Research Article

A Modified Mixed Ishikawa Iteration for Common Fixed Points of Two Asymptotically Quasi Pseudocontractive Type Non-Self-Mappings

Yuanheng Wang and Huimin Shi

Department of Mathematics, Zhejiang Normal University, Zhejiang 321004, China

Correspondence should be addressed to Yuanheng Wang; wangyuanhengmath@163.com

Received 3 January 2014; Accepted 21 February 2014; Published 26 March 2014

Academic Editor: Rudong Chen

Copyright © 2014 Y. Wang and H. Shi. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

A new modified mixed Ishikawa iterative sequence with error for common fixed points of two asymptotically quasi pseudocontractive type non-self-mappings is introduced. By the flexible use of the iterative scheme and a new lemma, some strong convergence theorems are proved under suitable conditions. The results in this paper improve and generalize some existing results.

1. Introduction

Let E be a real Banach space with its dual E^* and let C be a nonempty, closed, and convex subset of E . The mapping $J : E \rightarrow 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\| \cdot \|x^*\|, \|x\| = \|x^*\|\}, \quad x \in E. \quad (1)$$

Let $T : C \rightarrow E$ be a mapping. We denote the fixed point set of T by $F(T)$; that is, $F(T) = \{x \in C : x = Tx\}$. Recall that a mapping $T : C \rightarrow E$ is said to be nonexpansive if, for each $x, y \in C$,

$$\|Tx - Ty\| \leq \|x - y\|. \quad (2)$$

T is said to be asymptotically nonexpansive if there exists a sequence $k_n \subseteq [1, \infty)$ with $k_n \rightarrow 1$ as $n \rightarrow \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in C. \quad (3)$$

A sequence of self-mappings $\{T_i\}_{i=1}^\infty$ on C is said to be uniform Lipschitzian with the coefficient L if, for any $i = 1, 2, \dots$, the following holds:

$$\|T_i^n x - T_i^n y\| \leq L \|x - y\|, \quad \forall x, y \in C. \quad (4)$$

T is said to be asymptotically pseudocontractive if there exist $k_n \subseteq [1, \infty)$ with $k_n \rightarrow 1$ as $n \rightarrow \infty$ and $j(x - y) \in J(x - y)$ such that

$$\langle T^n x - T^n y, j(x - y) \rangle \leq k_n \|x - y\|^2, \quad \forall x, y \in C. \quad (5)$$

It is obvious to see that every nonexpansive mapping is asymptotically nonexpansive and every asymptotically nonexpansive mapping is asymptotically pseudocontractive. Goebel and Kirk [1] introduced the class of asymptotically nonexpansive mappings in 1972. The class of asymptotically pseudocontractive mappings was introduced by Schu [2] and has been studied by various authors for its generalized mappings in Hilbert spaces, Banach spaces, or generalized topological vector spaces by using the modified Mann or Ishikawa iteration methods (see, e.g., [3–21]).

In 2003, Chidume et al. [22] studied fixed points of an asymptotically nonexpansive non-self-mapping $T : C \rightarrow E$ and the strong convergence of an iterative sequence $\{x_n\}$ generated by

$$x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n T(P T)^{n-1} x_n), \quad n \geq 1, x_1 \in C, \quad (6)$$

where $P : E \rightarrow C$ is a nonexpansive retraction.

In 2011, Zegeye et al. [23] proved a strong convergence of Ishikawa scheme to a uniformly L-Lipschitzian and asymptotically pseudocontractive mappings in the intermediate sense which satisfies the following inequality (see [24]):

$$\limsup_{n \rightarrow \infty} \sup_{x, y \in C} (\langle T^n x - T^n y, x - y \rangle - k_n \|x - y\|^2) \leq 0, \quad (7)$$

$$\forall x, y \in C,$$

where $k_n \subseteq [1, \infty)$ with $k_n \rightarrow 1$ as $n \rightarrow \infty$.

Motivated and inspired by the above results, in this paper, we introduce a new modified mixed Ishikawa iterative sequence with error for common fixed points of two more generalized asymptotically quasi pseudocontractive type non-self-mappings. By the flexible use of the iterative scheme and a new lemma (i.e., Lemma 6 in this paper), under suitable conditions, we prove some strong convergence theorems. Our results extend and improve many results of other authors to a certain extent, such as [6, 8, 14–23].

2. Preliminaries

Definition 1. Let C be a nonempty closed convex subset of a real Banach space E . C is said to be a nonexpansive retract (with P) of E if there exists a nonexpansive mapping $P : E \rightarrow C$ such that, for all $x \in C$, $Px = x$. And P is called a nonexpansive retraction.

Let $T : C \rightarrow E$ be a non-self-mapping (maybe self-mapping). T is called uniformly L-Lipschitzian (with P) if there exists a constant $L > 0$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L \|x - y\|, \quad \forall x, y \in C, \quad n \geq 1. \quad (8)$$

T is said to be asymptotically pseudocontractive (with P) if there exist $k_n \subseteq [1, \infty)$ with $k_n \rightarrow 1$ as $n \rightarrow \infty$ and $\forall x, y \in C$, $\exists j(x - y) \in J(x - y)$ such that

$$\langle T(PT)^{n-1}x - T(PT)^{n-1}y, j(x - y) \rangle \leq k_n \|x - y\|^2. \quad (9)$$

