Research Article

On the Existence of Solutions for the Critical Fractional Laplacian Equation in $\mathbb{R}^N$

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Received 5 May 2013; Accepted 10 December 2013; Published 12 January 2014

Academic Editor: Wenming Zou

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We study existence of solutions for the fractional Laplacian equation

$$(-\Delta)^s u + V(x)u = |u|^{2^*_s - 2}u + f(x,u)$$

in $\mathbb{R}^N$, $u \in H^s(\mathbb{R}^N)$, with critical exponent $2^*_s = 2N/(N-2s)$, $N > 2s$, $s \in (0,1)$, where $V(x) \geq 0$ has a potential well and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a lower order perturbation of the critical power $|u|^{2^*_s - 2}u$. By employing the variational method, we prove the existence of nontrivial solutions for the equation.

1. Introduction

In the last 20 years, the classical nonlinear Schrödinger equation has been extensively studied by many authors [1–10] and the references therein. We just mention some earlier work about it. Brézis and Nirenberg [1] proved that the critical problem with small linear perturbations can provide positive solutions. In [3], Rabinowitz proved the existence of standing wave solutions of nonlinear Schrödinger equations. Making a standing wave ansatz reduces the problem to studying a class of semilinear elliptic equations. Floer and Weinstein [10] proved that Schrödinger equation with potential $V$ and cubic nonlinearity has standing wave solutions concentrated near each nondegenerate critical point of $V$.

However, a great attention has been focused on the study of problems involving the fractional Laplacian recently. This type of operator seems to have a prevalent role in physical situations such as combustion and dislocations in mechanical systems or in crystals. In addition, these operators arise in modelling diffusion and transport in a highly heterogeneous medium. This type of problems has been studied by many authors [11–18] and the references therein.

Servadei and Valdinoci [11–14] studied the problem

$$L_K u(\lambda) u + |u|^{2^*_s - 2} u + f(x,u) = 0 \quad \text{in } \Omega,$$

$$u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

where $s \in (0,1)$, $\Omega$ is an open bounded set of $\mathbb{R}^N$, $N > 2s$, with Lipschitz boundary, $\lambda > 0$ is a real parameter, and $2^*_s = 2N/(N-2s)$ is a fractional critical Sobolev exponent. $L_K$ is defined as follows:

$$L_K u(\lambda) x = \frac{1}{2} \int_{\mathbb{R}^N} (u(x+y) + u(x-y) - u(x)) K(y) \, dx \, dy,$$

$$x \in \mathbb{R}^N.$$ (2)

Here $K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ is a function such that

$$mK \in L^1(\mathbb{R}^N), \quad \text{where } m(x) = \min \{ |x|^2, 1 \};$$ (3)

there exists $\theta > 0$ such that $K(x) \geq \theta |x|^{-(N+2\alpha)}$ and $K(x) = K(-x)$ for any $x \in \mathbb{R}^N \setminus \{0\}$. They proved that problem (1) admits a nontrivial solution for any $\lambda > 0$. They also studied the case $f(x,u) \equiv 0$ and $K(x) = |x|^{-(N+2\alpha)}$, respectively.

Felmer et al. [15] studied the following nonlinear Schrödinger equation with fractional Laplacian:

$$(-\Delta)\alpha u + u = f(x,u) \quad \text{in } \mathbb{R}^N,$$

$$u > 0 \quad \text{in } \mathbb{R}^N, \quad \lim_{|x| \to \infty} u(x) = 0,$$ (4)

where $0 < \alpha < 1$, $N \geq 2$, and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is superlinear and has subcritical growth with respect to $u$. The
fractional Laplacian can be characterized as $\mathcal{F}((-\Delta)^s)\hat{\phi}(\xi) = |\xi|^{2s} \mathcal{F}(\phi)(\xi)$, where $\mathcal{F}$ is the Fourier transform. They gave the proof of existence of positive solutions and further analyzed regularity, decay, and symmetry properties of these solutions.

In this paper, we consider the following problem:

$$
(-\Delta)^s u + V(x)u = |u|^{2^*(s) - 2}u + f(x,u) \quad \text{in } \mathbb{R}^N,
$$

$$
u \in H^s(\mathbb{R}^N),
$$

with critical exponent $2^*(s) = \frac{2N}{N-2s}$, $N > 2s$, $s \in (0,1)$, where $V(x) \geq 0$ has a potential well, where $(-\Delta)^s$ is the fractional Laplace operator, which may be defined as

$$
\begin{aligned}
&\mathcal{F}((-\Delta)^s \phi)(\xi) = |\xi|^{2s} \mathcal{F}(\phi)(\xi), \\
&\text{where } \phi \text{ is the Fourier transform. They give the weak formulation of (5) by the following problem:}
\end{aligned}
$$

The aim of this paper is to find solutions for (5) by variational methods. For this, we give the weak formulation of (5) by the following problem:

$$
\begin{aligned}
\int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))}{|x-y|^{N+2s}} dx dy &+ \int_{\mathbb{R}^N} V(x)u(x) \varphi(x) dx \\
&= \int_{\mathbb{R}^N} |u(x)|^{2^*(s) - 2}u(x) \varphi(x) dx \\
&+ \int_{\mathbb{R}^N} f(x, u(x)) \varphi(x) dx, \quad \forall \varphi \in H^s(\mathbb{R}^N),
\end{aligned}
$$

Now, we give our results as follows.

