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Research Article

Existence of Positive Solutions for a Nonlinear Higher-Order Multipoint Boundary Value Problem

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We study the existence of positive solutions for a nonlinear higher-order multipoint boundary value problem. By applying a monotone iterative method, some existence results of positive solutions are obtained. The main result is illustrated with an example.

1. Introduction

We consider the following nonlinear higher-order differential equation:

$$u^{(n)}(t) + q(t) f\left(t, u(t), u'(t), \ldots, u^{(p)}(t)\right) = 0, \quad t \in (0,1), \tag{1}$$

with the multipoint boundary conditions

$$\begin{gathered} u^{(i)}(0) = 0, \quad 0 \leqslant i \leqslant n-2, \\ u^{(p)}(1) = \sum_{i=1}^{m} k_i u^{(p)}(\eta_i), \\ (1 \leqslant p \leqslant n-2, \text{ but fixed}). \end{gathered} \tag{2}$$

Throughout this paper, we assume that the following conditions are satisfied:

(H1) $m \geqslant 1$, $n \geqslant 3$, $p \in \{1, 2, \ldots, n-2\}$ are fixed integers, $k_i \geqslant 0$, $0 < \eta_i < 1$ $(1 \leqslant i \leqslant m)$ with $\Theta := \sum_{i=1}^{m} k_i \eta_i^{n-p-1} < 1$;

(H2) $q \in L^1[0,1]$ is nonnegative and $0 < \int_0^1 (1-s)^{n-p-1} q(s) ds < \infty$;

(H3) $f : [0,1] \times [0,\infty)^{p+1} \to [0,\infty)$ is continuous.

In this paper, by a positive solution u^* of problems (1) and (2), we mean a function u^* satisfying the differential equation (1) and the boundary conditions (2) with $u^*(t) > 0$ for all $t \in (0,1]$.

The multipoint boundary value problems for ordinary differential equations arise in a variety of different areas of applied mathematics and physics. In recent years, the existence and multiplicity of solutions of nonlinear higher-order differential equations with various multipoint boundary conditions have been studied extensively by numerous researchers using a variety of methods and techniques. For example, Graef and Yang [1] studied a higher-order multipoint boundary value problem

$$\begin{gathered} u^{(n)}(t) + \lambda g(t) f(u(t)) = 0, \quad t \in (0,1), \\ u^{(i)}(0) = 0, \quad 0 \leqslant i \leqslant n-2, \\ u^{(n-2)}(1) = \sum_{i=1}^{m} a_i u^{(n-2)}(\xi_i), \end{gathered} \tag{3}$$

where $n \geqslant 3$ and $m \geqslant 1$ are integers, $a_i > 0$ for $1 \leqslant i \leqslant m$, and $\sum_{i=1}^{m} a_i = 1$, $1/2 \leqslant \xi_1 < \xi_2 < \cdots < \xi_m < 1$, $\lambda > 0$, is a parameter. Some existence and nonexistence results of positive solutions were obtained by using Krasnosel'skii's fixed point theorem. In [2], by applying fixed point index theory, Pang et al. studied the expression and properties

of Green's function and obtained the existence of positive solutions for nth-order m-point boundary value problems

$$\begin{gathered} u^{(n)}(t)+f(t,u(t))=0, \quad t\in(0,1), \\ u^{(i)}(0)=0, \quad 0\leqslant i\leqslant n-2, \\ u(1)=\sum_{i=1}^{m-2}k_i u(\xi_i), \end{gathered} \tag{4}$$

where $n\geqslant 2$, $k_i>0$ $(i=1,2,\ldots,m-2)$, $0<\xi_1<\xi_2<\cdots<\xi_m<1$ with $0<\sum_{i=1}^{m-2}k_i\xi_i^{n-1}<1$. Guo et al. in [3] imposed growth conditions on the nonlinearity f which yield the existence of multiple positive solutions by using the Leggett-Williams fixed point theorem. Li and Wei in [4] and Yang and Wei in [5] improved and generalized the results of [2] by using different methods. Graef et al. [6] considered an nth-order multipoint boundary value problem