T is said to be an asymptotically pseudocontractive type (with P) if there exist $k_n \subseteq [1, \infty)$ with $k_n \rightarrow 1$ as $n \rightarrow \infty$ and $\forall x, y \in C$, $j(x - y) \in J(x - y)$ such that

$$\limsup_{n \rightarrow \infty} \sup_{x, y \in C} \liminf_{j(x-y) \in J(x-y)} (\langle T(PT)^{n-1}x - T(PT)^{n-1}y, j(x - y) \rangle - k_n \|x - y\|^2) \leq 0. \quad (10)$$

T is said to be an asymptotically quasi pseudocontractive type (with P) if $F(T) \neq \emptyset$, for $p \in F(T)$, there exist $k_n \subseteq [1, \infty)$ with $k_n \rightarrow 1$ as $n \rightarrow \infty$, and, $\forall x \in C$, $j(x - p) \in J(x - p)$ such that

$$\limsup_{n \rightarrow \infty} \sup_{x \in C} \liminf_{j(x-p) \in J(x-p)} (\langle T(PT)^{n-1}x - p, j(x - y) \rangle - k_n \|x - p\|^2) \leq 0. \quad (11)$$

Remark 2. It is clear that every asymptotically pseudocontractive mapping (with P) is asymptotically pseudocontractive type (with P) and every asymptotically pseudocontractive type (with P) is asymptotically quasi pseudocontractive type (with P). If $T : C \rightarrow C$ is a self-mapping, then we can choose $P = I$ as the identical mapping and we can get the usual definition of asymptotically pseudocontractive mapping, and so forth.

Definition 3. Let C be a nonexpansive retract (with P) of E , let $T_1, T_2 : C \rightarrow E$ be two uniformly L-Lipschitzian non-self-mappings and let T_1 be an asymptotically quasi pseudocontractive type (with P).

The sequence $\{x_n\}$ is called the new modified mixed Ishikawa iterative sequence with error (with P), if $\{x_n\}$ is generated by

$$\begin{aligned} x_{n+1} &= P \left((1 - \alpha_n - \gamma_n) x_n + \alpha_n T_1 (PT_1)^{n-1} \right. \\ &\quad \times \left. \left((1 - \beta_n) y_n + \beta_n T_1 (PT_1)^{n-1} y_n \right) + \gamma_n u_n \right), \\ y_n &= P \left((1 - \alpha'_n - \gamma'_n) x_n + \alpha'_n T_2 (PT_2)^{n-1} \right. \\ &\quad \times \left. \left((1 - \beta'_n) x_n + \beta'_n T_2 (PT_2)^{n-1} x_n \right) + \gamma'_n v_n \right), \end{aligned} \quad (12)$$

where $x_1 \in C$ is arbitrary, $\{u_n\}$ and $\{v_n\} \subset C$ are bounded, and $\alpha_n, \beta_n, \gamma_n, \alpha'_n, \beta'_n, \gamma'_n \in [0, 1]$, $n = 1, 2, \dots$

If $\alpha'_n = \beta'_n = \gamma'_n = 0$, (12) turns to

$$\begin{aligned} x_{n+1} &= P \left((1 - \alpha_n - \gamma_n) x_n + \alpha_n T_1 (PT_1)^{n-1} \right. \\ &\quad \times \left. \left((1 - \beta_n) x_n + \beta_n T_1 (PT_1)^{n-1} x_n \right) + \gamma_n u_n \right), \end{aligned} \quad (13)$$

and it is called the new modified mixed Mann iterative sequence with error (with P).

If $\gamma_n = \gamma'_n = 0$, (12) becomes

$$\begin{aligned} x_{n+1} &= P \left((1 - \alpha_n) x_n + \alpha_n T_1 (PT_1)^{n-1} \right. \\ &\quad \times \left. \left((1 - \beta_n) y_n + \beta_n T_1 (PT_1)^{n-1} y_n \right) \right), \\ y_n &= P \left((1 - \alpha'_n) x_n + \alpha'_n T_2 (PT_2)^{n-1} \right. \\ &\quad \times \left. \left((1 - \beta'_n) x_n + \beta'_n T_2 (PT_2)^{n-1} x_n \right) \right), \end{aligned} \quad (14)$$

and it is called the new modified mixed Ishikawa iterative sequence (with P).

If $\beta_n = \beta'_n = 0$, (14) turns to

$$\begin{aligned} x_{n+1} &= P \left((1 - \alpha_n) x_n + \alpha_n T_1 (PT_1)^{n-1} y_n \right), \\ y_n &= P \left((1 - \alpha'_n) x_n + \alpha'_n T_2 (PT_2)^{n-1} x_n \right), \end{aligned} \quad (15)$$

and it is called the new mixed Ishikawa iterative sequence (with P).

If $T_1 = T_2 = T : C \rightarrow C$ is a self-mapping and $P = I$ is the identical mapping, then (15) is just the modified Ishikawa iterative sequence

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n) x_n + \alpha_n T^n y_n, \\ y_n &= (1 - \alpha'_n) x_n + \alpha'_n T^n x_n. \end{aligned} \quad (16)$$

If $\alpha'_n = 0$, (15) becomes (6), obviously. So, iterative method (12) is greatly generalized.

The following lemmas will be needed in what follows to prove our main results.

Lemma 4 (see [19]). *Let E be a real Banach space. Then, for all $x, y \in E$, $j(x+y) \in J(x+y)$, the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle x, j(x+y) \rangle. \quad (17)$$

Lemma 5 (see [6, 7]). *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three sequences of nonnegative numbers satisfying the recursive inequality:*

$$a_{n+1} \leq (1+b_n)a_n + c_n, \quad \forall n \geq n_0, \quad (18)$$

where n_0 is some nonnegative integer. If $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} c_n < \infty$, then $\lim_{n \rightarrow \infty} a_n$ exists.

