Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 176736, 7 pages
http://dx.doi.org/10.1155/2014/176736

Research Article

The Structure of φ-Module Amenable Banach Algebras

Mahmood Lashkarizadeh Bami,[1] Mohammad Valaei,[1] and Massoud Amini[2,3]

[1] *Department of Mathematics, University of Isfahan, Isfahan, Iran*
[2] *Department of Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, Tehran 14115-134, Iran*
[3] *School of Mathematics, Institute for Research in Fundamental Sciences (IPM), Tehran 19395-5746, Iran*

Correspondence should be addressed to Mohammad Valaei; valaei@sci.ui.ac.ir

Received 14 November 2013; Accepted 4 February 2014; Published 2 April 2014

Academic Editor: Erdal Karapınar

We study the concept of φ-module amenability of Banach algebras, which are Banach modules over another Banach algebra with compatible actions. Also, we compare the notions of φ-amenability and φ-module amenability of Banach algebras. As a consequence, we show that, if S is an inverse semigroup with finite set E of idempotents and $l^1(S)$ is a commutative Banach $l^1(E)$-module, then $l^1(S)^{**}$ is φ^{**}-module amenable if and only if S is finite, when $\varphi \in \mathrm{Hom}_{l^1(E)}(l^1(S))$ is an epimorphism. Indeed, we have generalized a well-known result due to Ghahramani et al. (1996).

1. Introduction

The concept of amenability for Banach algebras was first introduced by Johnson in [1]. For a locally compact group G, Ghahramani et al. showed that $L^1(G)^{**}$ is amenable if and only if G is finite [2]. The notion of module amenability for a Banach algebra, $\mathscr{A}$ which is a Banach module over another Banach algebra $\mathfrak{A}$ with compatible actions, was introduced and studied by the third author in [3]. The notion of φ-module amenability was introduced by Bodaghi in [4]; he obtained some results for a specific compatible action (i.e., trivial left action). In [5, 6], the authors investigated the module amenability of the second dual $l^1(S)^{**}$ of the semigroup Banach algebra $l^1(S)$, for an inverse semigroup S with the set of idempotents E. They showed that $l^1(S)^{**}$ is module amenable, if and only if an appropriate group homomorphic image $S/\approx$ of S is finite, when $\mathfrak{A} := l^1(E)$ acts on $\mathscr{A} := l^1(S)$ by the compatible actions $\delta_e \cdot \delta_s = \delta_s$ and $\delta_s \cdot \delta_e = \delta_{se}$, for $s \in S$ and $e \in E$. Indeed, for the very specific compatible actions, they presented a generalization of the result due to Ghahramani et al. (in the discrete case).

The aim of this paper is to investigate the structure of φ-module amenable Banach algebras (we do not restrict ourselves to some specific compatible actions). In particular, we give the generalization of the result of Ghahramani et al. for arbitrary commutative compatible actions. The paper is organized as follows. In Section 1 we give the definitions which are needed throughout the paper. In Section 2 we introduce the notions of φ-module virtual diagonal and φ-module approximate diagonal and study the structure of φ-module amenable Banach algebras. We also find relations between φ-module amenability and φ-amenability (that generalize the concepts of module amenability and amenability, respectively) without the extra assumption that the compatible action is trivial from one direction, or the assumption that $\mathfrak{A}$ has a bounded approximate identity for $\mathscr{A}$. We assume that φ is either idempotent or surjective. The former is used to ensure that φ fixes points of its range. The latter is used in particular in Proposition 10 (and then in Theorem 13) to ensure that a φ-module approximate diagonal is also a module approximate diagonal.

In Section 3 we apply main results of Section 2 to semigroup Banach algebras.

2. Preliminaries

Let $\mathscr{A}$ be a Banach algebra and let σ be a endomorphism on $\mathscr{A}$. Suppose that X is a Banach $\mathscr{A}$-bimodule. A bounded linear map $D : \mathscr{A} \to X$ is called a σ-derivation if

$$D(ab) = \sigma(a) \cdot D(b) + D(a) \cdot \sigma(b) \quad (a, b \in \mathscr{A}). \tag{1}$$

For each $x \in X$, we define the σ-derivation ad_x^σ by

$$ad_x^\sigma(a) = \sigma(a) \cdot x - x \cdot \sigma(a) \quad (a \in \mathscr{A}). \tag{2}$$

These are called σ-inner derivations. The Banach algebra $\mathscr{A}$ is called σ-amenable if, for any Banach $\mathscr{A}$-bimodule X, every σ-derivation from $\mathscr{A}$ to X^* is σ-inner.

Throughout this paper, $\mathscr{A}$ and $\mathfrak{A}$ are Banach algebras such that $\mathscr{A}$ is a Banach $\mathfrak{A}$-bimodule with compatible actions; that is

$$\alpha \cdot (ab) = (\alpha \cdot a) b, \qquad (ab) \cdot \alpha = a (b \cdot \alpha) \qquad (\alpha \in \mathfrak{A},\ a, b \in \mathscr{A}). \tag{3}$$

Let X be a Banach $\mathscr{A}$-bimodule and a Banach $\mathfrak{A}$-bimodule with compatible actions; that is

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) x, \qquad a \cdot (\alpha \cdot x) = (a \cdot \alpha) \cdot x, \qquad (\alpha \cdot x) \cdot a = \alpha \cdot (x \cdot a), \tag{4}$$

for $\alpha \in \mathfrak{A}$, $a \in \mathscr{A}$, $x \in X$, and similarly for the right or two-side actions. Then X is called a Banach $\mathscr{A}$-$\mathfrak{A}$-module. If, moreover,

$$\alpha \cdot x = x \cdot \alpha \quad (\alpha \in \mathfrak{A},\ x \in X), \tag{5}$$

then X is called a commutative Banach $\mathscr{A}$-$\mathfrak{A}$-module.

It is obvious that, if X is a (commutative) Banach $\mathscr{A}$-$\mathfrak{A}$-module, then so is X^* under the following compatible actions:

$$\langle \alpha \cdot f, x \rangle = \langle f, x \cdot \alpha \rangle, \qquad \langle a \cdot f, x \rangle = \langle f, x \cdot a \rangle \qquad (\alpha \in \mathfrak{A},\ a \in \mathscr{A},\ f \in X^*), \tag{6}$$

and similarly for the right actions.

Note that, when $\mathscr{A}$ acts on itself by algebra multiplication, it need not be a Banach $\mathscr{A}$-$\mathfrak{A}$-module, as we have not assumed the compatibility condition $(a \cdot \alpha) \cdot b = a \cdot (\alpha \cdot b)$ for $\alpha \in \mathfrak{A}$ and $a, b \in \mathscr{A}$. But when $\mathscr{A}$ is a commutative $\mathfrak{A}$-module and acts on itself by multiplication from both sides, then it is a commutative Banach $\mathscr{A}$-$\mathfrak{A}$-module.