$$\begin{gathered} u^{(n)}+f\left(t,u,u',\ldots,u^{(n-1)}\right)=\lambda p(t), \quad t\in(0,1), \\ u^{(i)}(0)=A_i, \quad i=0,1,\ldots,n-3, \\ u^{(n-2)}(0)-\sum_{j=1}^{m}a_j u^{(n-2)}\left(t_j\right)=A_{n-2}, \\ u^{(n-2)}(1)-\sum_{j=1}^{m}b_j u^{(n-2)}\left(t_j\right)=A_{n-1}, \end{gathered} \tag{5}$$

where $n\geqslant 2$ and $m\geqslant 1$ are integers, $\lambda\in\mathbb{R}$ is a parameter, $f\in C([0,1]\times\mathbb{R}^n)$, $p\in C(0,1)$ with $p(t)>0$ on $[0,1]$, $A_i\in\mathbb{R}$ for $i=0,1,\ldots,n-1$, and $a_j,b_j\in\mathbb{R}^+:=[0,\infty)$ for $j=1,2,\ldots,m$. Sufficient conditions were obtained for the existence of one solution and two solutions of the problem for different values of λ. The analysis mainly relies on the lower and upper solution method and topological degree theory. The results extended and improved some recent work in the literature. In a recent paper [7], we study problems (1) and (2) with $m=1$ by a fixed point theorem of cone expansion and compression of functional type according to Avery et al. [8]. For other existence results of positive solutions for higher-order multipoint problems, for a small sample of such work, we refer the reader to Ahmad and Ntouyas [9], Anderson et al. [10], Davis et al. [11], Du et al. [12, 13], Eloe and Henderson [14], Fu and Du [15], Graef et al. [16, 17], Henderson and Luca [18], Ji and Guo [19], Jiang [20], Liu et al. [21], Liu and Ge [22], Liu et al. [23], Palamides [24], Su and Wang [25], Zhang et al. [26], and Zhang [27] and the references therein.

We noticed that the main tools employed in above-mentioned works are various fixed point theorems, such as Krasnosel'skii, Leggett-Williams, and Avery and Peterson. Recently, the monotone iterative method has been successfully employed to prove the existence of positive solutions of nonlinear boundary value problems for ordinary differential equations. For example, Ma et al. [28] proved the existence of positive solutions of some multipoint p-Laplacian boundary value problems via monotone iterative method. Ma and Yang [29] obtained the existence of positive solutions and established two corresponding iterative schemes for a third-order three-point boundary value problem with increasing homeomorphism and positive homomorphism. Sun and Ge [30] applied monotone iterative procedure to prove the existence of positive pseudosymmetric solutions for a three-point second-order p-Laplacian boundary value problem. Sun et al. [31] proved the existence of positive solutions for some fourth-order two-point boundary value problems via monotone iterative technique. Yao [32, 33] obtained a successively iterative scheme of positive solution of Lidstone boundary value problem and a beam equation with nonhomogeneous boundary condition, respectively. In this paper, we will study the existence and iteration of positive solutions for problems (1) and (2) by using the monotone iterative method. The monotone iterative scheme can be developed into a computational algorithm for numerical solutions.