The aim of this paper is to find solutions for (5) by variational methods. For this, we give the weak formulation of (5). We will prove the existence of the critical points of the functional $I$. It is convenient to define

$$
\begin{aligned}
S_r := &\inf_{u \in \mathbb{R} \setminus \{0\}} \sup_{x \in \mathbb{R}^N} f(x,t)t = 0 \text{ uniformly in } x \in \mathbb{R}^N; \\
S_{rs} := &\inf_{u \in H^s(\mathbb{R}^N) \setminus \{0\}} S_r(u),
\end{aligned}
$$

where $E$ will be defined in Section 2 and for any $u \in E \setminus \{0\}$

$$
S_r(u) := \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V(x)|u(x)|^2 dx
$$

and $H^s(\mathbb{R}^N) \setminus \{0\}$ will be defined in Section 2 and for any $u \in E \setminus \{0\}$

$$
S_{rs}(u) := \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} |u(x)|^2 dx
$$

$$
\left(\int_{\mathbb{R}^N} |u(x)|^{2^*(s)} dx\right)^{2/2^*(s)}
$$

Now, we give our results as follows.
Theorem 1. Let $N > 2s$, $s \in (0, 1)$, $V(x)$, and $f$ satisfy (V0) and (f0)–(f4), respectively. Then, (5) possesses at least one nontrivial solution.

We prove Theorem 1 applying the mountain pass theorem to the functional $I$. Although the Palais-Smale sequences might lose compactness in the whole space $\mathbb{R}^N$, we cannot apply the mountain pass theorem directly. In [15], they used a comparison argument to overcome this difficulty. But we will use it to us to overcome this problem related to the lack of compactness and to show that the Palais-Smale condition holds true in a suitable range related to $S_s$.

Theorem 2. Let $N > 2s$, $s \in (0, 1)$, $V(x)$, and $f$ satisfy (V0)–(f4), respectively. Then, (5) possesses at least one nontrivial radial symmetric solution.

For the case $V(x) \equiv 1$, the proof of the existence of solutions is similar to the proof of Theorem 1. To prove the existence of solutions, we consider the subspace $H^s_r(\mathbb{R}^N)$ of $H^s(\mathbb{R}^N)$. $H^s_r(\mathbb{R}^N)$ consists of radial symmetric functions of $H^s(\mathbb{R}^N)$ and has the same norm with $H^s(\mathbb{R}^N)$, and its norm is denoted by $\|u\|_{H^s_r}$. $H^s_r(\mathbb{R}^N)$ is continuously embedded in $H^s(\mathbb{R}^N)$.

The following two lemmas about the fractional Sobolev space $H^s(\mathbb{R}^N)$ are proved in [15].

Lemma 3. Let $2 \leq q \leq 2^*(s) = 2N/(N − 2s)$; then one has

$$\|u\|_{L^q(\mathbb{R}^N)} \leq C\|u\|_H \quad \forall u \in H^s(\mathbb{R}^N).$$

Consequently, the embedding $E \hookrightarrow L^{2^*(s)}(\mathbb{R}^N)$ and $H^s_r(\mathbb{R}^N) \hookrightarrow L^{2^*(s)}(\Omega)$ are continuous. If further $2 \leq q < 2^*(s)$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain, then the bound sequence $\{u_k\} \subset H^s(\mathbb{R}^N)$ has a convergent subsequence in $L^q(\Omega)$.

Thanks to Lemma 3, we can define the constants $S_s$ and $S_{s, R}$ and get that $S_{s, R} > 0$ and $S_{s, R} > 0$.

Lemma 4. Let $N \geq 2$. Assume that $\{u_k\}$ is bounded in $H^s(\mathbb{R}^N)$ and it satisfies

$$\lim_{k \to \infty} \sup_{x \in \mathbb{R}^N} \int_{B_4(x)} |u_k(x)|^2 \, dx = 0,$$

where $R > 0$. Then $u_k \to 0$ in $L^3(\mathbb{R}^N)$ for $2 < q < 2^*(s)$.

Lemma 5. (a) If (V0) holds true, then the embedding $E \hookrightarrow L^3(\mathbb{R}^N)$ is compact for any $\nu \in \{2, 2^*(s)\}$.

(b) The embedding $H^s_r(\mathbb{R}^N) \hookrightarrow L^3(\mathbb{R}^N)$ is compact for any $q \in (2, 2^* (s))$.

Proof. (a) We give the proof as in [19].