Let $\mathscr{A}$ and $\mathscr{B}$ be $\mathfrak{A}$-modules. A continuous mapping $T : \mathscr{A} \to \mathscr{B}$ is called an $\mathfrak{A}$-module morphism if

$$T(a \pm b) = T(a) \pm T(b), \qquad T(ab) = T(a) T(b), \tag{7}$$

also,

$$T(\alpha \cdot a) = \alpha \cdot T(a), \qquad T(a \cdot \alpha) = T(a) \cdot \alpha \qquad (\alpha \in \mathfrak{A},\ a, b \in \mathscr{A}). \tag{8}$$

We denote the space of all such $\mathfrak{A}$-module morphisms by $\mathrm{Hom}_{\mathfrak{A}}(\mathscr{A}, \mathscr{B})$ and denote $\mathrm{Hom}_{\mathfrak{A}}(\mathscr{A}, \mathscr{A})$ by $\mathrm{Hom}_{\mathfrak{A}}(\mathscr{A})$.

Let $\mathfrak{A}$, $\mathscr{A}$ and X be as above and let $\varphi \in \mathrm{Hom}_{\mathfrak{A}}(\mathscr{A})$. A bounded map $D : \mathscr{A} \to X$ is called a *φ-module derivation* if

$$D(a \pm b) = D(a) \pm D(b), \qquad D(ab) = \varphi(a) \cdot D(b) + D(a) \cdot \varphi(b), \tag{9}$$

also,

$$D(\alpha \cdot a) = \alpha \cdot D(a), \qquad D(a \cdot \alpha) = D(a) \cdot \alpha \qquad (\alpha \in \mathfrak{A}, a, b \in \mathscr{A}). \tag{10}$$

Note that $D : \mathscr{A} \to X$ is bounded if there exist $M > 0$ such that $\|D(a)\| \le M\|a\|$ $(a \in \mathscr{A})$, although D is not necessarily linear, but still its boundedness implies its norm continuity. Let $\varphi \in \mathrm{Hom}_{\mathfrak{A}}(\mathscr{A})$ and $x \in X$ if we define ad_x^φ as in (2); then ad_x^φ is a φ-module derivation that is called a φ-module inner derivation.

The Banach algebra $\mathscr{A}$ is called *φ-module amenable* if, for any commutative Banach $\mathscr{A}$-$\mathfrak{A}$-module X, each φ-module derivation form $\mathscr{A}$ to X^* is φ-module inner.

We note that, if φ is the identity map on $\mathscr{A}$, then $id_{\mathscr{A}}$-module amenability is the same as module amenability. Also, when $\mathfrak{A} := \mathbb{C}$, everything reduces to the classical case.

3. φ-Amenability and φ-Module Amenability

Throughout this section $\mathfrak{A}$ is a Banach algebra, $\mathscr{A}$ is a Banach $\mathfrak{A}$-module with compatible actions, and $\varphi \in \mathrm{Hom}_{\mathfrak{A}}(\mathscr{A})$, unless otherwise specified. We start this section by the following lemma, which is proved similar to Proposition 2.1.3 in [7].

Lemma 1. *Let X be a commutative Banach $\mathscr{A}$-$\mathfrak{A}$-module. Then every φ-module derivation from $\mathscr{A}$ to X^* is φ-module inner, when one of the following is satisfied.*

(i) *$\mathscr{A}$ has a bounded right approximate identity and $\varphi(\mathscr{A}) \cdot X = 0$.*

(ii) *$\mathscr{A}$ has a bounded left approximate identity and $X \cdot \varphi(\mathscr{A}) = 0$.*

Definition 2. Let $\mathscr{A}$ be a Banach algebra. A Banach $\mathscr{A}$-bimodule X is called φ-pseudo-unital if

$$X = \{\varphi(a) \cdot x \cdot \varphi(b) : a, b \in \mathscr{A},\ x \in X\}. \tag{11}$$

The proof of the following proposition is routine, but we give it for the sake of completeness.

Proposition 3. *Let φ be idempotent or surjective and let $\mathscr{A}$ have a bounded approximate identity. Suppose that, for any commutative Banach $\mathscr{A}$-$\mathfrak{A}$-module X which is φ-pseudo-unital, each φ-module derivation form $\mathscr{A}$ to X^* is φ-module inner. Then $\mathscr{A}$ is φ-module amenable.*

Proof. Let X be a commutative Banach $\mathscr{A}$-$\mathfrak{A}$-module and let $D : \mathscr{A} \to X^*$ be a φ-module derivation. Let $X_1 = \varphi(\mathscr{A}) \cdot X \cdot \varphi(\mathscr{A})$, $X_2 = X \cdot \varphi(\mathscr{A})$, and $X_3 = X$. Let $\pi_j : X_{j+1}^* \to X_j^*$ be the

restriction map ($j = 1, 2$). In the case where φ is idempotent, then we turn X into an another commutative Banach $\mathscr{A}$-$\mathfrak{A}$-module, by letting the same actions of $\mathfrak{A}$ and the following actions of $\mathscr{A}$:

$$a \bullet x := \varphi(a) \cdot x, \qquad x \bullet a := x \cdot \varphi(a) \qquad (a \in \mathscr{A}, x \in X). \tag{12}$$

Also, in the above actions, D is again a φ-module derivation. By Cohen's factorization theorem, X_1 and X_2 are closed $\mathscr{A}$-$\mathfrak{A}$-submodules of X (with respect to the module actions).

Let $d : \mathscr{A} \to X_2^*$ be a φ-module derivation; then so is $\pi_1 \circ d : \mathscr{A} \to X_1^*$. Since X_1 is φ-pseudo-unital, there is $f_1 \in X_1^*$ such that $\pi_1 \circ d = ad_{f_1}^{\varphi}$. Choose $f_2 \in X_2^*$ such that $f_2|_{X_1} = f_1$ and consider $\tilde{d} := d - ad_{f_2}^{\varphi}$. Then $\tilde{d} : \mathscr{A} \to X_2^* \cap X_1^{\perp} \cong (X_2/X_1)^*$ is a φ-module derivation. Therefore, there is $g_2 \in X_2^* \cap X_1^{\perp}$ such that $\tilde{d} = ad_{g_2}^{\varphi}$, by Lemma 1. Thus, $d = ad_{f_2+g_2}^{\varphi}$. Hence, any φ-module derivation from $\mathscr{A}$ into X_2^* is φ-module inner.