Lemma 6. *Suppose that $\phi : [0, +\infty) \rightarrow [0, +\infty)$ is a strictly increasing function with $\phi(0) = 0$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\lambda_n\}$ ($0 \leq \lambda_n \leq 1$) be four sequences of nonnegative numbers satisfying the recursive inequality:*

$$a_{n+1} \leq (1+b_n)a_n - \lambda_n \phi(a_{n+1}) + c_n, \quad \forall n \geq n_0, \quad (19)$$

where n_0 is some nonnegative integer. If $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Proof. From (19), we get

$$a_{n+1} \leq (1+b_n)a_n + c_n, \quad \forall n \geq n_0. \quad (20)$$

By Lemma 5, we know that $\lim_{n \rightarrow \infty} a_n = a \geq 0$ exists. Let $M = \sup_{1 \leq n \leq \infty} \{a_n\} < \infty$. Now we show $a = 0$. Otherwise, if $a > 0$, then $\exists n_1 \geq n_0$, such that $a_{n+1} \geq (1/2)a > 0$ when $n \geq n_1$. Because ϕ is a strictly increasing function, so $\phi(a_{n+1}) \geq \phi((1/2)a) > 0$. From (19) again, we have

$$\begin{aligned} 0 &< \phi\left(\frac{1}{2}a\right) \sum_{n=1}^{\infty} \lambda_n \\ &= \phi\left(\frac{1}{2}a\right) \sum_{n=1}^{n_1} \lambda_n + \phi\left(\frac{1}{2}a\right) \sum_{n=n_1+1}^{\infty} \lambda_n \\ &\leq \phi\left(\frac{1}{2}a\right) \sum_{n=1}^{n_1} \lambda_n + \sum_{n=n_1+1}^{\infty} \lambda_n \phi(a_{n+1}) \\ &\leq \phi\left(\frac{1}{2}a\right) \sum_{n=1}^{n_1} \lambda_n + \sum_{n=n_1+1}^{\infty} (a_n - a_{n+1}) \\ &\quad + \sum_{n=n_1+1}^{\infty} b_n a_n + \sum_{n=n_1+1}^{\infty} c_n \\ &\leq \phi\left(\frac{1}{2}a\right) \sum_{n=1}^{n_1} \lambda_n + a_{n_1+1} + M \sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} c_n < \infty. \end{aligned} \quad (21)$$

This is a contradiction with the given condition $\sum_{n=1}^{\infty} \lambda_n = \infty$. Therefore $\lim_{n \rightarrow \infty} a_n = 0$. $\square$

Lemma 7. *Suppose that $\phi : [0, +\infty) \rightarrow [0, +\infty)$ is a strictly increasing function with $\phi(0) = 0$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\lambda_n\}$ ($0 \leq \lambda_n \leq 1$), $\{\varepsilon_n\}$ be five sequences of nonnegative numbers satisfying the recursive inequality:*

$$a_{n+1} \leq (1+b_n)a_n - \lambda_n \phi(a_{n+1}) + c_n + \lambda_n \varepsilon_n, \quad \forall n \geq n_0, \quad (22)$$

where n_0 is some nonnegative integer. If $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, $\lim_{n \rightarrow \infty} \varepsilon_n = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Proof. Firstly, we show $\liminf_{n \rightarrow \infty} a_n = a = 0$. If $a > 0$, then, for arbitrary $r \in (0, a)$, $\exists n_1 \geq n_0$, such that $a_{n+1} \geq r > 0$ when $n \geq n_1$. Because ϕ is a strictly increasing function and $\lim_{n \rightarrow \infty} \varepsilon_n = 0$, so $\phi(a_{n+1}) \geq \phi(r) > 0$ and $\varepsilon_n \leq (1/2)\phi(r)$ when $n \geq n_1$. From (22), we have

$$\begin{aligned} a_{n+1} &\leq (1+b_n)a_n - \lambda_n \phi(a_{n+1}) + c_n + \lambda_n \frac{1}{2} \phi(a_{n+1}) \\ &= (1+b_n)a_n - \frac{1}{2} \lambda_n \phi(a_{n+1}) + c_n, \quad \forall n \geq n_1. \end{aligned} \quad (23)$$

By Lemma 6, we get $0 = \lim_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n = a > 0$. This is contradictory. So, $\liminf_{n \rightarrow \infty} a_n = 0$.

Secondly, $\forall \varepsilon > 0$, from the given conditions in Lemma 7, $\exists n_2 \geq n_0$, when $\forall n \geq n_2$, we have

$$\varepsilon_n \leq \phi(\varepsilon), \quad \sum_{n=n_2}^{\infty} b_n \leq \ln 2, \quad \sum_{n=n_2}^{\infty} c_n \leq \varepsilon. \quad (24)$$

On the other hand, since $\liminf_{n \rightarrow \infty} a_n = 0$, $\exists N \geq n_2$ such that $a_N \leq \varepsilon$. Now we claim

$$a_k \leq \left(\varepsilon + \sum_{n=N}^{k-1} c_n \right) \exp\left(\sum_{n=N}^{k-1} b_n \right), \quad \forall k \geq N. \quad (25)$$

In fact, when $k = N$, (25) holds. Suppose that (25) holds for k dose not for $k+1$. Then

$$a_{k+1} > \left(\varepsilon + \sum_{n=N}^k c_n \right) \exp\left(\sum_{n=N}^k b_n \right). \quad (26)$$