Case $\nu = 2$. We will show that $u_n \to 0$ strongly in $L^2(\mathbb{R}^N)$, whenever $u_n \to 0$ weakly in $E$. Indeed, let $C > 0$ be such that $\|u\|_E \leq C$. Given $\varepsilon > 0$, pick $R > 0$ such that $V(x) \geq 2C^2/q$ for all $|x| \geq R$ and denote by $B_R$ the ball of radius $R$ in $\mathbb{R}^N$.

Then we have that $u_n \to 0$ weakly in $H^s(\mathbb{R}^N)$. The compact embedding $H^s(\mathbb{R}^N) \hookrightarrow L^3(\mathbb{R}^N)$ implies that for some natural number $n_0$,

$$\int_{B_R} |u_n(x)|^2 \, dx \leq \frac{\varepsilon}{2}, \quad \forall n > n_0.$$
On the other hand, by our choice of $R > 0$, we have
\[
2 \int_{B_R} |u_n(x)|^2 \, dx \leq \frac{1}{c^2} \int_{B_R} V(x) |u_n(x)|^2 \, dx \leq \frac{1}{c^2} \|u_n\|_2^2 \leq 1. \tag{23}
\]
Combining (22) and (23), we obtain that $\|u_n\|_{L^2(R^N)}^2 < \varepsilon$ for all $n > n_0$.

Case 2 $2 < \nu < 2^*(s)$. Using Lemma 3, together with the interpolation inequality (where $1/\nu = (\sigma/2) + (1 - \sigma)/(2^*(s))$),
\[
\|u\|_{L^2(R^N)} \leq \|u\|_{L^2(R^N)}^{1-\sigma} |u|_{L^{2^*(s)}(R^N)}^{\sigma},
\]
for all $u \in L^2(R^N)$ and for all $\sigma > 0$, we have
\[
\int_{B_R(x)} |u|^2 \, dx \leq m(x, r)^{-1} \|u\|_{L^2(R^N)}^2. \tag{25}
\]
If $\{u_j\}$ is a bounded sequence in $H^s_{\nu}(R^N)$, for all $\varepsilon > 0$, $\exists R > 0$, we have
\[
\sup_{x \in \mathbb{R}^N} \left\{ \int_{B_R(x)} |u_j(x)|^2 \, dx \mid |x| \geq R \right\} < \varepsilon. \tag{26}
\]
We may assume that $u_j \rightharpoonup 0$ weakly in $H^s_{\nu}(R^N)$; then, by Lemma 3, after a subsequence $\int_{B_{R_j}} |u_j|^2 \, dx \to 0$, it follows that
\[
\sup_{x \in \mathbb{R}^N} \left\{ \int_{B_{R_j}(x)} |u_j(x)|^2 \, dx \mid |x| \leq R \right\} \to 0. \tag{27}
\]
By (26), (27), and Lemma 4, we have
\[
u_j \to 0 \quad \text{in} \quad L^\gamma \left( \mathbb{R}^N \right) \tag{28}
\]
for $q \in (2, 2^*(s))$.

In [17, Lemma 4], Servadei and Valdinoci proved the following result.

**Lemma 7.** Assume that $f$ satisfies (f0)–(f4). Then, there exist two positive measurable functions $m = m(x)$ and $M = M(x)$ such that a.e. $x \in \mathbb{R}^N$, and for any $t \in \mathbb{R}$,
\[
F(x, t) \geq m(x) |t|^\mu - M(x), \tag{31}
\]
where $F$ is defined as in (f4), $2 < \mu < 2^*(s)$, and $m, M \in L^\infty(R^N)$.

### 3. The Proof of Theorem 1

In this section we study the mountain pass geometry and Palais-Smale condition in a suitable energy range and finish the proof of Theorem 1. We consider the functional
\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |u(x) - u(y)|^2 \, dx dy + \frac{1}{2} \int_{\mathbb{R}^N} V(x) |u(x)|^2 \, dx
- \frac{1}{2^*(s) \mu} \int_{\mathbb{R}^N} |u(x)|^{2^*(s) \mu} \, dx - \int_{\mathbb{R}^N} F(x, u(x)) \, dx, \tag{32}
\]
where $F$ is defined as in (f4).

Then $I \in C^1(E, \mathbb{R})$ and critical points of $I$ are solutions of
\[
(-\Delta)^s u + V(x) u = |u|^{2^*(s) - 2} u + f(x, u), \tag{33}
\]
for $q \in (2, 2^*(s))$.