Now, by the above assertion, let $f_2 \in X_2^*$ such that $\pi_2 \circ D = ad_{f_2}^{\varphi}$. From Hahn-Banach theorem, we obtain an extension $f \in X^*$ of f_2, so that $D - ad_f^{\varphi} : \mathscr{A} \to X_2^{\perp} \cong (X/X_2)^*$ is a φ-module derivation. Since $(X/X_2) \cdot \varphi(\mathscr{A}) = 0$, there is $g \in X^*$ such that $D = ad_{f+g}^{\varphi}$. Let $h = f + g$. In the case where φ is an idempotent, we have

$$\begin{aligned} D(a) &= \varphi(a) \bullet h - h \bullet \varphi(a) = \varphi(\varphi(a)) \cdot h - h \cdot \varphi(\varphi(a)) \\ &= \varphi(a) \cdot h - h \cdot \varphi(a) \quad (a \in \mathscr{A}). \end{aligned} \tag{13}$$

Therefore $D = ad_h^{\varphi}$, where φ is idempotent or surjective (similarly). Consequently, D is φ-module inner. □

Let $\mathscr{B}$ be a Banach algebra with a bounded approximate identity and let $\sigma \in \mathrm{Hom}_{\mathbb{C}}(\mathscr{B})$ be idempotent or surjective. Consider $\mathfrak{A} := \mathbb{C}$; then $\mathscr{B}$ is automatically a commutative Banach $\mathbb{C}$-module. Also, σ-derivations and σ-module derivations are the same; hence σ-module amenability is the same as σ-amenability for $\mathscr{B}$. Consequently, $\mathscr{B}$ is σ-amenable if and only if, for any Banach $\mathscr{B}$-bimodule X which is φ-pseudo-unital, each σ-derivation form $\mathscr{B}$ to X^* is σ-inner, by Proposition 3.

Proposition 4. *Let $\mathscr{A}$ be a commutative Banach $\mathfrak{A}$-module. If $\mathscr{A}$ is φ-module amenable, then $\mathscr{A}$ has a bounded approximate identity for $\varphi(\mathscr{A})$.*

Proof. Consider $X := \mathscr{A}$, then X is a commutative Banach $\mathscr{A}$-$\mathfrak{A}$-module, with the same actions of $\mathfrak{A}$ and the following actions of $\mathscr{A}$:

$$a \cdot x := ax, \quad x \cdot a := 0 \quad (a \in \mathscr{A},\ x \in X). \tag{14}$$

Let $D : \mathscr{A} \to \mathscr{A}^{**}$ be the canonical embedding of $\mathscr{A}$ into its second dual. Then $D \circ \varphi$ is a φ-module derivation. Thus, there is E in X^{**} such that $\varphi(a) = \varphi(a) \cdot E$ for all $a \in \mathscr{A}$. Now, as the proof of Proposition 2.2.1 in [7], we can obtain a bounded net $(e_j)_j \subseteq \mathscr{A}$ such that it is an approximate identity for $\varphi(\mathscr{A})$. □

Lemma 5. *Let φ be linear and idempotent or surjective, let $\mathscr{A}$ be a commutative Banach $\mathfrak{A}$-module, and let $D : \mathscr{A} \to X^*$ be a φ-derivation for some φ-pseudo-unital Banach $\mathscr{A}$-bimodule X. If $\varphi(\mathscr{A})$ has a bounded approximate identity $(\varphi(e_j))_j$ such that $(D(a) \cdot \varphi(\alpha \cdot e_j))_j$ and $(\varphi(\alpha \cdot e_j) \cdot D(a))_j$ are convergent to $(\alpha \cdot a)(\alpha \in \mathfrak{A},\ a \in \mathscr{A})$, then there is a commutative Banach $\mathscr{A}$-$\mathfrak{A}$-module F such that $D : \mathscr{A} \to F^*$ is a φ-module derivation.*

Proof. Let E be the w^*-closed linear span of the following set:

$$Y = \{\varphi(a) \cdot D(b) \cdot \varphi(c) : a, b, c \in \mathscr{A}\}. \tag{15}$$

In the case where φ is idempotent, we turn X into an another Banach $\mathscr{A}$-bimodule via φ, as follows. Since X is φ-pseudo-unital, we conclude that E is a Banach $\mathscr{A}$-submodule of X^* such that $D(\mathscr{A}) \subseteq E$. Let F be a Banach $\mathscr{A}$-bimodule such that $F^* = E$, which exists by Exercise 2.1.2 of [7]. For $x \in X$, let $a, b \in \mathscr{A}$, and $z \in X$ be such that $x = \varphi(a) \cdot z \cdot \varphi(b)$. For $\alpha \in \mathfrak{A}$, define

$$\begin{aligned} \alpha \bullet x &:= \varphi(\alpha \cdot a) \cdot (z \cdot \varphi(b)), \\ x \bullet \alpha &:= (\varphi(a) \cdot z) \cdot \varphi(b \cdot \alpha). \end{aligned} \tag{16}$$

We claim that $\alpha \bullet x$ is well defined; that is, it is independent of the choices of a, b, and z. Let $c, d \in \mathscr{A}$, and $t \in X$ such that $x = \varphi(c) \cdot t \cdot \varphi(d)$. Then, for each $\alpha \in \mathfrak{A}$, we have

$$\begin{aligned} &\varphi(\alpha \cdot a) \cdot (z \cdot \varphi(b)) \\ &\quad = \lim_j \varphi(\alpha \cdot e_j) \cdot (\varphi(a) \cdot z \cdot \varphi(b)) \\ &\quad = \lim_j \varphi(\alpha \cdot e_j) \cdot (\varphi(c) \cdot t \cdot \varphi(d)) \\ &\quad = \varphi(\alpha \cdot c) \cdot (t \cdot \varphi(d)); \end{aligned} \tag{17}$$

similarly, $x \bullet \alpha$ is well defined. Clearly, by the above actions of $\mathfrak{A}$ and the given actions of $\mathscr{A}$, X is a Banach $\mathscr{A}$-$\mathfrak{A}$-module. For $\alpha \in \mathfrak{A}$ and $x \in F$, we have

$$\alpha \bullet x = \lim_j \alpha \bullet (\varphi(e_j) \cdot x) = \lim_j (\alpha \cdot \varphi(e_j)) \cdot x \in F; \tag{18}$$

similarly, $x \bullet \alpha \in F$. Thus F is a Banach $\mathfrak{A}$-submodule of X. So F is a Banach $\mathscr{A}$-$\mathfrak{A}$-module. For all b and c in $\mathscr{A}$, $D(b) \cdot \varphi(c)$ is an element of F^*, so for each $\alpha \in \mathfrak{A}$ and $a \in \mathscr{A}$, we have

$$\alpha \bullet (\varphi(a) \cdot D(b) \cdot \varphi(c)) = \varphi(\alpha \cdot a) \cdot (D(b) \cdot \varphi(c)); \tag{19}$$

similarly, $(\varphi(a)\cdot D(b)\cdot\varphi(c))\bullet\alpha=(\varphi(a)\cdot D(b))\cdot\varphi(c\cdot\alpha)$. Also, $\mathfrak{A}$-commutativity of $\mathscr{A}$, implies that

$$\begin{aligned}\varphi(\alpha\cdot a)\cdot(D(b)\cdot\varphi(c))&\\ &=\lim_j\varphi(a)\cdot\left[\varphi\left(\alpha\cdot e_j\right)\cdot D(b)\right]\cdot\varphi(c),\\ &=\varphi(a)\cdot[D(\alpha\cdot b)]\cdot\varphi(c),\\ &=\lim_j\varphi(a)\cdot\left[D(b)\cdot\varphi\left(\alpha\cdot e_j\right)\right]\cdot\varphi(c),\\ &=(\varphi(a)\cdot D(b))\cdot\varphi(c\cdot\alpha).\end{aligned}\tag{20}$$

