings of $B_n$, where $B_n$ is the unit ball in the $n$-dimensional complex space $\mathbb{C}^n$. Let $dv$ denote the Lebesgue measure on $B_n$ normalized so that $v(B_n) = 1$ and $d\sigma$ the normalized rotation invariant measure on the boundary $S = \partial B_n$ of $B_n$.

For $f \in H(B_n)$, let

$$\Re f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z)$$

be the radial derivative of $f$.

A positive continuous function $\mu$ on $[0,1)$ is called normal (see, e.g., [1]) if there exist three constants $0 \leq \delta < 1$, and $0 < a < b < \infty$, such that for $r \in [\delta,1)$

$$\frac{\mu(r)}{(1-r)^a} \downarrow 0, \quad \frac{\mu(r)}{(1-r)^b} \uparrow \infty, \quad r \to 1.$$

In the rest of this paper we always assume that $\mu$ is normal on $[0,1)$, and from now on if we say that a function $\mu : B_n \to [0,\infty)$ is normal we will also suppose that it is radial on $B_n$, that is, $\mu(z) = \mu(|z|)$ for $z \in B_n$.

Let $0 < p \leq \infty, 0 < q \leq \infty$, and $\mu$ be normal on $[0,1)$. $f$ is said to belong to the mixed norm space $L(p,q,\mu)$ if $f$ is a measurable function on $B_n$ and $\|f\|_{p,q,\mu} < \infty$, where

$$\|f\|_{p,q,\mu} = \left\{ \int_0^1 r^{2n-1}(1-r)^{-1} \mu^p(r) M^p_q(\mu,f) \ dr \right\}^{1/p}$$

$$= \left\{ \sup_{0 \leq r < 1} \mu(r) M_q(\mu,f) \right\}^{1/q}$$

$$M_q(\mu,f) = \sup_{\zeta \in S} \|f(r\zeta)\|,$$

$$M_q(\mu,f) = \left\{ \int_0^1 |f(r\zeta)|^q d\sigma(\zeta) \right\}^{1/q}, \quad (0 < q < \infty).$$

If $0 < p = q < \infty$, then $L(p,q,\mu)$ is just the space $L^p(\mu) = \{ f \}$ is measurable function on $B_n : \int_{B_n} |f(z)|^p(\mu^p(z)/(1-|z|))dv(z) < \infty$.

Let $H(p,q,\mu) = L(p,q,\mu) \cap H(B_n)$. If $0 < p = q < \infty$, then $H(p,q,\mu)$ is just the weighted Bergman space $L^p_a(\mu)$. In particular, $H(p,q,\mu)$ is Bergman space $L^p_a(\mu)$ if $0 < p = q < \infty$ and $\mu(r) = (1-r)^{1/p}$. Otherwise, if $p = q = 2$ and $\mu(r) = (1-r)^{\beta/2}$ ($\beta < 0$), then $H(p,q,\mu(r))$ is the Dirichlet-type space.
For \(0 < p, q < \infty, -1 < \gamma < 1\), let \(\mu(r) = r^{-(2n-1)/p}(1-r)^{(y+1)/p}\); it is easy to see that the mixed norm space \(H(p, q, \mu)\), written by \(H_{p,q,\mu}\), consists of all \(f \in H(B_n)\) such that
\[
\|f\|_{H_{p,q,\mu}} = \left\{ \int_0^1 M_q^p (f, r) (1-r)^y dr \right\}^{1/p} < \infty. \tag{4}
\]
Now \(f \in H(B_n)\) is said to belong to Bloch-type space \(\mathcal{B}_\mu\) if
\[
\|f\|_{\mathcal{B}_\mu} = \sup_{z \in B_n} \mu(z) |\nabla f(z)| < \infty,
\]
where \(\nabla f(z) = (\partial f(z)/\partial z_1, \ldots, \partial f(z)/\partial z_n)\) is the complex gradient of \(f\).

It is clear that \(\mathcal{B}_\mu\) is a Banach space with norm \(\|f\|_{\mathcal{B}_\mu} = |f(0)| + \|f\|_{\mathcal{B}_\mu}\). For \(f \in H(B_n)\), we denote
\[
\|f\|_{\mathcal{B}_{\mu,1}} = \sup_{z \in B_n} \mu(z) |f(z)|,
\]
where
\[
Q^f_\mu (z) = \sup_{u \in \mathbb{C} \setminus \{0\}} \frac{\|\nabla f(z), \overline{u}\|}{G^\mu_2 (u, u)},
\]
\[
G_\mu^2 (u, u) = \frac{1}{|\sigma_\mu (0)|} \left\{ \frac{\mu^2(z)}{\sigma^2_\mu (|z|)} |u|^2 + \left( 1 - \frac{\mu^2(z)}{\sigma^2_\mu (|z|)} \right) \frac{|z|^2}{|z|} \right\}^{1/2}.
\]

2. Some Lemmas

**Lemma 1.** Assume that \(0 < p, q < \infty, -1 < \gamma < 1\), and \(f \in H_{p,q,\gamma}\). Then there is a positive constant \(C\) which is independent of \(f\) such that
\[
|f(z)| \leq C \left( 1 - |z|^2 \right)^{(\gamma+1)/p}, \tag{10}
\]
\[
|\Re f(z)| \leq C \left( 1 - |z|^2 \right)^{\gamma/(\gamma+1)/p}. \tag{11}
\]

**Proof.** We first prove (10). By the monotonicity of the integral means and [20, Theorem 1.12] we have that
\[
\|f\|_{H_{p,q,\gamma}}^p \geq \int_{1+|z|/2}^{1+|z|/4} M_q^p (f, r) (1-r)^y dr
\]
\[
\geq CM_q^p \left( f, \frac{1+|z|}{2} \right) \int_{1+|z|/2}^{1+|z|/4} (1-r)^y dr
\]
\[
\geq CM_q^p \left( f, \frac{1+|z|}{2} \right) (1-|z|^2)^{\gamma+1},
\]
\[
\geq C(1-|z|^2)^{\gamma+1/p} |f(z)|^p,
\]
from which the desired result (10) follows.