The Fréchet derivative of $I$ is
\[
\langle I'(u), v \rangle = \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx dy
+ \int_{\mathbb{R}^N} V(x) u(x) v(x) \, dx
- \int_{\mathbb{R}^N} |u(x)|^{2^*(s) - 2} u(x) v(x) \, dx
- \int_{\mathbb{R}^N} f(x, u(x)) v(x) \, dx \quad \forall v \in E. \tag{34}
\]

**Proposition 8.** Let $N > 2s, s \in (0, 1), V(x)$, and $f$ satisfy (V0) and (f0)–(f4), respectively. Then, there exist $p > 0$ and $\beta > 0$ such that for any $u \in E$ with $\|u\|_E = \rho$ it results that $I(u) \geq \beta$. 

\[\]
Proof. Let \( u \) be a function in \( E \). By Lemma 3 and (30), we get that, for any \( \varepsilon > 0 \),
\[
I(u) \geq \frac{1}{2} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy + \frac{1}{2} \int_{\mathbb{R}^N} V(x) |u(x)|^2 \, dx
\]
\[
- \frac{1}{2s} \int_{\mathbb{R}^N} |u(x)|^{2s} \, dx - \varepsilon \int_{\mathbb{R}^N} |u(x)|^2 \, dx
\]
\[
- \delta(e) \int_{\mathbb{R}^N} |u(x)|^q \, dx
\]
\[
= \frac{1}{2} \|u\|_E^2 - \varepsilon \|u\|_{L^2(\mathbb{R}^N)}^2 - \frac{1}{2s} \|u\|_{L^{2s}(\mathbb{R}^N)}^{2s} - \delta(e) C_3 \|u\|_{E^q}^q
\]
\[
\geq \frac{1}{2} \|u\|_E^2 - \varepsilon C_4 \|u\|_{E^2}^2 - \frac{1}{2s} \|u\|_{L^{2s}(\mathbb{R}^N)}^{2s} - \delta(e) C_3 \|u\|_{E^q}^q
\]
Choosing \( \varepsilon > 0 \) such that \((1/2) - \varepsilon C_4 > 0\), by (35), it easily follows that
\[
I(u) \geq C_4 \|u\|_E^2 \left(1 - C_5 \|u\|_{E^2}^{2s} - C_6 \|u\|_{E^q}^q\right),
\]
for suitable positive constants \( C_4, C_5, \) and \( C_6 \).

Now, let \( u \in E \) be such that \( \|u\|_E = \rho > 0 \). Since \( 2^* > q > 2 \), we can choose \( \rho \) sufficiently small, so that
\[
\inf_{u \in E \setminus \{0\}} I(u) \geq C_4 \rho^2 \left(1 - C_5 \rho^{2s} - C_6 \rho^q\right) =: \beta > 0.
\]

Hence, Proposition 8 is proved.

**Proposition 9.** There exists \( u_0 \in E \setminus \{0\} \) with \( u_0 \geq 0 \) a.e. in \( \mathbb{R}^N \), such that
\[
\sup_{t \geq 0} I(tu_0) < \frac{s}{N} S_{N/2s}^{N/2s}.
\]

Proof. By definition of \( S_s \), we have that there exists \( u_0 \in E \setminus \{0\} \) such that
\[
\frac{\|u_0\|_E^2}{\|u_0\|_{L^{2s}(\mathbb{R}^N)}} < S_s + \varepsilon,
\]
for any \( \varepsilon > 0 \). We have \( u_0 \geq 0 \) a.e. in \( \mathbb{R}^N \), or else we can take \( |u_0| \in E \). Indeed, by triangle inequality, a.e. \( x, y \in \mathbb{R}^N \),
\[
|u_0(x) - u_0(y)| \leq |u_0(x) - u_0(y)|,
\]
and so
\[
\|u_0\|_E \leq \|u_0\|_{E^2}.
\]
Thus,
\[
\frac{\|u_0\|_E^2}{\|u_0\|_{L^{2s}(\mathbb{R}^N)}} < S_s + \varepsilon.
\]
It follows that there exists \( t_0 > 0 \) such that
\[
\max_{t \geq 0} I(tu_0) = \max_{t \geq 0} \left( \frac{t^2}{2} \left\|u_0\right\|_E^2 - \frac{t^{2s}}{2^s} \left\|u_0\right\|_{L^{2s}(\mathbb{R}^N)}^{2s} - \int_{\mathbb{R}^N} F(x, tu_0(x)) \, dx \right)
\]
\[
\leq \max_{t \geq 0} \left( \frac{t^2}{2} \left\|u_0\right\|_E^2 - \frac{t^{2s}}{2^s} \left\|u_0\right\|_{L^{2s}(\mathbb{R}^N)}^{2s} - \int_{\mathbb{R}^N} F(x, tu_0(x)) \, dx \right)
\]
\[
< \frac{s}{N} (S_s + \varepsilon)^{N/2s} - \int_{\mathbb{R}^N} F(x, tu_0(x)) \, dx,
\]
thanks to (39). Since \( \int_{\mathbb{R}^N} F(x, tu_0(x)) \, dx > 0 \), we can choose \( \varepsilon > 0 \) such that
\[
\sup_{t \geq 0} I(tu_0) < \frac{s}{N} S_{N/2s}^{N/2s}.
\]

Hence, Proposition 9 is proved.