Thus, by linearity and the w^*-continuity of the compatible actions, for $\alpha\in\mathfrak{A}$ and $x\in F$

$$\langle x\bullet\alpha,f\rangle=\langle x,\alpha\bullet f\rangle=\langle x,f\bullet\alpha\rangle=\langle\alpha\bullet x,f\rangle\quad(\varphi\in F^*);\tag{21}$$

thus $x\bullet\alpha=\alpha\bullet x$. Therefore, F is a commutative Banach $\mathscr{A}$-$\mathfrak{A}$-module. Also,

$$\begin{aligned}D(\alpha\cdot a)&=\lim_j\varphi\left(\alpha\cdot e_j\right)\cdot D(a),\\ &=\lim_j\alpha\bullet\left(\varphi\left(e_j\right)\cdot D(a)\right),\\ &=\alpha\bullet D(a)\quad(\alpha\in\mathfrak{A},\ a\in\mathscr{A}).\end{aligned}\tag{22}$$

Consequently, $D:\mathscr{A}\to F^*$ is a φ-module derivation. □

Proposition 6. *Let $\mathscr{A}$ be a commutative Banach $\mathfrak{A}$-module with a bounded approximate identity $(e_j)_j$. Suppose that $\mathscr{A}$ is φ-module amenable and has a $\varphi|_K$-amenable, closed subalgebra K such that $(\alpha\cdot e_j)\in K$ $(\alpha\in\mathfrak{A})$. Then $\mathscr{A}$ is φ-amenable, when φ is linear and idempotent or surjective.*

Proof. Let X be a Banach $\mathscr{A}$-bimodule and $D:\mathscr{A}\to X^*$ be a φ-derivation, without loss of generality; we may suppose that X is φ-pseudo-unital. By $\varphi|_K$-amenability of K, there is $f\in X^*$ such that

$$D(a)=\varphi(a)\cdot f-f\cdot\varphi(a)\quad(a\in K).\tag{23}$$

Let $\widetilde{D}=D-ad_f^{\varphi}$. Then $\widetilde{D}$ is a φ-derivation such that $\widetilde{D}(\alpha\cdot\varphi(e_j))=0$, for $\alpha\in\mathfrak{A}$ and $j\in J$. Let $e_j'=\varphi(e_j)$; then both $(e_j')_j$ and $(\varphi(e_j'))_j$ are bounded approximate identities for $\varphi(\mathscr{A})$.

In the case where φ is idempotent, for $a\in\mathscr{A}$, we have

$$\begin{aligned}\widetilde{D}(\varphi(a))&=\lim_j\left[\varphi(\varphi(a))\cdot\widetilde{D}\left(e_j'\right)+\widetilde{D}(\varphi(a))\cdot\varphi\left(e_j'\right)\right],\\ &=\lim_j\varphi(a)\cdot\widetilde{D}\left(e_j'\right)+\widetilde{D}(\varphi(a));\end{aligned}\tag{24}$$

thus, $\lim_j\varphi(a)\cdot\widetilde{D}(e_j')=0$. Moreover, since X is φ-pseudo-unital, we obtain that $\lim_j\widetilde{D}(ae_j')=\widetilde{D}(a)$. Similarly $\lim_j\widetilde{D}(e_j'a)=\widetilde{D}(a)$, for all $a\in\mathscr{A}$.

Therefore, if φ is idempotent or surjective, then $(\widetilde{D}(a)\cdot\varphi(\alpha\cdot e_j'))_j$ and $(\varphi(\alpha\cdot e_j')\cdot\widetilde{D}(a))_j$ are convergent to $\widetilde{D}(\alpha\cdot a)$, for all $\alpha\in\mathfrak{A}$ and $a\in\mathscr{A}$. Thus, by Lemma 5, there is a commutative Banach $\mathscr{A}$-$\mathfrak{A}$-module F such that $\widetilde{D}:\mathscr{A}\to F^*$ is a φ-module derivation. Hence, there is $g\in F^*$ such that $\widetilde{D}=ad_g^{\varphi}$. Since, by in the proof of Lemma 5, F^* is an $\mathscr{A}$-submodule of X^* (via φ, in the case φ is idempotent), we obtain that $D=ad_{f+g}^{\varphi}$. □

Theorem 7. *Let $\varphi\in Hom_{\mathfrak{A}}(\mathscr{A})$ be an epimorphism or an idempotent homomorphism. Suppose that $\mathscr{A}$ is a unital, commutative Banach $\mathfrak{A}$-module and $\mathfrak{A}$ is amenable. Then φ-module amenability of $\mathscr{A}$ implies its φ-amenability.*

Proof. Suppose that e is an identity for $\mathscr{A}$. Let K be the closed linear span of $\{\alpha\cdot\varphi(e):\alpha\in\mathfrak{A}\}$. Since $\varphi(e)$ is an identity for $\varphi(\mathscr{A})$, K is a closed subalgebra of $\mathscr{A}$ under the following multiplication:

$$(\alpha\cdot\varphi(e))\cdot(\beta\cdot\varphi(e)):=(\alpha\beta)\cdot\varphi(e)\quad(\alpha,\beta\in\mathfrak{A}).\tag{25}$$

Let $\theta:\mathfrak{A}\to K$ be defined by $\theta(\alpha)=\alpha\cdot\varphi(e)$, for $\alpha\in\mathfrak{A}$. Then θ is a continuous homomorphism and $\theta(\mathfrak{A})$ is dense in K. Hence K is amenable, by Proposition 2.3.1 of [7]. By definition of K, we have that $\varphi|_K$ is an endomorphism on K. Therefore, K is $\varphi|_K$-amenable (by Corollary 2.2 in [8]) and satisfies conditions of Proposition 6. □