Furthermore, $a_{k+1} > \varepsilon$, $\phi(a_{k+1}) > \phi(\varepsilon)$. But by (22), (24), and the inductive hypothesis, we have

$$\begin{aligned} a_{n+1} &\leq (1+b_n)a_n - \lambda_n \phi(a_{n+1}) + c_n + \lambda_n \varepsilon_n \\ &\leq (1+b_n)a_n - \lambda_n \phi(\varepsilon) + c_n + \lambda_n \phi(\varepsilon) \\ &\leq (1+b_n) \left(\varepsilon + \sum_{n=N}^{k-1} c_n \right) \exp\left(\sum_{n=N}^{k-1} b_n \right) + c_n \\ &\leq \left(\varepsilon + \sum_{n=N}^{k-1} c_n \right) \exp\left(\sum_{n=N}^k b_n \right) + c_n \\ &\leq \left(\varepsilon + \sum_{n=N}^k c_n \right) \exp\left(\sum_{n=N}^k b_n \right). \end{aligned} \quad (27)$$

This is a contradiction with (26). So, (25) holds. Whereupon,

$$\limsup_{k \rightarrow \infty} a_k \leq \left(\varepsilon + \sum_{n=N}^{\infty} c_n \right) \exp \left(\sum_{n=N}^{\infty} b_n \right) \leq 2(\varepsilon + \varepsilon) = 4\varepsilon. \quad (28)$$

Therefore, $\limsup_{k \rightarrow \infty} a_k = 0 = \lim_{n \rightarrow \infty} a_n$. $\square$

3. Main Results

Now, we are in a position to state and prove the main results of this paper.

Theorem 8. Let C be nonexpansive retract (with P) of a real Banach space E . Assume that $T_1, T_2 : C \rightarrow E$ are two uniformly L -Lipschitzian non-self-mappings (with P) and T_1 is an asymptotically quasi pseudocontractive type with coefficient numbers $\{k_n\} \subset [1, +\infty) : k_n \rightarrow 1$ satisfying $F = F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{u_n\}, \{v_n\} \subset C$ are two bounded sequences; $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\} \subset [0, 1]$ are six number sequences satisfying the following:

$$(C1) \quad \sum_{n=1}^{\infty} \alpha_n = +\infty, \sum_{n=1}^{\infty} \alpha_n^2 < +\infty, \sum_{n=1}^{\infty} \alpha_n(k_n - 1) < +\infty;$$

$$(C2) \quad \alpha_n + \gamma_n \leq 1, \alpha'_n + \gamma'_n \leq 1, \sum_{n=1}^{\infty} \gamma_n < +\infty;$$

$$(C3) \quad \sum_{n=1}^{\infty} \alpha_n \beta_n < +\infty, \sum_{n=1}^{\infty} \alpha_n \alpha'_n < +\infty, \sum_{n=1}^{\infty} \alpha_n \gamma'_n < +\infty.$$

If $x_1 \in C$ is arbitrary, then the iterative sequence $\{x_n\}$ generated by (12) converges strongly to the fixed point $x^* \in F$ if and only if there exists a strictly increasing function $\phi : [0, +\infty) \rightarrow [0, +\infty)$ with $\phi(0) = 0$ such that

$$\limsup_{n \rightarrow \infty} \inf_{j(x_{n+1}-x^*) \in J(x_{n+1}-x^*)} \left[\langle T_1(PT_1)^{n-1} x_{n+1} - x^*, j(x_{n+1} - x^*) \rangle - k_n \|x_{n+1} - x^*\|^2 + \phi(\|x_{n+1} - x^*\|) \right] \leq 0. \quad (29)$$

Proof. (Adequacy). Let

$$\begin{aligned} \varepsilon'_n &= \inf_{j(x_{n+1}-x^*) \in J(x_{n+1}-x^*)} \left[\langle T_1(PT_1)^{n-1} x_{n+1} - x^*, j(x_{n+1} - x^*) \rangle - k_n \|x_{n+1} - x^*\|^2 \right. \\ &\quad \left. + \phi(\|x_{n+1} - x^*\|) \right], \\ \varepsilon_n &= \max \{ \varepsilon'_n, 0 \} + \frac{1}{n}. \end{aligned} \quad (30)$$

Then there exists $j(x_{n+1} - x^*) \in J(x_{n+1} - x^*)$ such that

$$\begin{aligned} &\langle T_1(PT_1)^{n-1} x_{n+1} - x^*, j(x_{n+1} - x^*) \rangle \\ &- k_n \|x_{n+1} - x^*\|^2 + \phi(\|x_{n+1} - x^*\|) \leq \varepsilon_n. \end{aligned} \quad (31)$$

From (29), we know that $\limsup_{n \rightarrow \infty} \varepsilon'_n \leq 0$. So, $\lim_{n \rightarrow \infty} \varepsilon_n = 0$.