**Proposition 10.** Let \( N > 2s \), \( s \in (0, 1) \), \( V(x) \), and \( f \) satisfy (V0) and (f0)-(f4), respectively. Then, there exists \( e \in E \) such that \( e \geq 0 \) a.e. in \( \mathbb{R}^N \), \( \|e\|_E > \rho \), and \( I(e) < \beta \), where \( \rho \) and \( \beta \) are given in Proposition 8.

In particular, we can construct \( e \) as follows:
\[
e = t_0 u_0
\]
with \( u_0 \) as in (38) and \( t_0 > 0 \) large enough.

Proof. We fix \( u \in E \) such that \( \|u\|_E \neq 0 \). By (31), we get
\[
I(tu) \leq \frac{1}{2} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy
\]
\[
+ \frac{1}{2} \int_{\mathbb{R}^N} V(x) |u(x)|^2 \, dx
\]
\[
- \frac{1}{2^s} \int_{\mathbb{R}^N} |u(x)|^{2s} \, dx
\]
\[
- t^\alpha \int_{\mathbb{R}^N} m(x) |u(x)|^\mu - M(x) \, dx
\]
\[
= \frac{1}{2} \|u\|_E^2 - \frac{1}{2^s} \|u\|_{L^{2s}(\mathbb{R}^N)}^{2s} - t^{\delta} C_1 \|u\|_{L^\delta(\mathbb{R}^N)}^\delta + C_2,
\]
where \( C_1, C_2 \geq 0 \) are constant. Since \( 2^s > 2 \) and \( \mu > 2 \), passing to the limit as \( t \to +\infty \), we get that \( I(tu) \to -\infty \).
where \(\beta \leq c < \frac{S_{\gamma}}{N}\), \(S_{\gamma}\) is defined in formula (12).

By [6, Theorem 2.2], we have a sequence \(u_j \in E\) such that

\[
I(u_j) \longrightarrow c, \quad \sup \left\{ \left\langle I'(u_j) \right\rangle, \|u_j\|_E \right\} \longrightarrow 0
\]

as \(j \rightarrow +\infty\).

**Proposition 11.** There exists \(u_\infty \in E\) such that, up to a subsequence, \(\|u_j - u_\infty\|_E \rightarrow 0\) as \(j \rightarrow +\infty\).

**Proof.** We proceed by steps.

**Step 1.** The sequence \(u_j\) is bounded in \(E\).

**Proof.** For any \(j \in \mathbb{N}\) by (50) and (51) it easily follows that there exists \(C_1 > 0\) such that

\[
\left| I(u_j) \right| \leq C_1, \quad \left| \left\langle I'(u_j), \frac{u_j}{\|u_j\|_E} \right\rangle \right| \leq C_1.
\]

As a consequence of (52), we have

\[
I(u_j) - \frac{1}{\mu} \left\langle I'(u_j), u_j \right\rangle \leq C_1 \left(1 + \|u_j\|_E\right). \tag{53}
\]

By (f2) and (f4), we have \(\mu < 2^*(s)\) and

\[
I(u_j) - \frac{1}{\mu} \left\langle I'(u_j), u_j \right\rangle = \left(1 - \frac{1}{\mu}\right) \left( \int_{\mathbb{R}^N} \frac{|u_j(x) - u_j(y)|^2}{|x - y|^{N+2s}} \, dx \, dy + \int_{\mathbb{R}^N} V(x) |u_j(x)|^2 \, dx \right) + \left(1 - \frac{1}{\mu}\right) \int_{\mathbb{R}^N} |u_j(x)|^2 \, dx.
\]

We obtain that \(u_j\) is bounded in \(E\) by \(2 < \mu < 2^*(s)\), (53), and (54).

**Step 2.** Problem (10) admits a solution \(u_\infty \in E\).

**Proof.** By Step 1 and since \(E\) is a reflexive space, up to a subsequence, still denoted by \(u_j\), there exists \(u_\infty \in E\) such that \(u_j \rightarrow u_\infty\) weakly in \(E\); that is,

\[
\int_{\mathbb{R}^N} \frac{(u_j(x) - u_j(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy + \int_{\mathbb{R}^N} V(x) u_j(x) \varphi(x) \, dx \rightarrow \int_{\mathbb{R}^N} V(x) u_\infty(x) \varphi(x) \, dx \tag{55}
\]

as \(j \rightarrow +\infty\). By \(2 < \mu < 2^*(s)\), (53), and (54), we have that \(u_j\) is bounded in \(L^{2^*(s)}(\mathbb{R}^N)\). Since \(L^{2^*(s)}(\mathbb{R}^N)\) is a reflexive space, up to a subsequence,

\[
u_j \rightarrow u_\infty\ \text{weakly in } L^{2^*(s)}(\mathbb{R}^N) \tag{56}
\]

as \(j \rightarrow +\infty\), while by Lemma 5(a), up to a subsequence,

\[
u_j \rightarrow u_\infty \quad \text{in } L^v(\mathbb{R}^N), \tag{57}
\]

\[
u_j \rightarrow u_\infty \quad \text{a.e. in } \mathbb{R}^N \tag{58}
\]