Let $\mathscr{A}\widehat{\otimes}\mathscr{A}$ be the projective tensor product of $\mathscr{A}$ by itself. Then $\mathscr{A}\widehat{\otimes}\mathscr{A}$ is a Banach $\mathscr{A}$-$\mathfrak{A}$-module with the canonical actions [5]. Consider the closed ideal $\mathscr{I}$ of $\mathscr{A}\widehat{\otimes}\mathscr{A}$ generated by elements of the form $a\cdot\alpha\otimes b-a\otimes\alpha\cdot b$, for $a,b\in\mathscr{A}$ and $\alpha\in\mathfrak{A}$. Let $\mathscr{J}$ be the closed ideal of $\mathscr{A}$ generated by elements of the form $(a\cdot\alpha)b-a(\alpha\cdot b)$, for $a,b\in\mathscr{A}$ and $\alpha\in\mathfrak{A}$. It is clear that $\mathscr{J}$ and $\mathscr{I}$ are both $\mathfrak{A}$-submodules and $\mathscr{A}$-submodules of $\mathscr{A}$ and $\mathscr{A}\widehat{\otimes}\mathscr{A}$, respectively. Hence, the module projective tensor product $\mathscr{A}\widehat{\otimes}_{\mathfrak{A}}\mathscr{A}\cong(\mathscr{A}\widehat{\otimes}\mathscr{A})/\mathscr{I}$ [9] and the quotient Banach algebra $\mathscr{A}/\mathscr{J}$ are both Banach $\mathfrak{A}$-modules and Banach $\mathscr{A}$-modules. Define $\omega:\mathscr{A}\widehat{\otimes}\mathscr{A}\to\mathscr{A}$ by $\omega(a\otimes b)=ab$ and $\widetilde{\omega}:\mathscr{A}\widehat{\otimes}_{\mathfrak{A}}\mathscr{A}\to\mathscr{A}/\mathscr{J}$ by $\widetilde{\omega}(a\otimes b+\mathscr{I})=\omega(a\otimes b)+\mathscr{J}$, extended by linearity and continuity. Clearly, $\widetilde{\omega}$ is an $\mathscr{A}$-module homomorphism and an $\mathfrak{A}$-module homomorphism.

Suppose that $\varphi\in\mathrm{Hom}_{\mathfrak{A}}(\mathscr{A})$ and I is a closed ideal of $\mathscr{A}$ such that $\varphi(I)\subseteq I$. Then we may define $\varphi_I:\mathscr{A}/I\to\mathscr{A}/I$ by $\varphi_I(a+I)=\varphi(a)+I$. In particular, for all $a,b\in\mathscr{A}$ and $\alpha\in\mathfrak{A}$

$$\begin{aligned}&\varphi((a\cdot\alpha)b-a(\alpha\cdot b))\\ &\quad=(\varphi(a)\cdot\alpha)\varphi(b)-\varphi(a)(\alpha\cdot\varphi(b))\in\mathscr{J};\end{aligned}\tag{26}$$

that is, $\varphi(\mathscr{J})\subseteq\mathscr{J}$. Therefore, we can define $\varphi_{\mathscr{J}}:\mathscr{A}/\mathscr{J}\to\mathscr{A}/\mathscr{J}$.

In the remainder of this section, we use $\overline{a}$ to denote the coset of $a\in\mathscr{A}$ in $\mathscr{A}/\mathscr{J}$.

Lemma 8. *$\mathscr{A}$ is φ-module amenable if and only if $\mathscr{A}/\mathscr{J}$ is $\varphi_{\mathscr{J}}$-module amenable.*

Proof. Let $\mathscr{A}/\mathscr{J}$ be $\varphi_{\mathscr{J}}$-module amenable. Suppose that X is a commutative Banach $\mathscr{A}$-$\mathfrak{A}$-module and $D:\mathscr{A}\to X^*$ is a φ-module derivation. Clearly $\mathscr{J}\cdot X=X\cdot\mathscr{J}=0$, so X is

a commutative Banach $\mathscr{A}/\mathscr{J}$-$\mathfrak{A}$-module, by the same actions of $\mathfrak{A}$ and $\overline{a}\bullet x = a\cdot x$ and $x\bullet\overline{a} = x\cdot a$ $(a \in \mathscr{A})$. For $a, b \in \mathscr{A}$ and $\alpha \in \mathfrak{A}$, we have $D((a\cdot\alpha)b) = D(a(\alpha\cdot b))$. Hence, D vanishes on $\mathscr{J}$ and induces a map $\widetilde{D}$ from $\mathscr{A}/\mathscr{J}$ into X^* which is clearly a $\varphi_{\mathscr{J}}$-module derivation. Hence $\widetilde{D} = ad_f^{\varphi_{\mathscr{J}}}$, for some f in X^*. Thus,

$$\begin{aligned} D(a) &= \widetilde{D}(\overline{a}), \\ &= \overline{\varphi(a)}\bullet f - f\bullet\overline{\varphi(a)}, \\ &= \varphi(a)\cdot f - f\cdot\varphi(a) \quad (a\in\mathscr{A}). \end{aligned} \tag{27}$$

Consequently, $D = ad_f^{\varphi}$.

The converse follows from Proposition 2.5 in [4]. □

Now, we define the concepts of φ-module virtual diagonal and φ-module approximate diagonal as a generalization of the earlier notions of virtual diagonal and approximate diagonal

Definition 9. Let $\varphi \in \mathrm{Hom}_{\mathfrak{A}}(\mathscr{A})$.

(i) An element $\mathbf{M} \in (\mathscr{A}\widehat{\otimes}_{\mathfrak{A}}\mathscr{A})^{**}$ is called a φ-module virtual diagonal for $\mathscr{A}$ if

$$\varphi(a)\cdot\mathbf{M} = \mathbf{M}\cdot\varphi(a), \qquad \varphi(a)\cdot\widetilde{\omega}^{**}\mathbf{M} = \varphi(a) + \mathscr{J}^{\perp\perp} \quad (a\in\mathscr{A}). \tag{28}$$

(ii) A bounded net $(\mathbf{m}_j)_j$ in $\mathscr{A}\widehat{\otimes}_{\mathfrak{A}}\mathscr{A}$ is called a φ-module approximate diagonal for $\mathscr{A}$ if $(\widetilde{\omega}\mathbf{m}_j)_j$ is a bounded approximate identity for $\varphi(\mathscr{A})/\mathscr{J}$ and

$$\varphi(a)\cdot\mathbf{m}_j - \mathbf{m}_j\cdot\varphi(a) \longrightarrow 0 \quad (a\in\mathscr{A}). \tag{29}$$

We note that, if φ is the identity map, then $\mathrm{id}_{\mathscr{A}}$-module virtual (or approximate) diagonal is the same as module virtual (or approximate) diagonal [6]. Moreover, in the case where $\mathfrak{A} := \mathbb{C}$, $\mathrm{id}_{\mathscr{A}}$-module virtual (or approximate) diagonal and virtual (or approximate) diagonal coincide.