2. Basic Lemmas

In this section, we present two lemmas, related to the following higher-order differential equation with multipoint boundary conditions:

$$u^{(n)}(t)+h(t)=0, \quad t\in(0,1), \tag{6}$$

$$\begin{gathered} u^{(i)}(0)=0, \quad 0\leqslant i\leqslant n-2, \\ u^{(p)}(1)=\sum_{i=1}^{m}k_i u^{(p)}(\eta_i). \end{gathered} \tag{7}$$

Lemma 1. *Let $h\in C[0,1]$ be a given function; then the solution of problems* (6) *and* (7) *is given by*

$$u(t)=\int_0^1 G(t,s)\,h(s)\,ds, \tag{8}$$

where

$$G(t,s)=G_1(t,s)+G_2(t,s), \tag{9}$$

$$G_1(t,s)=\frac{1}{(n-1)!}\begin{cases} t^{n-1}(1-s)^{n-p-1}-(t-s)^{n-1}, & s\leqslant t, \\ t^{n-1}(1-s)^{n-p-1}, & t\leqslant s, \end{cases} \tag{10}$$

$$G_2(t,s)=\frac{t^{n-1}}{(n-1)!\,(1-\Theta)}\sum_{i=1}^{m}k_i H(\eta_i,s), \tag{11}$$

$$H(\eta_i,s)=\begin{cases} \eta_i^{n-p-1}(1-s)^{n-p-1}-(\eta_i-s)^{n-p-1}, & 0\leqslant s\leqslant\eta_i, \\ \eta_i^{n-p-1}(1-s)^{n-p-1}, & \eta_i\leqslant s\leqslant 1. \end{cases} \tag{12}$$

Proof. The solution of (6) is

$$u(t)=-\frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}h(s)\,ds+At^{n-1}+\sum_{i=0}^{n-2}A_i t^i, \tag{13}$$

for some $A, A_i \in \mathbb{R}$ $(i = 0, 1, 2, \ldots, n-2)$. Noting that the conditions are $u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0$, we obtain $A_0 = A_1 = \cdots = A_{n-2} = 0$. Consequently, the general solution of problems (6) and (7) is

$$u(t) = -\frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} h(s)\,ds + At^{n-1}. \tag{14}$$

Therefore, by (14), we have

$$\begin{aligned} u^{(p)}(t) = &-\frac{1}{(n-p-1)!}\int_0^t (t-s)^{n-p-1} h(s)\,ds \\ &+ \frac{(n-1)!}{(n-p-1)!} At^{n-p-1}, \end{aligned} \tag{15}$$

which implies that

$$\begin{aligned} u^{(p)}(1) = &-\frac{1}{(n-p-1)!}\int_0^1 (1-s)^{n-p-1} h(s)\,ds \\ &+ \frac{(n-1)!}{(n-p-1)!} A, \end{aligned} \tag{16}$$

$$\begin{aligned} u^{(p)}(\eta_i) = &-\frac{1}{(n-p-1)!}\int_0^{\eta_i} (\eta_i - s)^{n-p-1} h(s)\,ds \\ &+ \frac{(n-1)!}{(n-p-1)!} A\eta_i^{n-p-1}. \end{aligned} \tag{17}$$

By the condition $u^{(p)}(1) = \sum_{i=1}^m k_i u^{(p)}(\eta_i)$, (16) and (17), we deduce

$$\begin{aligned} A &= \frac{1}{(n-1)!\,(1-\Theta)}\left[\int_0^1 (1-s)^{n-p-1} h(s)\,ds \right. \\ &\qquad \left. - \sum_{i=1}^m k_i \int_0^{\eta_i} (\eta_i - s)^{n-p-1} h(s)\,ds\right] \\ &= \frac{1}{(n-1)!}\int_0^1 (1-s)^{n-p-1} h(s)\,ds \\ &\quad + \frac{\sum_{i=1}^m k_i \eta_i^{n-p-1}}{(n-1)!\,(1-\Theta)}\int_0^1 (1-s)^{n-p-1} h(s)\,ds \\ &\quad - \frac{1}{(n-1)!\,(1-\Theta)}\sum_{i=1}^m k_i \int_0^{\eta_i} (\eta_i - s)^{n-p-1} h(s)\,ds \\ &= \frac{1}{(n-1)!}\int_0^1 (1-s)^{n-p-1} h(s)\,ds \\ &\quad + \frac{1}{(n-1)!\,(1-\Theta)}\sum_{i=1}^m k_i \int_0^1 H(\eta_i, s)\, h(s)\,ds. \end{aligned} \tag{18}$$