Now, from the given conditions and (12), we can let

$$\begin{aligned} \sigma_n &= (1 - \beta_n) \gamma_n + \beta_n T_1 (PT_1)^{n-1} \gamma_n, \\ \delta_n &= (1 - \beta'_n) x_n + \beta'_n T_2 (PT_2)^{n-1} x_n, \end{aligned} \quad (32)$$

and $M = \sup_{n \geq 1} \{\|\mu_n - x^*\|, \|\gamma_n - x^*\|\} < \infty$. Then

$$\begin{aligned} \|\delta_n - x^*\| &\leq \beta'_n \|T_2 (PT_2) x_n - x^*\| + (1 - \beta'_n) \|x_n - x^*\| \\ &\leq \beta'_n L \|x_n - x^*\| + \|x_n - x^*\|; \\ \|\gamma_n - x^*\| &\leq (1 - \alpha'_n - \gamma'_n) \|x_n - x^*\| \\ &\quad + \alpha'_n L \|\delta_n - x^*\| + \gamma'_n \|\gamma_n - x^*\| \\ &\leq \|x_n - x^*\| + \alpha'_n \beta'_n L^2 \|x_n - x^*\| \\ &\quad + \alpha'_n L \|x_n - x^*\| + \gamma'_n M \\ &= (1 + \alpha'_n \beta'_n L^2 + \alpha'_n L) \|x_n - x^*\| + \gamma'_n M \\ &\leq (1 + L + L^2) \|x_n - x^*\| + M; \\ \|\sigma_n - x^*\| &\leq \beta_n \|T_1 (PT_1)^{n-1} \gamma_n - x^*\| \\ &\quad + (1 - \beta_n) \|\gamma_n - x^*\| \\ &\leq \beta_n L \|\gamma_n - x^*\| + \|\gamma_n - x^*\| \\ &\leq (1 + L) (1 + L + L^2) \|x_n - x^*\| + (1 + L) M; \\ \|\gamma_n - x_{n+1}\| &\leq \alpha_n L \|\sigma_n - x^*\| + \alpha_n \|x_n - x^*\| \\ &\quad + \alpha'_n L \|\delta_n - x^*\| + \alpha'_n \|x_n - x^*\| \\ &\quad + (\gamma_n + \gamma'_n) \|x_n - x^*\| + (\gamma_n + \gamma'_n) M \\ &\leq \alpha_n L [(1 + L) (1 + L + L^2) \|x_n - x^*\| \\ &\quad + (1 + L) M] \\ &\quad + \alpha'_n L [(1 + \beta'_n L) \|x_n - x^*\|] \\ &\quad + (\alpha_n + \alpha'_n + \gamma_n + \gamma'_n) \|x_n - x^*\| + (\gamma_n + \gamma'_n) M \\ &\leq [\alpha_n L (1 + L) (1 + L + L^2) + \alpha'_n L (1 + \beta'_n L) \\ &\quad + \alpha_n + \alpha'_n + \gamma_n + \gamma'_n] \|x_n - x^*\| \\ &\quad + (\alpha_n L (1 + L) + \gamma_n + \gamma'_n) M; \\ \|\sigma_n - x_{n+1}\| &\leq \|\gamma_n - x_{n+1}\| + \beta_n \|T_1 (PT_1)^{n-1} \gamma_n - \gamma_n\| \\ &\leq s_n \|x_n - x^*\| + t_n, \end{aligned} \quad (33)$$

where

$$\begin{aligned} s_n &= \alpha_n L(1+L) \left(1+L+L^2\right) + \alpha'_n L \left(1+\beta'_n L\right) + \alpha_n \\ &\quad + \alpha'_n + \gamma_n + \gamma'_n + \beta_n (1+L) \left(1+L+L^2\right); \\ t_n &= \left[\alpha_n L(1+L) + \gamma_n + \gamma'_n + \beta_n (1+L)\right] M. \end{aligned} \quad (34)$$

So, by Lemma 4,

$$\begin{aligned} 2\alpha_n \left\langle T_1(PT_1)^{n-1} \sigma_n - T_1(PT_1)^{n-1} x_{n+1}, j(x_{n+1} - x^*) \right\rangle \\ \leq 2\alpha_n L \|x_{n+1} - x^*\| \|\sigma_n - x_{n+1}\| \\ \leq 2\alpha_n L \|x_{n+1} - x^*\| [s_n \|x_n - x^*\| + t_n]; \end{aligned} \quad (35)$$

$$\begin{aligned} \|x_{n+1} - x^*\|^2 \\ \leq (1 - \alpha_n - \gamma_n)^2 \|x_n - x^*\|^2 \\ + 2\alpha_n \left\langle T_1(PT_1)^{n-1} \sigma_n - x^*, j(x_{n+1} - x^*) \right\rangle \\ + 2\gamma_n \left\langle \mu_n - x^*, j(x_{n+1} - x^*) \right\rangle \\ \leq (1 - \alpha_n - \gamma_n)^2 \|x_n - x^*\|^2 \\ + 2\alpha_n \left\langle T_1(PT_1)^{n-1} \sigma_n - T_1(PT_1)^{n-1} x_{n+1}, \right. \\ \quad \left. j(x_{n+1} - x^*) \right\rangle \\ + 2\alpha_n \left\langle T_1(PT_1)^{n-1} x_{n+1} - x^*, j(x_{n+1} - x^*) \right\rangle \\ + 2\gamma_n M \|x_{n+1} - x^*\|. \end{aligned} \quad (36)$$

For the third in (36), we have

$$\begin{aligned} 2\alpha_n \left\langle T_1(PT_1)^{n-1} x_{n+1} - x^*, j(x_{n+1} - x^*) \right\rangle \\ = 2\alpha_n d_n + 2\alpha_n \left[k_n \|x_{n+1} - x^*\|^2 - \phi(\|x_{n+1} - x^*\|) \right] \\ \leq 2\alpha_n \varepsilon_n + 2\alpha_n \left[k_n \|x_{n+1} - x^*\|^2 - \phi(\|x_{n+1} - x^*\|) \right], \end{aligned} \quad (37)$$

where

$$\begin{aligned} d_n &= \left\langle T_1(PT_1)^{n-1} x_{n+1} - x^*, j(x_{n+1} - x^*) \right\rangle \\ &\quad - k_n \|x_{n+1} - x^*\|^2 + \phi(\|x_{n+1} - x^*\|) \leq \varepsilon_n. \end{aligned} \quad (38)$$