Next we prove (11). By the monotonicity of the integral means, using the well-known asymptotic formula (e.g., [21, Theorem 2]), we obtain that
\[
\int_0^1 M_q^p (f, r) (1-r)^y dr
\]
\[
\leq |f(0)|^p + \int_0^1 M_q^p (\Re f, r) (1-r)^{\gamma/p} dr.
\]

References therein. In particular, Stević [18] gave some conditions of weighted composition operators between mixed-norm spaces and \(H^\infty_{\mu}\) spaces on the unit ball. Zhou and Chen [19] discussed weighted composition operators from \(F(p, q, s)\) to Bloch-type spaces on the unit ball. More recently, the weighted composition operator from Bers-type space to Bloch-type space on the unit ball was studied in [6]. Now in this paper, we will continue this line of research and characterize the boundedness and compactness of the weighted composition operator \(T_{\psi,\varphi}\), acting from mixed-norm spaces \(H_{p,q,\gamma}\) to Bloch-type space \(\mathcal{B}_\mu\) on the unit ball of \(\mathbb{C}^n\). The paper is organized as follows. In Section 2, we give some lemmas. The main results are given in Section 3.

Throughout the remainder of this paper, \(C\) will denote a positive constant; the exact value of which will vary from one appearance to the next. The notation \(A = B\) means that there is a positive constant \(C\) such that \(B/C \leq A \leq CB\).
By [20, Theorem 1.12], it follows that
\[
\|f\|_{H_{p,q,\gamma}}^p \geq \int_{|z|/2}^1 M_{\gamma}^p (f, r) (1 - r)^{\gamma} dr \\
\geq C \int_{|z|/2}^1 M_{\gamma}^p \left( \Re f, r \right) (1 - r)^{\gamma+p} dr \\
\geq CM_{\gamma}^p \left( \Re f, \frac{1 + |z|}{2} \right) \int_{|z|/2}^1 (1 - r)^{\gamma+p} dr \tag{14}
\]
\[
\geq CM_{\gamma}^p \left( \Re f, \frac{1 + |z|}{2} \right) (1 - |z|^2)^{\gamma+1+p} \\
\geq C (1 - |z|^2)^{\gamma+1+p+(\gamma+1)/p} |\Re f (z)|^p.
\]

Then the desired result (11) follows. This completes the proof. \( \square \)

From the above lemma, when \( f \in H_{p,q,\gamma} \), then
\[
f \in \mathcal{B}^{n/\gamma+1+(\gamma+1)/p} \quad \Rightarrow \quad \|f\|_{\mathcal{B}^{n/\gamma+1+(\gamma+1)/p}} \leq C \|f\|_{H_{p,q,\gamma}}. \tag{15}
\]

For \( z \in B_n \), \( u \in C^n \), denote the Bergman metric of \( B_n \) by
\[
H_z (u,u) = \frac{(1 - |z|^2)|u|^2 + \langle z, u \rangle^2}{(1 - |z|^2)^2}. \tag{16}
\]

**Lemma 2.** Let \( v(r) = (1-r^3)^{n\gamma/(n+1)+1/p} \) and \( \varphi \in S(B_n) \). Then
\[
G_{\varphi(z)} (\varphi(z), \varphi(z)) \leq \frac{CH_{\varphi(z)} (\varphi(z), \varphi(z))}{(1 - |\varphi(z)|^2)^{\gamma}} \tag{17}
\]
for all \( z \in B_n \), where \( \varphi(z) \) denotes the Jacobian matrix of \( \varphi(z) \) and
\[
\varphi(z) = \begin{pmatrix}
\frac{\partial \varphi_1}{\partial z_1} & \cdots & \frac{\partial \varphi_n}{\partial z_1} \\
\vdots & \ddots & \vdots \\
\frac{\partial \varphi_1}{\partial z_n} & \cdots & \frac{\partial \varphi_n}{\partial z_n}
\end{pmatrix}^T. \tag{18}
\]