Substituting (18) into (14), we obtain the unique solution $u(t)$ of problems (6) and (7) as

$$\begin{aligned} u(t) = &-\frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} h(s)\,ds \\ &+ \frac{t^{n-1}}{(n-1)!}\int_0^1 (1-s)^{n-p-1} h(s)\,ds \\ &+ \frac{t^{n-1}}{(n-1)!\,(1-\Theta)}\sum_{i=1}^m k_i \int_0^1 H(\eta_i, s)\, h(s)\,ds \\ = &\int_0^1 G_1(t,s)\, h(s)\,ds + \int_0^1 G_2(t,s)\, h(s)\,ds \\ = &\int_0^1 G(t,s)\, h(s)\,ds. \end{aligned} \tag{19}$$

The proof is completed. □

Lemma 2. *Green's function $G(t,s)$ defined by (9) has the following properties:*

(i) *$\partial^j G(t,s)/\partial t^j$ is continuous on $[0,1]\times[0,1]$, $j = 0, 1, 2, \ldots, n-2$;*

(ii) *$0 \leqslant \partial^j G(t,s)/\partial t^j \leqslant (t^{n-j-1}/(n-j-1)!(1-\Theta))(1-s)^{n-p-1}$, for all $t, s \in [0,1]$, $j = 0, 1, 2, \ldots, p$;*

(iii) *$t^{n-1}G(1,s) \leqslant G(t,s) \leqslant G(1,s)$, for all $t, s \in [0,1]$.*

Proof. The statement (i) is obvious. For the proof of the statement (ii), we note that, for all $t, s \in [0,1]$, if $t \leqslant s$, from definition, it is clear that $\partial^j G_1(t,s)/\partial t^j \geqslant 0$ for $j = 0, 1, 2, \ldots, n-1$. If $s \leqslant t$, from (10), we obtain that

$$\begin{aligned} \frac{\partial^j G_1(t,s)}{\partial t^j} &= \frac{1}{(n-j-1)!}\left[t^{n-j-1}(1-s)^{n-p-1} - (t-s)^{n-j-1}\right] \\ &\geqslant \frac{1}{(n-j-1)!}\left[t^{n-j-1}(1-s)^{n-j-1} - (t-s)^{n-j-1}\right] \\ &\geqslant \frac{1}{(n-j-1)!}\left[(t-ts)^{n-j-1} - (t-s)^{n-j-1}\right] \\ &\geqslant 0, \quad j = 0, 1, 2, \ldots, p. \end{aligned} \tag{20}$$

For any $i = 1, 2, \ldots, m$, if $s \geqslant \eta_i$, from (12), it is obvious that $H(\eta_i, s) \geqslant 0$. If $s \leqslant \eta_i$, from (12), we have

$$\begin{aligned} &\eta_i^{n-p-1}(1-s)^{n-p-1} - (\eta_i - s)^{n-p-1} \\ &\quad = (\eta_i - \eta_i s)^{n-p-1} - (\eta_i - s)^{n-p-1} \geqslant 0, \end{aligned} \tag{21}$$

which implies that

$$H(\eta_i, s) \geqslant 0, \quad s \in [0,1],\ i = 1, 2, \ldots, m. \tag{22}$$

By (20) and (22), we get

$$\begin{aligned} \frac{\partial^j G(t,s)}{\partial t^j} &= \frac{\partial^j G_1(t,s)}{\partial t^j} + \frac{\partial^j G_2(t,s)}{\partial t^j} \\ &= \frac{\partial^j G_1(t,s)}{\partial t^j} + \frac{t^{n-j-1}}{(n-j-1)!\,(1-\Theta)}\sum_{i=1}^{m-2} k_i H(\eta_i, s) \\ &\geqslant 0, \quad t, s \in [0,1],\ j = 0, 1, 2, \ldots, p. \end{aligned} \tag{23}$$