The next proposition follows from Corollary 2.3 of [4] and Theorem 2.1 of [3].

Proposition 10. *If $\mathscr{A}$ has a φ-module approximate diagonal such that φ is surjective, then $\mathscr{A}$ is φ-module amenable.*

Proposition 11. *Let $\mathscr{A}\widehat{\otimes}_{\mathfrak{A}}\mathscr{A}$ be a commutative Banach $\mathscr{A}$-$\mathfrak{A}$-module. If $\mathscr{A}$ is φ-module amenable and $\mathscr{A}/\mathscr{J}$ has a bounded approximate identity, then $\mathscr{A}$ has a φ-module virtual diagonal.*

Proof. Let $(\overline{e_j})_j$ be a bounded approximate identity for $\mathscr{A}/\mathscr{J}$ and let $E + \mathscr{I}^{\perp\perp}$ in $(\mathscr{A}\widehat{\otimes}\mathscr{A})^{**}/\mathscr{I}^{\perp\perp} \cong (\mathscr{A}\widehat{\otimes}_{\mathfrak{A}}\mathscr{A})^{**}$ be a w^*-accumulation point of $(e_j\otimes e_j + \mathscr{I})_j$. Hence,

$$\widetilde{\omega}^{**}\left(\varphi(a)\cdot E + \mathscr{I}^{\perp\perp} - E\cdot\varphi(a) + \mathscr{I}^{\perp\perp}\right) = 0_{(\mathscr{A}/\mathscr{J})^{**}} \quad (a\in\mathscr{A}). \tag{30}$$

Thus, $ad^{\varphi}_{E+\mathscr{I}^{\perp\perp}} : \mathscr{A} \to (\mathscr{A}\widehat{\otimes}_{\mathfrak{A}}\mathscr{A})^{**}$ is a φ-module derivation into $\ker\widetilde{\omega}^{**} \cong (\ker\widetilde{\omega})^{**}$. Since $\mathscr{A}\widehat{\otimes}_{\mathfrak{A}}\mathscr{A}$ is a commutative Banach $\mathscr{A}$-$\mathfrak{A}$-module, so is $\ker\widetilde{\omega}$. By φ-module amenability of $\mathscr{A}$, there is $N + \mathscr{I}^{\perp\perp} \in (\ker\widetilde{\omega})^{**}$ such that $ad^{\varphi}_{E+\mathscr{I}^{\perp\perp}} = ad^{\varphi}_{N+\mathscr{I}^{\perp\perp}}$. Consequently, it is clear that $(E - N) + \mathscr{I}^{\perp\perp}$ is a φ-module virtual diagonal for $\mathscr{A}$. □

Lemma 12. *$\mathscr{A}$ has a φ-module virtual diagonal if and only if it has a φ-module approximate diagonal.*

Proof. This is essentially the same as the proof of Lemma 2.9.64 of [10]. □

In Proposition 2.1 of [3] and Proposition 3.3 of [6], the authors proved that module amenability of $\mathscr{A}$ follows from amenability of $\mathscr{A}$ and $\mathscr{A}/\mathscr{J}$, respectively, under the strong condition that $\mathfrak{A}$ has a bounded approximate identity for $\mathscr{A}$. According to Lemma 8, we present the generalization of Proposition 2.1 of [3] and Proposition 3.3 of [6] without the extra assumption that $\mathfrak{A}$ has a bounded approximate identity for $\mathscr{A}$. Indeed, (as an application of the following theorem) we show that the class of amenable Banach algebras is contained in the class of module amenable Banach algebras.

Theorem 13. *Let $\mathscr{A}$ be a Banach $\mathfrak{A}$-module and let $\varphi \in Hom_{\mathfrak{A}}(\mathscr{A})$ be linear. Then φ-module amenability of $\mathscr{A}$ follows from its φ-amenability, when one of the following is satisfied:*

(i) *φ is surjective.*

(ii) *φ is an idempotent and $\mathscr{A}$ is unital.*

Proof. (i) Since φ is linear, φ-amenability of $\mathscr{A}$ implies that $\mathscr{A}$ is φ-module amenable as a commutative Banach $\mathbb{C}$-module. Also, automatically $\mathscr{A}\widehat{\otimes}_{\mathbb{C}}\mathscr{A} = \mathscr{A}\widehat{\otimes}\mathscr{A}$ is a commutative Banach $\mathscr{A}$-$\mathbb{C}$-module. Therefore from Propositions 4 and 11, and Lemma 12, there is a bounded net $(\mathbf{m}_j)_j$ in $\mathscr{A}\widehat{\otimes}\mathscr{A}$ such that $(\omega\mathbf{m}_j)_j$ is a bounded approximate identity for $\varphi(\mathscr{A})$ and $\varphi(a)\cdot\mathbf{m}_j - \mathbf{m}_j\cdot\varphi(a) \to 0$, for all $a \in \mathscr{A}$. Now define $(\mathbf{m}'_j)_j$ in $\mathscr{A}\widehat{\otimes}_{\mathfrak{A}}\mathscr{A}$ by $\mathbf{m}'_j = \mathbf{m}_j + \mathscr{I}$. Then it is clear that $(\mathbf{m}'_j)_j$ is a φ-module approximate diagonal for $\mathscr{A}$. Consequently $\mathscr{A}$ is φ-module amenable, by Proposition 10.

(ii) Let X be a commutative Banach $\mathscr{A}$-$\mathfrak{A}$-module which is φ-pseudo-unital and let $D : \mathscr{A} \to X^*$ be a φ-module derivation. Let e be an identity for $\mathscr{A}$. Clearly $D(e)$ is zero and for $n \in \mathbb{N}$, additivity of D implies that $nD((1/n)e) = D(e) = 0$. Thus, $(re) = 0$ $(r \in \mathbb{Q})$. Hence, by continuity of D, we have $D(re) = 0$ $(r \in \mathbb{R})$. Moreover,

$$\begin{aligned} 0 = D(-e) = D\left(i^2 e\right) &= \varphi(ie)\cdot D(ie) + D(ie)\cdot\varphi(ie) \\ &= 2iD(ie). \end{aligned} \tag{31}$$

Thus, $D(ie) = 0$ and therefore $D(\lambda e) = 0$ $(\lambda \in \mathbb{C})$. Now, it is routinely checked that D is linear. Consequently D is φ-module inner, by φ-amenability of $\mathscr{A}$. □