**Proof.** Let \( \alpha = n/\gamma + (\gamma + 1)/p \). If \( \varphi(z) = 0 \), the desired result is obvious. If \( \varphi(z) \neq 0 \), from the definition of \( \sigma_v \),
\[
\frac{1}{\sigma_v (r)} = 1 + \int_0^r \frac{dt}{(1-t)^{1/2} (1-t^2)^{\alpha+1}} = \frac{(1 - r^2)^{1/2}}{\sqrt{v(r)}}, \quad 0 \leq r < 1.
\]
Thus
\[
G_{\varphi(z)} (\varphi(z), \varphi(z)) \geq \frac{1}{v^2 (\varphi(z))} \times \left[ \frac{v^2 (\varphi(z))}{\sigma_v^2 (\varphi(z))} \right] \left| \varphi(z), \varphi(z) \right|^2 \\
+ \left( 1 - \frac{v^2 (\varphi(z))}{\sigma_v^2 (\varphi(z))} \right) \left| \varphi(z), \varphi(z) \right|^2 \frac{\left| \varphi(z), \varphi(z) \right|^2}{|\varphi(z)|^2} \\
\leq \frac{C}{v^2 (\varphi(z))} \times \left[ (1 - |\varphi(z)|^2) \left( |\varphi(z)|^2 - \frac{|\varphi(z), \varphi(z)|^2}{|\varphi(z)|^2} \right) \\
+ \left| \varphi(z), \varphi(z) \right|^2 \frac{|\varphi(z), \varphi(z)|^2}{|\varphi(z)|^2} \right] \\
= \frac{C}{v^2 (\varphi(z))} \times \left[ (1 - |\varphi(z)|^2) \left( |\varphi(z)|^2 + |\varphi(z), \varphi(z)|^2 \right) \right] \\
= C \frac{(1 - |\varphi(z)|^2)^2}{v^2 (\varphi(z))} H_{\varphi(z)} (\varphi(z), \varphi(z)) \\
= \frac{CH_{\varphi(z)} (\varphi(z), \varphi(z))}{(1 - |\varphi(z)|^2)^{2(n\gamma/(n+1)+1/p)}}.
\]

The desired result follows from (20). The proof is completed. \( \square \)

The proof of the next lemma is standard; see, for example, [4, Proposition 3.11]. Hence, it is omitted.

**Lemma 3.** Assume that \( 0 < p,q < \infty \), \( -1 < \gamma < \infty \), \( \mu \) is a normal function, and \( \varphi \in S(B_n) \), \( \psi \in H(B_n) \). Then \( T_{\varphi,\psi} : H_{p,q,\gamma} \rightarrow \mathcal{B}_p^\mu \) is compact and only if only if for any bounded sequence \( \{f_k\}_{k \in \mathbb{N}} \) in \( H_{p,q,\gamma} \) which converges to zero uniformly on compact subsets of \( B_n \) as \( k \to \infty \); then \( \|T_{\varphi,\psi} f_k\|_{\mathcal{B}_p^\mu} \to 0 \), as \( k \to \infty \).
Lemma 4. For $\beta > -1$ and $m > 1 + \beta$, one has
\[
\int_0^1 \frac{(1-r)^\beta}{(1-\rho r)^m} dr \leq C(1-\rho)^{1+\beta-m}, \quad 0 < \rho < 1.
\] (21)

Proof.
\[
\int_0^1 \frac{(1-r)^\beta}{(1-\rho r)^m} dr = \int_0^1 \frac{(1-r)^\beta}{(1-\rho r)^{m-\beta}(1-r)^\beta} dr \\
\leq \int_0^1 \frac{(1-r)^\beta}{(1-\rho r)^{m-\beta}} dr \\
= \int_0^1 \frac{1}{p(1+\beta-m)(1-r)^{1+\beta-m}} dr \\
= C(1-\rho)^{1+\beta-m}.
\] (22)

This completes the proof. \qed

3. The Boundedness and Compactness of $T_{\psi,\varphi} : H_{p,q,\gamma} \to B_\mu$

Theorem 5. Assume that $0 < p, q < \infty$, $-1 < \gamma < \infty$, $\mu$ is a normal function, and $\varphi \in S(B_n)$, $\psi \in H(B_n)$. Then $T_{\psi,\varphi} : H_{p,q,\gamma} \to B_\mu$ is bounded if and only if
\[
M_1 := \sup_{z \in B_n} \frac{\mu(z) |\Re \psi(z)|}{(1-|\varphi(z)|^2)^{n/q+(\gamma+1)/p}} < \infty, \quad \text{(23)}
\]
\[
M_2 := \sup_{z \in B_n} \frac{\mu(z) |\varphi(z)|}{(1-|\varphi(z)|^2)^{n/q+(\gamma+1)/p}} \times \left| H_{\psi(z)}(\varphi(z) z, J\varphi(z) z) \right|^{1/2} < \infty. \quad \text{(24)}
\]

Proof

Sufficiency. Assume that (23) and (24) hold. Then for any $f \in H_{p,q,\gamma}$, if $J\varphi(z)z \neq 0$ for $z \in B_n$, by Lemma 1 and Lemma 2, it follows that
\[
\|T_{\psi,\varphi} f(z)\|_{B_\mu} = \sup_{z \in B_n} \mu(z) |\Re (T_{\psi,\varphi} f)(z)| \\
\leq \sup_{z \in B_n} \mu(z) |\Re \psi(z)| \left| f(\varphi(z)) \right| \\
\leq \sup_{z \in B_n} \frac{\mu(z) |\Re \psi(z)|}{(1-|\varphi(z)|^2)^{n/q+(\gamma+1)/p}} \left( H_{\psi(z)}(\varphi(z) z, J\varphi(z) z) \right) \\
\leq M_1 \|f\|_{H_{p,q,\gamma}} + C M_2 \|f\|_{B_{(1-|\varphi(z)|^2)^{-n/q+(\gamma+1)/p}}} \leq C \|f\|_{H_{p,q,\gamma}}.
\]

When $J\varphi(z)z = 0$ for $z \in B_n$. From (23) we can easily obtain
\[
\mu(z) |\Re (T_{\psi,\varphi} f)(z)| < M_1 \|f\|_{H_{p,q,\gamma}}. \quad \text{(26)}
\]

Combining (25) and (26), the boundedness of $T_{\psi,\varphi} : H_{p,q,\gamma} \to B_\mu$ follows.