On the other hand, by (10) and (12), we find that

$$\frac{\partial^j G_1(t,s)}{\partial t^j} \leqslant \frac{t^{n-j-1}}{(n-j-1)!}(1-s)^{n-p-1},$$
$$t,s \in [0,1], \quad j = 0,1,2,\ldots,n-2,$$
$$H(\eta_i, s) \leqslant \eta_i^{n-p-1}(1-s)^{n-p-1}, \quad s \in [0,1], \ i = 1,2,\ldots,m. \tag{24}$$

Therefore,

$$\begin{aligned}
\frac{\partial^j G(t,s)}{\partial t^j} &= \frac{\partial^j G_1(t,s)}{\partial t^j} + \frac{\partial^j G_2(t,s)}{\partial t^j} \\
&\leqslant \frac{t^{n-j-1}}{(n-j-1)!}(1-s)^{n-p-1} \\
&\quad + \frac{t^{n-j-1}}{(n-j-1)!(1-\Theta)}\sum_{i=1}^{m} k_i H(\eta_i, s) \\
&\leqslant \frac{t^{n-j-1}}{(n-j-1)!}(1-s)^{n-p-1} \\
&\quad + \frac{t^{n-j-1}}{(n-j-1)!(1-\Theta)}\sum_{i=1}^{m} k_i \eta_i^{n-p-1}(1-s)^{n-p-1} \\
&\leqslant \frac{t^{n-j-1}}{(n-j-1)!}(1-s)^{n-p-1} \\
&\quad + \frac{\Theta t^{n-j-1}}{(n-j-1)!(1-\Theta)}(1-s)^{n-p-1} \\
&= \frac{t^{n-j-1}}{(n-j-1)!(1-\Theta)}(1-s)^{n-p-1}, \\
&\quad t,s \in [0,1], \quad j = 0,1,2,\ldots,n-2.
\end{aligned} \tag{25}$$

In view of (23) and (25), we have the assertion.

Now, we prove the statement (iii). In fact, from the statement (ii), we know that $\partial G(t,s)/\partial t \geqslant 0$ for any $t,s \in [0,1]$. Thus, $G(t,s)$ is nondecreasing with respect to t for any $s \in [0,1]$. Consequently,

$$G(t,s) \leqslant G(1,s), \quad \text{for any } (t,s) \in [0,1]\times[0,1]. \tag{26}$$

If $s \leqslant t$, then, from (10), we have

$$\begin{aligned}
G_1(t,s) &= \frac{1}{(n-1)!}\left[t^{n-1}(1-s)^{n-p-1} - (t-s)^{n-1}\right] \\
&= \frac{t^{n-1}}{(n-1)!}\left[(1-s)^{n-p-1} - (1-s)^{n-1}\right] \\
&\quad + \frac{1}{(n-1)!}\left[(t-ts)^{n-1} - (t-s)^{n-1}\right] \\
&\geqslant \frac{t^{n-1}}{(n-1)!}\left[(1-s)^{n-p-1} - (1-s)^{n-1}\right] \\
&= t^{n-1}G_1(1,s).
\end{aligned} \tag{27}$$

Also, if $s \geqslant t$, from (10), we have

$$\begin{aligned}
G_1(t,s) &= \frac{1}{(n-1)!}t^{n-1}(1-s)^{n-p-1} \\
&= \frac{t^{n-1}}{(n-1)!}\left[(1-s)^{n-p-1} - (1-s)^{n-1}\right] \\
&\quad + \frac{1}{(n-1)!}(t-ts)^{n-1} \\
&\geqslant \frac{t^{n-1}}{(n-1)!}\left[(1-s)^{n-p-1} - (1-s)^{n-1}\right] \\
&= t^{n-1}G_1(1,s).
\end{aligned} \tag{28}$$