as \(j \rightarrow +\infty\), for any \(v \in [2, 2^*(s)]\). By (56) and since \(|u_j|^{2^*(s)-2} u_j\) is bounded in \(L^{2^*(s)/(2^*(s)-1)}(\mathbb{R}^N)\), we have

\[|u_j|^{2^*(s)-2} u_j \rightarrow |u_\infty|^{2^*(s)-2} u_\infty \quad \text{weakly in } L^{2^*(s)/(2^*(s)-1)}(\mathbb{R}^N) \tag{59}\]

as \(j \rightarrow +\infty\). By the proof of Lemma 6 [11, Lemma 6], we get

\[
\int_{\mathbb{R}^N} \varphi(x) \, dx \leq 2 \epsilon \int_{\mathbb{R}^N} |u(x)| \, dx + q \delta(x) \int_{\mathbb{R}^N} |u(x)|^{q-1} \, dx. \tag{60}
\]
Moreover, we have, by taking \( \varepsilon = 1 \),
\[
\int_{\mathbb{R}^N} f(x, u(x)) \, dx \leq C_1 + C_2 \int_{\mathbb{R}^N} |u(x)|^{q-1} \, dx.
\]  
(61)

By (61), \( u_j \) being bounded in \( E \), and Lemma 5(a), we obtain
\[
f(\cdot, u_j (\cdot)) \text{ is bounded in } L^{q/(q-1)} (\mathbb{R}^N).
\]
(62)

Since \( L^{q/(q-1)} (\mathbb{R}^N) \) is a reflexive space, we get
\[
f(\cdot, u_j (\cdot)) \rightharpoonup f(\cdot, u_\infty (\cdot)) \text{ weakly in } L^{q/(q-1)} (\mathbb{R}^N)
\]
as \( j \to +\infty \). It is easily seen that
\[
\int_{\mathbb{R}^N} f(x, u_j(x)) \varphi(x) \, dx \to \int_{\mathbb{R}^N} f(x, u_\infty(x)) \varphi(x) \, dx
\]
\[\forall \varphi \in L^q (\mathbb{R}^N)
\]
as \( j \to +\infty \) and so, in particular,
\[
\int_{\mathbb{R}^N} f(x, u_j(x)) \varphi(x) \, dx \to \int_{\mathbb{R}^N} f(x, u_\infty(x)) \varphi(x) \, dx
\]
\[\forall \varphi \in E
\]
as \( j \to +\infty \). Since (51) holds true, for any \( \varphi \in E \),
\[
0 \leftarrow \langle I'(u_j) , \varphi \rangle
\]
\[
= \int_{\mathbb{R}^N} \frac{(u_j(x) - u_j(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dx \, dy
\]
\[+ \int_{\mathbb{R}^N} V(x)u_j(x)\varphi(x) \, dx
\]
\[- \int_{\mathbb{R}^N} |u_j(x)|^{2(\sigma-2)}u_j(x)\varphi(x) \, dx
\]
\[- \int_{\mathbb{R}^N} f(x, u_j(x))\varphi(x) \, dx.
\]
(63)

Passing to the limit in this expression as \( j \to +\infty \) and taking into account (55), (57), (59), and (65), we get
\[
\int_{\mathbb{R}^N} \frac{(u_\infty(x) - u_\infty(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dx \, dy
\]
\[+ \int_{\mathbb{R}^N} V(x)u_\infty(x)\varphi(x) \, dx
\]
\[- \int_{\mathbb{R}^N} |u_\infty(x)|^{2(\sigma-2)}u_\infty(x)\varphi(x) \, dx
\]
\[- \int_{\mathbb{R}^N} f(x, u_\infty(x))\varphi(x) \, dx = 0
\]
for any \( \varphi \in E \); that is, \( u_\infty \) is a solution of problem (10).

\textbf{Step 3.} The following equality holds true:
\[
I(u_\infty) = \frac{s}{N} \int_{\mathbb{R}^N} |u_\infty(x)|^{2^*(\sigma)} \, dx
\]
\[+ \frac{1}{2} \int_{\mathbb{R}^N} f(x, u_\infty(x))u_\infty(x) \, dx
\]
\[- \int_{\mathbb{R}^N} F(x, u_\infty(x)) \, dx \geq 0.
\]
(68)

\textbf{Proof.} By Step 2, taking \( \varphi = u_\infty \in E \) as a test function in (10), we have
\[
\int_{\mathbb{R}^{2N}} \frac{|u_\infty(x) - u_\infty(y)|^2}{|x-y|^{N+2s}} \, dx \, dy + \int_{\mathbb{R}^N} V(x)|u_\infty(x)|^2 \, dx
\]
\[= \int_{\mathbb{R}^N} |u_\infty(x)|^{2^*(\sigma)} \, dx + \int_{\mathbb{R}^N} f(x, u_\infty(x))u_\infty(x) \, dx
\]
so that
\[
I(u_\infty) = \frac{s}{N} \int_{\mathbb{R}^N} |u_\infty(x)|^{2^*(\sigma)} \, dx
\]
\[+ \frac{1}{2} \int_{\mathbb{R}^N} f(x, u_\infty(x))u_\infty(x) \, dx
\]
\[- \int_{\mathbb{R}^N} F(x, u_\infty(x)) \, dx \geq 0.
\]
(69)

The last inequality follows from assumption (f4).