Necessity. Suppose that $T_{\psi,\varphi} : H_{p,q,\gamma} \to B_\mu$ is bounded. Firstly, we assume that $w \in B_n$ and $\varphi(w) = r_w e_1$, where $r_w = |\varphi(w)|$ and $e_1 = (1, 0, 0, \ldots, 0)$. If $\sqrt{(1-r_w^2)} |\eta_1|^2 + \cdots + |\eta_n|^2 \leq |\eta_1|$, where $J\varphi(w)w = (\eta_1, \ldots, \eta_n)^T$, choose the function
\[
f_w(z) = \frac{z_1-r_w}{1-r_w z_1} \left( \frac{1-r_w^2}{1-r_w z_1} \right)^{n/q+(\gamma+1)/p}.
\] (27)

By [20, Theorem 1.12] and Lemma 4 we have that
\[
M_q (f_w, r) = \left( \int_S |f_w(r \zeta)|^q d\sigma(\zeta) \right)^{1/q} \\
\leq \left( \int_S \left( \frac{1-r_w^2}{1-r_w r_w^2 \zeta_1^2} \right)^{n/q+(\gamma+1)/p} d\sigma(\zeta) \right)^{1/q} \\
\leq C \left( \frac{1-r_w^2}{1-r_w^2} \right)^{n/q+(\gamma+1)/p},
\]

where $\zeta = (\zeta_1, \ldots, \zeta_n)^T$ and $\sigma$ is the Lebesgue measure on $S(\mathbb{C})$.
\[
\|f_w\|_{H^{p,q,\gamma}}^p = \int_0^1 M_{\gamma}^p (f_w, r) (1 - r)^\gamma dr \\
\leq C (1 - r_w^2)^{\frac{mpq}{2} + \frac{\gamma}{2}} \int_0^1 \frac{(1 - r)^\gamma}{(1 - rr_w^2)^{\frac{mpq}{2} + \frac{\gamma}{2}}} dr \\
\leq C (1 - r_w^2)^{\frac{mpq}{2} + \frac{\gamma}{2}} \frac{1}{(1 - r_w^2)^{\frac{mpq}{2} + \frac{\gamma}{2}}} \leq C. 
\]  
\[\text{(28)}\]

Then \(f_w \in H_{p,q,\gamma}\) and \(\|f_w\|_{H^{p,q,\gamma}} \leq C\). Moreover, \(f_w(\varphi(w)) = 0\) and

\[
\nabla f_w (\varphi (w)) = \left( \frac{1}{(1 - r_w^2)^{\frac{m}{q} + \frac{\gamma}{p} + 1}}, 0, \ldots, 0 \right). 
\]  
\[\text{(29)}\]

Thus

\[
\left\|T_{\varphi f} f_w \right\|_{B_\mu} \geq \mu (w) |\mathfrak{R} (f \circ \varphi) (w)| \\
\geq \mu (w) |\varphi (w)| \left| \mathfrak{R} \right| (f \circ \varphi) (w) | - \mu (w) |\mathfrak{R} \varphi (w)| \left| f_w (\varphi (w)) \right| \\
= \mu (w) |\varphi (w)| \left| \nabla f_w (\varphi (w), J\varphi (w) w) \right| \\
= \mu (w) |\varphi (w)| |\eta| \\
(1 - r_w^2)^{\frac{m}{q} + \frac{\gamma}{p} + 1}. 
\]

By the definition of \(H_{\varphi \omega}(J\varphi (w) w, J\varphi (w) w)\) and (30) it follows that

\[
\mu (w) |\varphi (w)| \left\{ H_{\varphi \omega} (J\varphi (w) w, J\varphi (w) w) \right\}^{1/2} \\
= \left( \mu (w) |\varphi (w)| \right) \left\{ (1 - |\varphi (w)|^2) J\varphi (w) w^2 + |\varphi (w), J\varphi (w) w|^2 \right\}^{1/2} \\
\times \left( \frac{1 - |\varphi (w)|^2}{(1 - |\varphi (w)|^2)^{\frac{m}{q} + \frac{\gamma}{p} + 1}} \right) \\
= \mu (w) |\varphi (w)| \left\{ (1 - r_w^2)^2 \left| \eta_2 \right|^2 + \cdots + \left| \eta_n \right|^2 \right\}^{1/2} \\
\leq \sqrt{\mu (w) |\varphi (w)| |\eta|} \\
\leq \sqrt{2} \mu (w) |\varphi (w)| |\eta| \\
(1 - r_w^2)^{\frac{m}{q} + \frac{\gamma}{p} + 1} \leq C \left\| T_{\varphi f} f_w \right\|_{B_\mu} \leq C. 
\]  
\[\text{(31)}\]

This shows that when \(\sqrt{(1 - r_w^2)(\left| \eta_2 \right|^2 + \cdots + \left| \eta_n \right|^2)} \leq |\eta_1|,\) (24) follows.

On the other hand, if \(\sqrt{(1 - r_w^2)(\left| \eta_2 \right|^2 + \cdots + \left| \eta_n \right|^2)} > |\eta_1|\).