Lemma 14. *Let I be a closed ideal and an $\mathfrak{A}$-submodule of $\mathscr{A}$ such that $\varphi(I) \subseteq I$. If I is $\varphi|_I$-module amenable and $\mathscr{A}/I$ is φ_I-module amenable, then $\mathscr{A}$ is φ-module amenable.*

Proof. Let X be a commutative Banach $\mathscr{A}$-$\mathfrak{A}$-module. Suppose that E is the space of all elements $\psi \in X^*$ such that

$\psi \cdot \varphi(I) = \varphi(I) \cdot \psi = 0$ and F is the subspace of X generated by $\varphi(I) \cdot X + X \cdot \varphi(I)$. Since $\varphi(I) \cdot (X/F) = (X/F) \cdot \varphi(I) = 0$, the following module actions are well defined

$$\alpha \cdot (x + F) := \alpha \cdot x + F, \qquad (a + I) \cdot (x + F) := \varphi(a) \cdot x + F \qquad (\alpha \in \mathfrak{A}, a \in \mathscr{A}, x \in X), \tag{32}$$

and similar for the right actions. Therefore, X/F is a commutative Banach $\mathscr{A}/I$-$\mathfrak{A}$-module and so is $E \cong (X/F)^*$.

Now, let $D : \mathscr{A} \to X^*$ be a φ-module derivation. Consider $f \in X^*$ such that $D|_I = ad_f^{\varphi}$ and let $\widetilde{D} := D - ad_f^{\varphi}$. Since $\widetilde{D}$ vanishes on I, so it induces a φ-module derivation from $\mathscr{A}/I$ to X^*, which we denote likewise by $\widetilde{D}$. Also, for all $a \in \mathscr{A}$ and $b \in I$ we obtain that $\varphi(b) \cdot \widetilde{D}(a) = \widetilde{D}(a) \cdot \varphi(b) = 0$. Hence, $\widetilde{D}(\mathscr{A}/I) \subseteq E$. Therefore, $\widetilde{\varphi}$-module amenability of $\mathscr{A}/I$ implies that $\widetilde{D} = ad_g^{\varphi_I}$ for some $g \in E$. Consequently, $D = ad_{f+g}^{\varphi}$. □

Now we are ready to prove the main result of this section. In Theorem 13 we obtained sufficient conditions that φ-amenability of $\mathscr{A}$ implies φ-module amenability of $\mathscr{A}$. The next corollary together with Theorem 7 may be considered as the converse of Theorem 13.

Theorem 15. *Let $\mathscr{A}$ be a commutative Banach $\mathfrak{A}$-module and let $\varphi \in \mathrm{Hom}_{\mathfrak{A}}(\mathscr{A})$ be an epimorphism. Then φ-module amenability of $\mathscr{A}$ implies its φ-amenability, when $\mathfrak{A}$ is commutative and amenable.*

Proof. First we suppose that $\mathfrak{A}$ has an identity e for itself and $\mathscr{A}$. Consider $\mathscr{B} = \mathscr{A} \oplus \mathfrak{A}$ with the following multiplication:

$$(a + \alpha) \cdot (b + \beta) := ab + \alpha \cdot b + \beta \cdot a + \alpha\beta \qquad (\alpha, \beta \in \mathfrak{A}, a, b \in \mathscr{A}). \tag{33}$$

It is straightforward that $\mathscr{B}$ is a unital Banach algebra with the norm algebra $\|a + \alpha\| := \|a\| + \|\alpha\|$ and $\mathscr{A}$ is a closed ideal of $\mathscr{B}$. Also, $\mathscr{B}$ is a commutative Banach $\mathfrak{A}$-module with the following compatible actions:

$$\gamma \cdot (a + \alpha) = (a + \alpha) \cdot \gamma := \gamma \cdot a + \gamma\alpha \quad (\alpha, \gamma \in \mathfrak{A}, a \in \mathscr{A}). \tag{34}$$

Define

$$\psi : \mathscr{B} \longrightarrow \mathscr{B}, \quad \psi(a + \alpha) := \varphi(a) + \alpha \quad (a \in \mathscr{A}, \alpha \in \mathfrak{A}); \tag{35}$$

then $\psi \in \mathrm{Hom}_{\mathfrak{A}}(\mathscr{B})$ such that $\psi|_{\mathscr{A}} = \varphi$. Since $\mathscr{B}/\mathscr{A} \cong \mathfrak{A}$ is amenable, it is $\psi_{\mathscr{A}}$-amenable and $\psi_{\mathscr{A}}$-module amenability of $\mathscr{B}/\mathscr{A}$ follows from Theorem 13. Therefore $\mathscr{B}$ is ψ-module amenable, by Lemma 14. Hence, Theorem 7 implies that $\mathscr{B}$ is ψ-amenable. Now, by Proposition 3.1 of [11] and Proposition 4, we obtain that $\mathscr{A}$ is φ-amenable.

In the case $\mathfrak{A}$ is not unital we consider $\mathfrak{A}^\sharp$ as the unitization of $\mathfrak{A}$. We also define the compatible actions of $\mathfrak{A}^\sharp$ on $\mathscr{A}$ that extend the compatible actions of $\mathfrak{A}$ on $\mathscr{A}$, by letting

$$(\alpha + \lambda e) \cdot a = a \cdot (\alpha + \lambda e) := \alpha \cdot a + \lambda a \qquad (\alpha \in \mathfrak{A}, \lambda \in \mathbb{C}, a \in \mathscr{A}). \tag{36}$$

Then $\mathscr{A}$ is a commutative Banach $\mathfrak{A}^\sharp$-module and e is an identity for the actions on $\mathscr{A}$. Also, $\varphi \in \mathrm{Hom}_{\mathfrak{A}^\sharp}(\mathscr{A})$. Since $\mathfrak{A} \subseteq \mathfrak{A}^\sharp$, any φ-module derivation on $\mathscr{A}$ where $\mathscr{A}$ is a Banach $\mathfrak{A}^\sharp$-module is a φ-module derivation on $\mathscr{A}$ where $\mathscr{A}$ is a Banach $\mathfrak{A}$-module. Therefore, if $\mathscr{A}$ is φ-module amenable as a Banach $\mathfrak{A}$-module, then it is φ-module amenable as a Banach $\mathfrak{A}^\sharp$-module. Consequently $\mathscr{A}$ is φ-amenable, by the first case. □

Let $S = (\mathbb{N}, \vee)$ be the inverse semigroup of positive integers with maximum operation. Then $\mathscr{A} := l^1(S)$ is not amenable, by Theorem 2 of [12]. On the other hand, as in the proof of the last example of [13], we obtain that $\mathscr{A}$ is module amenable on itself by the multiplication algebra. Consequently, Theorem 7 and Theorem 15 are not valid, when $\mathfrak{A}$ is not amenable.