Now, we conclude the proof of Proposition 12.

We write \( v_j := u_j - u_\infty \); then, \( v_j \rightharpoonup 0 \) weakly in \( E \). Moreover, since (58) holds true, by the Brézis-Lieb Lemma, we get
\[
\int_{\mathbb{R}^{2N}} \frac{|u_j(x) - u_j(y)|^2}{|x-y|^{N+2s}} \, dx \, dy
\]
\[+ \int_{\mathbb{R}^N} V(x)|u_j(x)|^2 \, dx
\]
\[= \int_{\mathbb{R}^{2N}} \frac{|v_j(x) - v_j(y)|^2}{|x-y|^{N+2s}} \, dx \, dy
\]
\[+ \int_{\mathbb{R}^N} V(x)|v_j(x)|^2 \, dx
\]
\[+ \int_{\mathbb{R}^{2N}} \frac{|u_\infty(x) - u_\infty(y)|^2}{|x-y|^{N+2s}} \, dx \, dy
\]
\[+ \int_{\mathbb{R}^N} V(x)|u_\infty(x)|^2 \, dx + o(1),
\]
\[
\int_{\mathbb{R}^N} |u_j(x)|^{2^*(\sigma)} \, dx = \int_{\mathbb{R}^N} |v_j(x)|^{2^*(\sigma)} \, dx
\]
\[+ \int_{\mathbb{R}^N} |u_\infty(x)|^{2^*(\sigma)} \, dx + o(1).
\]
(71)
Then,
\[
c \to \langle I(u_j), u_j \rangle = \frac{1}{2} \int_{\mathbb{R}^N} \frac{|u_j(x) - u_j(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \\
+ \frac{1}{2} \int_{\mathbb{R}^N} V(x) |u_j(x)|^2 \, dx \\
- \frac{1}{2^* (s)} \int_{\mathbb{R}^N} |u_j(x)|^{2^*(s)} \, dx \\
- \int_{\mathbb{R}^N} F(x, u_j(x)) \, dx \\
= \frac{1}{2} \int_{\mathbb{R}^N} \frac{|v_j(x) - v_j(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \\
+ \frac{1}{2} \int_{\mathbb{R}^N} V(x) |v_j(x)|^2 \, dx \\
+ \frac{1}{2} \int_{\mathbb{R}^N} |u_{\infty}(x) - u_{\infty}(y)|^2 \, dx \, dy \\
+ \frac{1}{2} \int_{\mathbb{R}^N} V(x) |v_j(x)|^2 \, dx \\
- \frac{1}{2^* (s)} \int_{\mathbb{R}^N} |v_j(x)|^{2^*(s)} \, dx \\
- \int_{\mathbb{R}^N} F(x, u_{\infty}(x)) \, dx + o(1) \\
= I(u_{\infty}) + \frac{1}{2} \int_{\mathbb{R}^N} \frac{|v_j(x) - v_j(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \\
+ \frac{1}{2} \int_{\mathbb{R}^N} V(x) |v_j(x)|^2 \, dx \\
- \frac{1}{2^* (s)} \int_{\mathbb{R}^N} |v_j(x)|^{2^*(s)} \, dx \\
- \int_{\mathbb{R}^N} F(x, u_{\infty}(x)) \, dx + o(1) ,
\]

By \( \langle I'(u_{\infty}), u_{\infty} \rangle = 0 \) and \( \langle I'(u_j), u_j \rangle \to 0 \), we get
\[
\int_{\mathbb{R}^N} \frac{|v_j(x) - v_j(y)|^2}{|x - y|^{N+2s}} \, dx \, dy + \int_{\mathbb{R}^N} V(x) |v_j(x)|^2 \, dx \to b ,
\int_{\mathbb{R}^N} |v_j(x)|^{2^*(s)} \, dx \to b.
\]

By the definition of \( S_1 \), we have
\[
\|v_j\|_E^2 \geq S_1 \|v_j\|_{2^* (s)}^{2^* (s)} (\mathbb{R}^N)
\]
and so \( b \geq S_1 b^{2^*(s)} \). Either \( b = 0 \) or \( b \geq S_1^{N/2s} \). If \( b = 0 \), the proof is complete. Assuming that \( b \geq S_1^{N/2s} \), we obtain, from (49), (68), and (72),
\[
\frac{s}{N} S_1^{N/2s} \leq \left( \frac{1}{2} - \frac{1}{2^* (s)} \right) b \leq c < \frac{s}{N} S_1^{N/2s}
\]
which is a contradiction. Thus \( b = 0 \) and
\[
\|u_j - u_{\infty}\|_E \to 0
\]
as \( j \to +\infty \). This ends the proof of Proposition 12. 