For \(j = 2, \ldots, n\), let \(\theta_j = \arg \eta_j\) and \(a_j = e^{-\theta_j}\), when \(\eta_j \neq 0\); otherwise \(a_j = 0\) when \(\eta_j = 0\). Take

\[
f_w (z) = \frac{a_2 z_2 + \cdots + a_n z_n}{(1 - r_w z_1)^{\frac{m}{q} + \frac{\gamma}{p} + 1}}. 
\]  
\[\text{(32)}\]

By [20, Theorem 1.12] and Lemma 4 we obtain that

\[
M_{\gamma} (f_w, r) \leq C \left\{ \int_S \frac{(k_n)^q}{|1 - r_w K_{\gamma}^1|^{n+q/2} + q \frac{q}{2}} d\sigma(\zeta) \right\}^{1/q} \\
\leq C \left\{ \int_S \frac{C(k_n)^q}{|1 - r_w K_{\gamma}^1|^{n+q/2} + q \frac{q}{2}} d\sigma(\zeta) \right\}^{1/q} \\
= \left\{ \int_S \frac{C(1 - k_n)^q}{|1 - r_w K_{\gamma}^1|^{n+q/2} + q \frac{q}{2}} d\sigma(\zeta) \right\}^{1/q} \\
\leq \frac{C}{(1 - r_w^2)^{\frac{m}{q} + \frac{\gamma}{p} + 1}}. 
\]

\[\|f_w\|_{H^{p,q,\gamma}}^p = \int_0^1 M_{\gamma}^p (f_w, r) (1 - r)^\gamma dr \\
= C \int_0^1 \frac{(1 - r)^\gamma}{(1 - r_r^2)^{\frac{mpq}{2} + \frac{\gamma}{2}}} dr \\
\leq C (1 - r_w^2)^{\frac{mpq}{2} + \frac{\gamma}{2}} \frac{1}{(1 - r_w^2)^{\frac{mpq}{2} + \frac{\gamma}{2}}} \leq C. 
\]  
\[\text{(33)}\]

Hence \(f_w \in H_{p,q,\gamma}\) and \(\|f_w\|_{H^{p,q,\gamma}} \leq C\). Moreover \(f_w(\varphi(w)) = 0\) and

\[
\nabla f_w (\varphi (w)) \\
= \left( 0, \frac{a_2}{(1 - r_w^2)^{\frac{m}{q} + \frac{\gamma}{p} + 1}}, \ldots, \frac{a_n}{(1 - r_w^2)^{\frac{m}{q} + \frac{\gamma}{p} + 1}} \right). 
\]  
\[\text{(34)}\]

Similar to the proof of (30), we obtain that

\[
\mu (w) |\varphi (w)| \left\{ \eta_2 + \cdots + |\eta_n| \right\} \leq C \left\| T_{\varphi f} f_w \right\|_{B_\mu}. 
\]  
\[\text{(35)}\]

It follows from (35) that

\[
\mu (w) |\varphi (w)| \left\{ (1 - |\varphi (w)|^2)^{\frac{m}{q} + \frac{\gamma}{p} + 1} \right\}^{1/2} \\
= \left( \mu (w) |\varphi (w)| \right) \left\{ (1 - |\varphi (w)|^2)^{\frac{m}{q} + \frac{\gamma}{p} + 1} \right\}^{1/2} \\
\times \left\{ (1 - |\varphi (w)|^2) |J\varphi (w) w|^2 + |\varphi (w), J\varphi (w) w|^2 \right\}^{1/2}. 
\]
\[ \left( 1 - |\psi(w)|^2 \right)^{n/(q+(y+1)/p)+1} \]
\[ \mu(w) \left| \psi(w) \right| \left( \left( 1 - r_w^2 \right) \left( |\eta_1|^2 + \cdots + |\eta_n|^2 \right) + |\eta|^2 \right)^{1/2} \]
\[ \leq \mu(w) \left| \psi(w) \right| \left( \sqrt{(1 - r_w^2) \left( |\eta_1|^2 + \cdots + |\eta_n|^2 \right)} \right)^{1/2} \]
\[ \leq C \left( \mu(w) \right) \left( \sqrt{(1 - r_w^2) \left( |\eta_1|^2 + \cdots + |\eta_n|^2 \right)} \right)^{1/2} \]
\[ \leq \mu(w) \left| \psi(w) \right| \frac{\sqrt{(1 - r_w^2) \left( \eta_1^2 + \cdots + \eta_n^2 \right)}}{(1 - r_w^2)^{n/(q+(y+1)/p)+1}} \]
\[ \leq \mu(w) \left| \psi(w) \right| \left( \sqrt{(1 - r_w^2) \left( |\eta_1|^2 + \cdots + |\eta_n|^2 \right)} \right)^{1/2} \]
\[ \leq \mu(w) \left| \psi(w) \right| \left( \sqrt{(1 - r_w^2) \left( |\eta_1|^2 + \cdots + |\eta_n|^2 \right)} \right)^{1/2} \]

By [20, Theorem 1.12], it follows that
\[ M_p \left( h_w(z) , r \right) \leq \frac{\left( 1 - |w|^2 \right)^{b-(y+1)/p}}{(1 - r|w|^2)^b} \] (40)