Substituting (35) into (36), we get

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \varepsilon_n \\ &\quad + 2\alpha_n k_n \|x_{n+1} - x^*\|^2 - 2\alpha_n \phi(\|x_{n+1} - x^*\|) \\ &\quad + 2\alpha_n L (s_n \|x_n - x^*\| + t_n) \|x_{n+1} - x^*\| \\ &\quad + 2\gamma_n M \|x_{n+1} - x^*\|. \end{aligned} \quad (39)$$

Let $a_n = \|x_n - x^*\|^2$, $\varphi(t) = 2\phi(\sqrt{t})$, and

$$\begin{aligned} \xi_n &= L\alpha_n s_n \\ &= L^2 \alpha_n^2 (1+L) \left(1+L+L^2\right) \\ &\quad + \alpha_n \alpha'_n L^2 \left(1+\beta'_n L\right) + \alpha_n^2 L + \alpha_n \alpha'_n L + L\alpha_n \gamma_n \\ &\quad + L\alpha_n \gamma'_n + L\alpha_n \beta_n (1+L) \left(1+L+L^2\right), \\ \rho_n &= L\alpha_n t_n + M\gamma_n \\ &= \left[\alpha_n^2 L^2 (1+L) + L\alpha_n \gamma_n + L\alpha_n \gamma'_n + \alpha_n \beta_n (L+L^2) \right] M \\ &\quad + \gamma_n M. \end{aligned} \quad (40)$$

Then (39) becomes

$$\begin{aligned} a_{n+1} &\leq (1 - \alpha_n)^2 a_n + 2\alpha_n \varepsilon_n + 2\alpha_n k_n a_{n+1} - \alpha_n \varphi(a_{n+1}) \\ &\quad + 2(\xi_n \|x_n - x^*\| + \rho_n) \|x_{n+1} - x^*\|. \end{aligned} \quad (42)$$

By using $2ab \leq a^2 + b^2$, we have

$$\begin{aligned} a_{n+1} &\leq (1 - \alpha_n)^2 a_n + 2\alpha_n \varepsilon_n + 2\alpha_n k_n a_{n+1} \\ &\quad - \alpha_n \varphi(a_{n+1}) + \xi_n (a_n + a_{n+1}) + \rho_n (1 + a_{n+1}) \\ &= (1 - 2\alpha_n + \alpha_n^2 + \xi_n) a_n + (2\alpha_n k_n + \xi_n + \rho_n) a_{n+1} \\ &\quad - \alpha_n \varphi(a_{n+1}) + 2\alpha_n \varepsilon_n + \rho_n. \end{aligned} \quad (43)$$

From (40), (41), and the given conditions, we know

$$\sum_{n=1}^{\infty} \alpha_n^2 < +\infty, \quad \sum_{n=1}^{\infty} \xi_n < +\infty, \quad \sum_{n=1}^{\infty} \rho_n < +\infty. \quad (44)$$

Then, $\lim_{n \rightarrow \infty} (2\alpha_n k_n + \xi_n + \rho_n) = 0$. Therefore $\exists n_0$, when $n \geq n_0$, $2\alpha_n k_n + \xi_n + \rho_n \leq 1/2$. Let

$$\begin{aligned} b_n &= \frac{1 - 2\alpha_n + \alpha_n^2 + \xi_n}{1 - 2\alpha_n k_n - \xi_n - \rho_n} - 1 = \frac{2\alpha_n (k_n - 1) + \alpha_n^2 + 2\xi_n + \rho_n}{1 - 2\alpha_n k_n - \xi_n - \rho_n}; \\ c_n &= \frac{\rho_n}{1 - 2\alpha_n k_n - \xi_n - \rho_n}. \end{aligned} \quad (45)$$

So, when $n \geq n_0$, we get

$$0 \leq b_n \leq 2 \left[2\alpha_n (k_n - 1) + \alpha_n^2 + 2\xi_n + \rho_n \right], \quad 0 \leq c_n \leq 2\rho_n. \quad (46)$$

From (44) and the given conditions, we have $\sum_{n=n_0}^{\infty} b_n < +\infty$, $\sum_{n=n_0}^{\infty} c_n < +\infty$. On the other hand, from (43), we have

$$a_{n+1} \leq (1 + b_n) a_n - \alpha_n \varphi(a_{n+1}) + 4\alpha_n \varepsilon_n + c_n, \quad \forall n \geq n_0. \quad (47)$$

By Lemma 7, we at last get

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \|x_n - x^*\|^2 = 0; \quad (48)$$

for example, $\lim_{n \rightarrow \infty} x_n = x^* \in F = F(T_1) \cap F(T_2)$.

(Necessity). Suppose that $\lim_{n \rightarrow \infty} x_n = x^* \in F$. Then we can choose an arbitrary continuous strictly increasing function $\phi : [0, +\infty) \rightarrow [0, +\infty)$ with $\phi(0) = 0$, such as $\phi(t) = t$. We can get $\lim_{n \rightarrow \infty} \phi(\|x_{n+1} - x^*\|) = 0$.