We have finished the proof of Theorem 1 by Propositions 8, 10, and 12 and the mountain pass theorem.

4. The Proof of Theorem 2

In this section we consider the case \( V(x) \equiv 1 \), study the mountain pass geometry and Palais-Smale condition in a
suitable energy range, and finish the proof of Theorem 2. We consider the functional
\[
I_1 (u) = \frac{1}{2} \int_{\mathbb{R}^N} |u(x) - u(y)|^2 |x-y|^{N+2s} \, dx \, dy + \frac{1}{2} \int_{\mathbb{R}^N} |u(x)|^2 \, dx
- \frac{1}{2s} \int_{\mathbb{R}^N} |u(x)|^{2s} \, dx - \int_{\mathbb{R}^N} F(x, u(x)) \, dx,
\]
(78)
where \( F \) is defined as in (f4). Then \( I_1 \in C^1(E, \mathbb{R}) \) and critical points of \( I_1 \) are solutions of
\[
(-\Delta)^{s} u + u = |u|^{2s-2} u + f(x, u), \quad u \in H^s(\mathbb{R}^N).
\]
(79)
The Fréchet derivative of \( I_1 \) is
\[
\langle I_1'(u), \varphi \rangle = \int_{\mathbb{R}^N} (u(x) - u(y)) (\varphi(x) - \varphi(y)) |x-y|^{N+2s} \, dx \, dy
+ \int_{\mathbb{R}^N} u(x) \varphi(x) \, dx
- \int_{\mathbb{R}^N} |u(x)|^{2s} u(x) \varphi(x) \, dx
- \int_{\mathbb{R}^N} f(x, u(x)) \varphi(x) \, dx
\]
for any \( \varphi \in H^s(\mathbb{R}^N) \).

Similar to the proof of Theorem 1, we have the following conclusions.

**Proposition 13.** Let \( N > 2s \), \( s \in (0,1) \), and \( f \) satisfy (f0)–(f4). Then, there exist \( \rho > 0 \) and \( \beta > 0 \) such that for any \( u \in H^s(\mathbb{R}^N) \) with \( \|u\|_{H^s} = \rho \) it results that \( I_1(u) \geq \beta \).

**Proposition 14.** There exists \( u_0 \in H^s(\mathbb{R}^N) \setminus \{0\} \) with \( u_0 \geq 0 \) a.e. in \( \mathbb{R}^N \), such that
\[
\sup_{t \geq 0} I_1(tu_0) < \frac{s}{N} S_{N/2s}^N.
\]
(81)

**Proposition 15.** Let \( N > 2s \), \( s \in (0,1) \), and \( f \) satisfy (f0)–(f4). Then, there exists \( e \in H^s(\mathbb{R}^N) \) such that \( e \geq 0 \) a.e. in \( \mathbb{R}^N \), \( \|e\|_{H^s} > \rho \), and \( I_1(e) < \beta \), where \( \rho \) and \( \beta \) are given in Proposition 13.

In particular, we can construct \( e \) as follows:
\[
e = t_0 u_0 \quad (82)
\]
with \( u_0 \) as in (81) and \( t_0 \) large enough.

We easily see that \( I_1(0) = 0 < \beta \), with \( \beta \) given in Proposition 13. Now, set
\[
c = \inf_{T \in \Gamma} \sup_{u \in \Gamma([0,1])} I_1(u),
\]
(83)
where
\[
\Gamma = \{\Gamma \in C([0,1]; H^s_0(\mathbb{R}^N)) : \Gamma(0) = 0, \Gamma(1) = e\},
\]
(84)
with \( e = t_0 u_0 \) in Proposition 13.

**Proposition 16.** The constant \( c \) is given in (83) such that
\[
\beta \leq c < \frac{s}{N} S_{N/2s}^N,
\]
(85)
where \( \beta \) is given in Proposition 13 and \( S_{N/2s} \) is defined in formula (13).

By [6, Theorem 2.2], we have a sequence \( u_j \) in \( H^s(\mathbb{R}^N) \) such that
\[
I_1(u_j) \longrightarrow c,
\]
\[
\sup \{|\langle I_1'(u_j), \varphi \rangle : \varphi \in H^s_0(\mathbb{R}^N), \|\varphi\|_{H^s} = 1\} \longrightarrow 0
\]
(86)
as \( j \to +\infty \).

**Proposition 17.** There exists \( u_{\infty} \in H^s_0(\mathbb{R}^N) \) such that, up to a subsequence, \( \|u_j - u_{\infty}\|_{H^s} \to 0 \) as \( j \to +\infty \).

We have finished the proof of Theorem 2 by Propositions 13, 15, and 17 and the mountain pass theorem.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

**Acknowledgment**

The work is supported by the National Nature Science Foundation of China (11271331).

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