Applying Lemma 4 we have that
\[ \| h_w \|_{H_{p,q,y}}^q \leq \int_0^1 M_p^q \left( h_w, r \right) (1 - r)^y \, dr \]
\[ \leq C \int_0^1 \left( 1 - |w|^2 \right)^{b-(y+1)} (1 - r)^y \, dr \]
\[ = C \left( 1 - |w|^2 \right)^{b-(y+1)} \int_0^1 (1 - r)^y \, dr \]
\[ \leq C \left( 1 - |w|^2 \right)^{b-(y+1)} (1 - |w|^2) = C. \]

Therefore \( h_w \in H_{p,q,y} \), and \( \sup_{w \in \mathbb{B}_q} \| h_w \|_{H_{p,q,y}} \leq C \). Besides,
\[ h_{\psi(w)} (\varphi(w)) = \left( \frac{1}{1 - |\psi(w)|^2} \right)^{n/(q+(y+1)/p)} \]
\[ \nabla h_{\psi(w)} (\varphi(w)) = \left( \frac{n}{q} + b \right) \left( \frac{\varphi_1(w)}{(1 - |\psi(w)|^2)^{n/(q+(y+1)/p)+1}} \right) \]
\[ \left( \frac{\varphi_n(w)}{(1 - |\psi(w)|^2)^{n/(q+(y+1)/p)+1}} \right) \)

Therefore,
\[ \infty \geq \| T_{\psi} (h_{\psi(w)}(\varphi)) \|_{\mathbb{B}_p} \geq \mu(w) \left| \Re (h_{\psi(w)} \ast \varphi)(w) \right| \]
\[ = \mu(w) \left| \Re (\varphi_1) \right| h_{\psi(w)} (\varphi(w)) + \psi(w) \Re \left( h_{\psi(w)} \ast \varphi \right)(w) \]
\[ \geq \mu(w) \left| \Re (\varphi_1) \right| \left( 1 - |\psi(w)|^2 \right)^{n/(q+(y+1)/p)} \]
\[ - \mu(w) \left| \psi(w) \right| \left| \Re \left( h_{\psi(w)} \ast \varphi \right)(w) \right|. \] (44)
It follows from (43) and (24) that
\[
\mu(w) |\psi(w)| \Re (h_{\psi(w)} \circ \varphi)(w) = \mu(w) |\psi(w)| \left| \langle \nabla h_{\psi(w)}(\varphi(w)), \varphi(w) \rangle \right|
\]
\[
= \left( \frac{n+b}{q} \right) \frac{\mu(w) |\psi(w)| |\varphi(w)|}{(1 - |\varphi(w)|^2)^{n/\gamma(q+1) + p}}
\]
\[
\leq \left( \frac{n+b}{q} \right) \frac{\mu(w) |\psi(w)|}{(1 - |\varphi(w)|^2)^{n/\gamma(q+1) + p}}
\]
\[
\times \left\{ H_{\psi(w)} \left( J\varphi(w) \varphi, J\varphi(w) \varphi \right) \right\}^{1/2}
\]
\[
\leq CM_2 < \infty.
\]
Combining (44) and (45), the desired result (23) holds. This completes the proof.

**Theorem 6.** Assume that \( 0 < p, q < \infty, -1 < \gamma < \infty, \mu \) is a normal function, and \( \varphi \in \mathbb{S}(B_n), \psi \in H(B_n). \) Then \( \mathcal{T}_{\psi, \varphi} : H_{p,q,\gamma} \rightarrow \mathcal{R}_\mu \) is compact if and only if the followings are all satisfied:

(a) \( \psi \in \mathcal{R}_\mu \) and \( \psi \varphi_l \in \mathcal{R}_{\mu_l} \) for \( l = 1, \ldots, n; \)

(b) \( \lim_{\varphi(z) \rightarrow 1} \frac{\mu(z) |\psi(z)|}{(1 - |\varphi(z)|^2)^{n/\gamma(q+1) + p}} = 0; \)

(c) \( \lim_{|\varphi(z)| \rightarrow 1} \frac{\mu(z) |\psi(z)|}{(1 - |\varphi(z)|^2)^{n/\gamma(q+1) + p}} \times \left\{ H_{\psi(z)} \left( J\varphi(z) \varphi, J\varphi(z) \varphi \right) \right\}^{1/2} = 0. \)

**Proof**

**Sufficiency.** Suppose that (a), (b), and (c) hold. Then for any \( \varepsilon > 0, \) there is \( \delta > 0, \) such that

\[
\frac{\mu(z) |\psi(z)|}{(1 - |\varphi(z)|^2)^{n/\gamma(q+1) + p}} < \varepsilon,
\]

\[
\frac{\mu(z) |\psi(z)|}{(1 - |\varphi(z)|^2)^{n/\gamma(q+1) + p}} \times \left\{ H_{\psi(z)} \left( J\varphi(z) \varphi, J\varphi(z) \varphi \right) \right\}^{1/2} < \varepsilon,
\]

when \( |\varphi(z)| > \delta. \)

Let \( \{ f_k \}_{k \in \mathbb{N}} \) be any sequence which converges to 0 uniformly on compact subsets of \( B_n \) satisfying \( \| f_k \|_{H_{p,q,\gamma}} \leq 1. \)

Then \( f_k \) and \( \mathcal{R}_\mu f_k \) converge to 0 uniformly on \( K = \{ w \in B_n : |w| \leq \delta \}. \) Hence

\[
\sup_{z \in B_n} \mu(z) \Re \left( \mathcal{T}_{\psi, \varphi} f_k(z) \right) = \sup_{z \in B_n} \mu(z) \Re \left( \mathcal{T}_{\psi, \varphi} f_k(z) \right)
\]

\[
+ \sup_{z \in B_n} \mu(z) \Re \left( \mathcal{T}_{\psi, \varphi} f_k(z) \right).
\]