Because T_1 is an asymptotically quasi pseudocontractive type (with P), by (11) in Definition 1, for any $p \in F(T_1) \supseteq F$, we have

$$\limsup_{n \rightarrow \infty} \sup_{x \in C} \liminf_{j(x-p) \in J(x-p)} \left(\langle T_1(PT_1)^{n-1}x - p, j(x-p) \rangle - k_n \|x - p\|^2 \right) \leq 0. \quad (49)$$

So,

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \inf_{j(x_{n+1}-x^*) \in J(x_{n+1}-x^*)} \left[\langle T_1(PT_1)^{n-1}x_{n+1} - x^*, \right. \\ & \quad \left. j(x_{n+1} - x^*) \rangle - k_n \|x_{n+1} - x^*\|^2 \right. \\ & \quad \left. + \phi(\|x_{n+1} - x^*\|) \right] \\ & = \limsup_{n \rightarrow \infty} \inf_{j(x_{n+1}-x^*) \in J(x_{n+1}-x^*)} \left[\langle T_1(PT_1)^{n-1}x_{n+1} - x^*, \right. \\ & \quad \left. j(x_{n+1} - x^*) \rangle \right. \\ & \quad \left. - k_n \|x_{n+1} - x^*\|^2 \right] \\ & + \lim_{n \rightarrow \infty} \phi(\|x_{n+1} - x^*\|) \leq 0 + 0 = 0; \end{aligned} \quad (50)$$

that is, (29) holds. This completes the proof of Theorem 8. $\square$

Combining with Theorem 8 and Definition 3, we have some results as follows.

Theorem 9. Let C be nonexpansive retract (with P) of a real Banach space E . Assume that $T_1, T_2 : C \rightarrow E$ are two uniformly L -Lipschitzian non-self-mappings (with P) and T_1 is an asymptotically quasi pseudocontractive type with coefficient numbers $\{k_n\} \subset [1, +\infty) : k_n \rightarrow 1$ satisfying $F = F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\alpha'_n\}, \{\beta'_n\} \subset [0, 1]$ are four number sequences satisfying the following:

- (C1) $\sum_{n=1}^{\infty} \alpha_n = +\infty, \sum_{n=1}^{\infty} \alpha_n^2 < +\infty, \sum_{n=1}^{\infty} \alpha_n(k_n - 1) < +\infty;$
- (C2) $\sum_{n=1}^{\infty} \alpha_n \beta_n < +\infty, \sum_{n=1}^{\infty} \alpha_n \alpha'_n < +\infty.$

If $x_1 \in C$ is arbitrary, then the iterative sequence $\{x_n\}$ generated by (14) converges strongly to the fixed point $x^* \in F$ if and only if there exists a strictly increasing function $\phi : [0, +\infty) \rightarrow [0, +\infty)$ with $\phi(0) = 0$ such that (29) holds.

Theorem 10. Let C be nonexpansive retract (with P) of a real Banach space E . Assume that $T_1, T_2 : C \rightarrow E$ are two

uniformly L -Lipschitzian non-self-mappings (with P) and T_1 is an asymptotically quasi pseudocontractive type with coefficient numbers $\{k_n\} \subset [1, +\infty) : k_n \rightarrow 1$ satisfying $F = F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}, \{\alpha'_n\} \subset [0, 1]$ are two number sequences satisfying the following:

- (C1) $\sum_{n=1}^{\infty} \alpha_n = +\infty, \sum_{n=1}^{\infty} \alpha_n^2 < +\infty, \sum_{n=1}^{\infty} \alpha_n(k_n - 1) < +\infty;$
- (C2) $\sum_{n=1}^{\infty} \alpha_n \alpha'_n < +\infty.$

If $x_1 \in C$ is arbitrary, then the iterative sequence $\{x_n\}$ generated by (15) converges strongly to the fixed point $x^* \in F$ if and only if there exists a strictly increasing function $\phi : [0, +\infty) \rightarrow [0, +\infty)$ with $\phi(0) = 0$ such that (29) holds.

Theorem 11. Let C be a nonempty closed convex subset of a real Banach space E . Assume that $T : C \rightarrow C$ is uniformly L -Lipschitzian self-mappings and asymptotically quasi pseudocontractive type with coefficient numbers $\{k_n\} \subset [1, +\infty) : k_n \rightarrow 1$ satisfying $F = F(T) \neq \emptyset$. Suppose that $\{\alpha_n\}, \{\alpha'_n\} \subset [0, 1]$ are two number sequences satisfying the following:

- (C1) $\sum_{n=1}^{\infty} \alpha_n = +\infty, \sum_{n=1}^{\infty} \alpha_n^2 < +\infty, \sum_{n=1}^{\infty} \alpha_n(k_n - 1) < +\infty;$
- (C2) $\sum_{n=1}^{\infty} \alpha_n \alpha'_n < +\infty.$

If $x_1 \in C$ is arbitrary, then the iterative sequence $\{x_n\}$ generated by (16) converges strongly to the fixed point $x^* \in F$ if and only if there exists a strictly increasing function $\phi : [0, +\infty) \rightarrow [0, +\infty)$ with $\phi(0) = 0$ such that (29) holds.

Remark 12. Our research and results in this paper have the following several advantaged characteristics. (a) The iterative scheme is the new modified mixed Ishikawa iterative scheme with error on two mappings T_1, T_2 . (b) The common fixed point $x^* \in F = F(T_1) \cap F(T_2)$ is studied. (c) The research object is the very generalized asymptotically quasi pseudocontractive type (with P) non-self-mapping. (d) The tool used by us is the very powerful tool: Lemma 7. So, our results here extend and improve many results of other authors to a certain extent, such as [6, 8, 14–23], and the proof methods are very different from the previous.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors would like to thank the editors and referees for their many useful comments and suggestions for the improvement of the paper. This work was supported by the National Natural Science Foundations of China (Grant no. 11271330) and the Natural Science Foundations of Zhejiang Province (Grant no. Y6100696).