4. Semigroup Algebras

Recall that a discrete semigroup S is called an *inverse semigroup* if for each $s \in S$ there is a unique element $s^* \in S$ such that $s^*ss^* = s^*$ and $ss^*s = s$. Elements of the form ss^* are called idempotents of S and form a commutative subsemigroup E. An inverse semigroup whose idempotents are in the center is called a *Clifford semigroup*.

The Banach algebra $l^1(E)$ could be regarded as a subalgebra of $l^1(S)$ (see [14]) and thereby $l^1(S)$ is a Banach algebra and a Banach $l^1(E)$-module with proper compatible actions. It is possible to consider arbitrary actions of $l^1(E)$ on $l^1(S)$ and prove certain module amenability results. Here we do not restrict ourselves to any particular action.

In the following theorem, we generalize the well-known result of Ghahramani et al. (in the case discrete), which assert that G is finite if and only if $l^1(G)^{**}$ is amenable, when G is a locally compact group.

Theorem 16. *Let S be an inverse semigroup with set of idempotents E, let $l^1(S)$ be a commutative Banach $l^1(E)$-module, and let $\varphi \in \mathrm{Hom}_{l^1(E)}(l^1(S))$ be an epimorphism. Assume that $l^1(E)$ is amenable as a Banach algebra. Then $l^1(S)^{**}$ is φ^{**}-module amenable if and only if S is finite.*

Proof. Since $l^1(S)$ is w^*-dense in $l^1(S)^{**}$ and the compatible actions are w^*-continuous, $l^1(S)^{**}$ is a commutative Banach $l^1(E)$-module and $\varphi^{**} \in \mathrm{Hom}_{l^1(E)}(l^1(S)^{**})$ is an epimorphism. If $l^1(S)^{**}$ is φ^{**}-module amenable, then φ^{**}-amenability of $l^1(S)^{**}$ follows from Theorem 15 and its amenability follows from Proposition 2.3 of [8]. Therefore S is finite, by Theorem 11.8 of [15].

Conversely, if S is finite, then $l^1(S) \cong l^1(S)^{**}$ is amenable and so it is φ^{**}-amenable. Consequently, φ^{**}-module amenability of $l^1(S)^{**}$ follows from Theorem 13. □

In the main results of [5, 6] (see Theorem 3.4 and Theorem 2.11, resp.), the authors studied the module amenability of $l^1(S)^{**}$, when $l^1(S)$ is a Banach $l^1(E)$-module with very specific compatible actions. Also, in [13] we studied the super module amenability of $l^1(S)^{**}$, when $l^1(S)$ is a commutative Banach $l^1(E)$-module with some commutative compatible actions that $l^1(S)$ is pseudo-unital (see Corollary 3.5 of [13]). Now, in the following corollary, we investigate the module amenability of $l^1(S)^{**}$, when $l^1(S)$ is a commutative Banach $l^1(E)$-module with arbitrary commutative compatible actions.

Corollary 17. *Let S be an inverse semigroup with finitely many idempotents. If $l^1(S)$ is a commutative Banach $l^1(E)$-module, then $l^1(S)^{**}$ is module amenable if and only if S is finite.*

Proof. This is immediate from Theorem 16, when φ is the identity map on $l^1(S)$. □

Let S be a Clifford semigroup. Given $k \in \mathbb{C}$, consider the following commutative compatible actions:

$$\delta_e \cdot \delta_s = \delta_s \cdot \delta_e := k\delta_{es}, \tag{37}$$

or

$$\delta_e \cdot \delta_s = \delta_s \cdot \delta_e := k\delta_{es} - k\delta_s \quad (e \in E, s \in S). \tag{38}$$

Consequently, there are large extra numbers of commutative compatible actions that turn $l^1(S)$ into a commutative Banach $l^1(E)$-module (note that, with the above second actions, $l^1(S)$ is not necessary pseudo-unital. For instance, if we let S be a discrete group, then the second actions above are zero).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

Massoud Amini was partly supported by a Grant from IPM (no. 90430215).

References

[1] B. E. Johnson, *Cohomology in Banach Algebras*, Memoirs of the American Mathematical Society, no. 127, American Mathematical Society, Providence, RI, USA, 1972.

[2] F. Ghahramani, R. J. Loy, and G. A. Willis, "Amenability and weak amenability of second conjugate Banach algebras," *Proceedings of the American Mathematical Society*, vol. 124, no. 5, pp. 1489–1497, 1996.

[3] M. Amini, "Module amenability for semigroup algebras," *Semigroup Forum*, vol. 69, no. 2, pp. 243–254, 2004.

[4] A. Bodaghi, "Module (φ, ψ)-amenability of Banach algebras," *Archivum Mathematicum*, vol. 46, no. 4, pp. 227–235, 2010.

[5] H. Pourmahmood-Aghababa, "(Super) module amenability, module topological centre and semigroup algebras," *Semigroup Forum*, vol. 81, no. 2, pp. 344–356, 2010.

[6] M. Amini, A. Bodaghi, and D. E. Bagha, "Module amenability of the second dual and module topological center of semigroup algebras," *Semigroup Forum*, vol. 80, no. 2, pp. 302–312, 2010.

[7] V. Runde, *Lectures on Amenability*, vol. 1774 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2002.

[8] M. S. Moslehian and A. N. Motlagh, "Some notes on (σ, τ)-amenability of Banach algebras," *Studia Universitatis Babeş-Bolyai Mathematica*, vol. 53, no. 3, pp. 57–68, 2008.

[9] M. A. Rieffel, "Induced Banach representations of Banach algebras and locally compact groups," *Journal of Functional Analysis*, vol. 1, no. 4, pp. 443–491, 1967.

[10] H. G. Dales, *Banach Algebras and Automatic Continuity*, Oxford University Press, Oxford, UK, 2000.

[11] Z. Ghorbani and M. L. Bami, "φ-approximate biflat and φ-amenable Banach algebras," *Proceedings of the Romanian Academy A: Mathematics, Physics, Technical Sciences, Information Science*, vol. 13, no. 1, pp. 3–10, 2012.

[12] J. Duncan and A. L. T. Paterson, "Amenability for discrete convolution semigroup algebras," *Mathematica Scandinavica*, vol. 66, no. 1, pp. 141–146, 1990.

[13] M. L. Bami, M. Valaei, and M. Amini, "Super module amenability of inverse semigroup algebras," *Semigroup Forum*, vol. 86, no. 2, pp. 279–288, 2013.

[14] J. M. Howie, *An Introduction to Semigroup Theory*, Academic Press, London, UK, 1976.

[15] H. G. Dales, A. T.-M. Lau, and D. Strauss, "Banach algebras on semigroups and on their compactifications," *Memoirs of the American Mathematical Society*, vol. 205, no. 966, pp. 1–171, 2010.