If \( \varphi(z) \in B_n \setminus K \) and \( J\varphi(z) \varphi \neq 0, \) by Lemma 1 and Lemma 2, we have

\[
\mu(z) \Re \left( \mathcal{T}_{\psi, \varphi} f_k(z) \right) \leq \mu(z) |\psi(z)| |\Re \left( f_k \varphi \right)(z) + \mu(z) |\Re \left( \psi \right)(z)| \left| f_k \varphi \right|(z)\]

\[
\leq \left| \mu(z) \Re \left( f_k \varphi \right)(z) \right| \left( n \right) \frac{n/\gamma(q+1) + p}{n/\gamma(q+1) + p}
\]

\[
\times \left( 1 - |\varphi(z)|^2 \right)^{n/\gamma(q+1) + p} \left( H_{\psi(z)} \left( J\varphi(z) \varphi, J\varphi(z) \varphi \right) \right)^{1/2}
\]

\[
\leq C \varepsilon \| f_k \|_{H_{p,q,\gamma}} \leq C \varepsilon.
\]

When \( J\varphi(z) \varphi = 0, \)

\[
\mu(z) \Re \left( \mathcal{T}_{\psi, \varphi} f_k(z) \right) \leq \varepsilon \| f_k \|_{H_{p,q,\gamma}} \leq \varepsilon.
\]

Combining (50) and (51) we obtain that

\[
\sup_{\varphi(z) \in B_n, K} \mu(z) \Re \left( \mathcal{T}_{\psi, \varphi} f_k(z) \right) \leq C \varepsilon.
\]

If \( \varphi(z) \in K, \) by (a), we have that

\[
\mu(z) \Re \left( \mathcal{T}_{\psi, \varphi} f_k(z) \right)
\]

\[
\leq \mu(z) |\psi(z)| |\Re \left( f_k \varphi \right)(z) + \mu(z) |\Re \left( \psi \right)(z)| \left| f_k \varphi \right|(z)
\]

\[
\leq \mu(z) |\psi(z)| \left| \left\{ \nabla f_k \varphi(z), J\varphi(z) \right\} \right| + \left| f_k \varphi(z) \right| \left| \psi \right| \left| \Re \left( \psi \right)(z) \right|
\]

\[
\leq \left| \nabla f_k \varphi(z) \right| \left( \sum_{l=1}^{n/\gamma(q+1) + p} \left( \mu(z) |\psi(z)| \left| \Re \left( \psi \right)(z) \right| \right)
\]

\[
+ \left| f_k \varphi(z) \right| \left| \psi \right| \left| \Re \left( \psi \right)(z) \right|
\]
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If \( \sqrt{1-r_k^2}(|\eta_2|^2 + \cdots + |\eta_n|^2) \leq |\eta_1| \), where \( J\varphi(z_k)z_k = (\eta_1, \ldots, \eta_n)^T \). Let

\[
f_k(z) = \frac{z_k - r_k}{1 - r_k^2} \left( \frac{1 - r_k^2}{(1 - r_k z_k^2)^{n/2}} \right)^{n/2}.
\]

From Theorem 5 we know that \( f_k \in H_{p,q,\gamma} \) and we notice that \( f_k \) converges to 0 uniformly on compact subsets of \( B_n \) when \( k \to \infty \). By Lemma 3 we have \( \lim_{k \to \infty} \| T_{w,w}(f_k) \|_{B_\mu} = 0 \). Then by a similar proof of (30) in Theorem 5 we have

\[
\frac{\mu(z_k)}{\nu_{n+1}(p+1)} \leq \| T_{w,w}(f_k) \|_{B_\mu} \to 0, \quad k \to \infty.
\]

And similar to the proofs of (31) and (57) we get that

\[
\frac{\mu(z_k)}{\nu_{n+1}(p+1)} \leq \| T_{w,w}(f_k) \|_{B_\mu} \to 0, \quad k \to \infty.
\]

On the other hand, we consider the case of

\[
\sqrt{1-r_k^2}(|\eta_2|^2 + \cdots + |\eta_n|^2) > |\eta_1|.
\]

For \( j = 2, \ldots, n \), let \( \theta_j = \arg \eta_j \) and \( e_j = e^{i \theta_j} \), when \( \eta_j \neq 0 \); otherwise \( e_j = 0 \) when \( \eta_j = 0 \). Take

\[
f_k(z) = \left( \frac{a_2 z_2 + \cdots + a_n z_n}{1 - r_k^2} \right) \left( \frac{1 - r_k^2}{(1 - r_k z_k^2)^{n/2}} \right)^{n/2}
\]

Then \( f_k \in H_{p,q,\gamma} \), \( k \in \mathbb{N} \), and \( f_k \) converges to 0 uniformly on compact subsets of \( B_n \) when \( k \to \infty \). By Lemma 3 we have \( \lim_{k \to \infty} \| T_{w,w}(f_k) \|_{B_\mu} = 0 \). Notice that \( f_k(q(z_k)) = 0 \) and

\[
\nabla f_{\varphi}(q(z_k)) = 0.
\]

By a similar proof of (30), it follows that

\[
\frac{\mu(z_k)}{\nu_{n+1}(p+1)} \leq \| T_{w,w}(f_k) \|_{B_\mu} \to 0, \quad k \to \infty.
\]
And similar to the proofs of (31) and (61), we obtain
\[
\frac{\mu(z_k)|\psi(z_k)|}{(1 - |\psi(z_k)|^2)^{\alpha}} \left[ H_{\psi(z_k)} (J\psi(z_k) z_k, J\psi(z_k) z_k) \right]^{1/2}
\leq C \frac{\mu(z_k)|\psi(z_k)|}{(1 - |z_k|^2)^{n/2q + (\nu + 1)/p}} \rightarrow 0
\]
\[
k \rightarrow \infty.
\]
Combining (58) and (62), (47) holds under the two cases.