References

- [1] K. Goebel and W. A. Kirk, "A fixed point theorem for asymptotically nonexpansive mappings," *Proceedings of the American Mathematical Society*, vol. 35, pp. 171–174, 1972.

- [2] J. Schu, "Approximation of fixed points of asymptotically nonexpansive mappings," *Proceedings of the American Mathematical Society*, vol. 112, pp. 143–151, 1991.
- [3] W. Takahashi, N.-C. Wong, and J.-C. Yao, "Attractive point and weak convergence theorems for new generalized hybrid mappings in Hilbert spaces," *Journal of Nonlinear and Convex Analysis*, vol. 13, pp. 745–757, 2012.
- [4] Y. Kurokawa and W. Takahashi, "Weak and strong convergence theorems for nonspreading mappings in Hilbert spaces," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 73, no. 6, pp. 1562–1568, 2010.
- [5] H. K. Xu, "Vistosity approximation methods for nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 298, pp. 279–291, 2004.
- [6] D. R. Sahu, H.-K. Xu, and J.-C. Yao, "Asymptotically strict pseudocontractive mappings in the intermediate sense," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 70, no. 10, pp. 3502–3511, 2009.
- [7] J. K. Kim, D. R. Sahu, and Y. M. Nam, "Convergence theorem for fixed points of nearly uniformly L-Lipschitzian asymptotically generalized Φ -hemicontractive mappings," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 71, no. 12, pp. e2833–e2838, 2009.
- [8] S. S. Chang, "Some results for asymptotically pseudocontractive mappings and asymptotically nonexpansive mappings," *Proceedings of the American Mathematical Society*, vol. 129, no. 3, pp. 845–853, 2001.
- [9] P. Cholamjiak and S. Suantai, "A new hybrid algorithm for variational inclusions, generalized equilibrium problems, and a finite family of quasi-nonexpansive mappings," *Fixed Point Theory and Applications*, vol. 2009, Article ID 350979, 7 pages, 2009.
- [10] C. Klin-Eam and S. Suantai, "Strong convergence of composite iterative schemes for a countable family of nonexpansive mappings in Banach spaces," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 73, no. 2, pp. 431–439, 2010.
- [11] S. Plubtieng and K. Ungchittrakool, "Approximation of common fixed points for a countable family of relatively nonexpansive mappings in a Banach space and applications," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 72, no. 6, pp. 2896–2908, 2009.
- [12] T. Suzuki, "Strong convergence theorems for infinite families of nonexpansive mappings in general Banach spaces," *Fixed Point Theory and Applications*, vol. 2005, no. 1, pp. 103–123, 2005.
- [13] L.-C. Zeng and J.-C. Yao, "Implicit iteration scheme with perturbed mapping for common fixed points of a finite family of nonexpansive mappings," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 64, no. 11, pp. 2507–2515, 2006.
- [14] G. Marino and H.-K. Xu, "A general iterative method for nonexpansive mappings in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 318, no. 1, pp. 43–52, 2006.
- [15] Y.-H. Wang and Y.-H. Xia, "Strong convergence for asymptotically pseudocontractions with the demiclosedness principle in Banach spaces," *Fixed Point Theory and Applications*, vol. 2012, 8 pages, 2012.
- [16] Y.-H. Wang and W. F. Xuan, "Convergence theorems for common fixed points of a finite family of relatively nonexpansive mappings in Banach spaces," *Abstract and Applied Analysis*, vol. 2013, Article ID 259470, 7 pages, 2013.
- [17] L.-C. Zeng and J.-C. Yao, "Stability of iterative procedures with errors for approximating common fixed points of a couple of q -contractive-like mappings in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 321, no. 2, pp. 661–674, 2006.
- [18] L.-C. Zeng, T. Tanaka, and J.-C. Yao, "Iterative construction of fixed points of nonself-mappings in Banach spaces," *Journal of Computational and Applied Mathematics*, vol. 206, no. 2, pp. 814–825, 2007.
- [19] L. C. Zeng, N. C. Wong, and J. C. Yao, "Convergence analysis of iterative sequences for a pair of mappings in Banach spaces," *Acta Mathematica Sinica*, vol. 24, no. 3, pp. 463–470, 2008.
- [20] X. Wu, J. C. Yao, and L. C. Zeng, "Uniformly normal structure and strong convergence theorems for asymptotically pseudocontractive mappings," *Journal of Nonlinear and Convex Analysis*, vol. 6, no. 3, pp. 453–463, 2005.
- [21] L.-C. Zeng, N.-C. Wong, and J.-C. Yao, "Fixed point solutions of variational inequalities for a finite family of asymptotically nonexpansive mappings without common fixed point assumption," *Computers and Mathematics with Applications*, vol. 56, no. 9, pp. 2312–2322, 2008.
- [22] C. E. Chidume, E. U. Ofoedu, and H. Zegeye, "Strong and weak convergence theorems for asymptotically nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 280, no. 2, pp. 364–374, 2003.
- [23] H. Zegeye, M. Robdera, and B. Choudhary, "Convergence theorems for asymptotically pseudocontractive mappings in the intermediate sense," *Computers and Mathematics with Applications*, vol. 62, no. 1, pp. 326–332, 2011.
- [24] S.-S. Chang, "Some problems and results in the study of nonlinear analysis," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 30, no. 7, pp. 4197–4208, 1997.