For the general situation, if there exists \( \varphi(z_k) \) such that \( \varphi(z_k) \neq |\varphi(z_k)|e_1 \), then there is a unitary transformation \( U_k \) such that \( \varphi(z_k) = r_k e_1 U_k, k \in \{1, 2, \ldots, n\} \). And we can prove (47) by taking the function sequence \( g_k = f_k \circ U_k^{-1} \) and the details are omitted.

Next we prove (46). Let \( \{z_k \} \) be a sequence in \( B_n \) such that \( |\varphi(z_k)| \rightarrow 1 \) as \( k \rightarrow \infty \). Choose
\[
h_k(z) = \frac{1 - |\varphi(z_k)|^2}{1 - \langle z, \varphi(z_k) \rangle} \left( b - \frac{(\nu + 1)}{p} \right) \frac{1}{n/\sqrt{q} + b^2}.
\]
Then \( h_k \in H_{p,q,y} \), \( k \in \mathbb{N} \), and \( \|h_k\|_{H_{p,q,y}} \leq C \). It is obvious that \( h_k \rightarrow 0 \) uniformly on compact subsets of \( B_n \) as \( k \rightarrow \infty \). By Lemma 3 we have that \( \lim_{k \rightarrow \infty} \|T_{\psi h_k}(z)\|_{\mathcal{B}_u} = 0 \).

Then by the similar proof of (44) we obtain
\[
\|T_{\psi h_k}(z)\|_{\mathcal{B}_u} \geq \frac{\mu(z_k)|R\psi(z_k)|}{(1 - |\psi(z_k)|^2)^{n/2q + (\nu + 1)/p}}
- \frac{\mu(z_k)|\psi(z_k)|}{(1 - |\psi(z_k)|^2)^{n/2q + (\nu + 1)/p}} \left| R(h_k \circ \varphi)(z_k) \right|.
\]

From the similar proof of (45) it follows that
\[
\mu(z_k)|\psi(z_k)| \left| R(h_k \circ \varphi)(z_k) \right|
\leq \left( \frac{n}{q} + b \right) \frac{\mu(z_k)|\psi(z_k)|}{(1 - |\psi(z_k)|^2)^{n/2q + (\nu + 1)/p}}
\times \left\{ H_{\psi(z_k)} (J\psi(z_k) z_k, J\psi(z_k) z_k) \right\}^{1/2} \rightarrow 0,
\]
k \rightarrow \infty.

Combining (64) and (65) we obtain (46). This completes the proof.

**Corollary 8.** Assume that \( 0 < p, q < \infty, -1 < \gamma < \infty, \mu \) is a normal function, and \( \varphi \in S(B_n) \). Then \( C_{\varphi} : H_{p,q,y} \rightarrow \mathcal{B}_u \) is compact if and only if
\[
\lim_{|z| \rightarrow 1} \frac{\mu(z) \left\{ H_{\varphi(z)} (J\psi(z) z, J\varphi(z) z) \right\}^{1/2}}{(1 - |\varphi(z)|^2)^{n/2q + (\nu + 1)/p}} = 0.
\]

And \( \varphi_i \in \mathcal{B}_u \) for \( l \in \{1, \ldots, n\} \).

**Corollary 9.** Assume that \( 0 < p, q < \infty, -1 < \gamma < \infty, \mu \) is a normal function, and \( \psi \in H(B_n) \). Then \( M_{\psi} : H_{p,q,y} \rightarrow \mathcal{B}_u \) is compact if and only if
\[
\sup_{z \in B_n} \frac{\mu(z) |R\psi(z)|}{(1 - |z|^2)^{n/2q + (\nu + 1)/p}} < \infty,
\]
\[
\sup_{z \in B_n} \frac{\mu(z) |\psi(z)|}{(1 - |z|^2)^{n/2q + (\nu + 1)/p + 1}} < \infty.
\]

**Corollary 10.** Assume that \( 0 < p, q < \infty, -1 < \gamma < \infty, \mu \) is a normal function, and \( \psi \in H(B_n) \). Then \( M_{\psi} : H_{p,q,y} \rightarrow \mathcal{B}_u \) is compact if and only if the following are all satisfied:

(a) \( \psi \in \mathcal{B}_u \) and \( \varphi_l \in \mathcal{B}_u \) for any \( l \in \{1, \ldots, n\} \);

(b) \[
\lim_{|z| \rightarrow 1} \left( \frac{\mu(z) |R\psi(z)|}{(1 - |z|^2)^{n/2q + (\nu + 1)/p}} \right) = 0;
\]

(c) \[
\lim_{|z| \rightarrow 1} \left( \frac{\mu(z) |\psi(z)|}{(1 - |z|^2)^{n/2q + (\nu + 1)/p + 1}} \right) = 0.
\]

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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**References**