Thus, from (27) and (28), we obtain

$$G_1(t,s) \geqslant t^{n-1}G_1(1,s), \tag{29}$$

which together with (9) and (11) implies

$$\begin{aligned}
G(t,s) &= G_1(t,s) \\
&\quad + \frac{t^{n-1}}{(n-1)!\left(1-\sum_{i=1}^{m-2} k_i\eta_i^{n-p-1}\right)}\sum_{i=1}^{m} k_i H(\eta_i, s) \\
&\geqslant t^{n-1}G_1(1,s) \\
&\quad + \frac{t^{n-1}}{(n-1)!\left(1-\sum_{i=1}^{m-2} k_i\eta_i^{n-p-1}\right)}\sum_{i=1}^{m} k_i H(\eta_i, s) \\
&= t^{n-1}G(1,s).
\end{aligned} \tag{30}$$

Inequations (26) and (30) show that the statement (iii) is true. Then, the proof is completed. □

3. Main Results

In this section, we consider the existence of positive solutions for problems (1) and (2) by using the monotone iterative method. In the sequel, for any $u \in C[0,1]$, we define $\|u\|_\infty = \max_{0\leqslant t\leqslant 1}|u(t)|$. Let $E = C^p[0,1]$ be a Banach space with the norm

$$\|u\| = \max\left\{\|u\|_\infty, \left\|u'\right\|_\infty, \ldots, \left\|u^{(p)}\right\|_\infty\right\}. \tag{31}$$

We define a cone $P \subset E$ by

$$P = \left\{u \in C^p[0,1] : u(t) \geqslant t^{n-1}\|u\|_\infty, u^{(j)}(t) \geqslant 0, t \in [0,1], \right.$$
$$\left. j = 0,1,2,\ldots,p\right\}, \tag{32}$$

and an integral operator $T : P \to E$ by

$$(Tu)(t) = \int_0^1 G(t,s)\,q(s)\,f\left(s, u(s), u'(s), \ldots, u^{(p)}(s)\right)ds,$$
$$u \in C^p[0,1]. \tag{33}$$

Obviously, the fixed points of T are nonnegative solutions of problems (1) and (2). Applying Ascoli-Arzelà theorem and a standard argument, we can prove that T is completely continuous.

For any $u \in P$, it flows from Lemma 2 (iii) that

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,s)\,q(s)\,f\left(s,u(s),u'(s),\ldots,u^{(p)}(s)\right)ds \\ &\leqslant \int_0^1 G(1,s)\,q(s)\,f\left(s,u(s),u'(s),\ldots,u^{(p)}(s)\right)ds, \\ &\qquad t\in[0,1],\end{aligned} \tag{34}$$

which implies that

$$\|Tu\|_\infty \leqslant \int_0^1 G(1,s)\,q(s)\,f\left(s,u(s),u'(s),\ldots,u^{(p)}(s)\right)ds. \tag{35}$$

On the other hand, by Lemma 2 (iii), we have

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,s)\,q(s)\,f\left(s,u(s),u'(s),\ldots,u^{(p)}(s)\right)ds \\ &\geqslant t^{n-1}\int_0^1 G(1,s)\,q(s) \\ &\qquad \times f\left(s,u(s),u'(s),\ldots,u^{(p)}(s)\right)ds, \\ &\qquad t\in[0,1],\end{aligned} \tag{36}$$

which together with (35) implies

$$(Tu)(t) \geqslant t^{n-1}\|Tu\|_\infty, \quad t\in[0,1]. \tag{37}$$

In addition, it follows from Lemma 2 (ii) that

$$\begin{aligned}(Tu)^{(j)}(t) &= \int_0^1 \frac{\partial^j G(t,s)}{\partial t^j} q(s) \\ &\qquad \times f\left(s,u(s),u'(s),\ldots,u^{(p)}(s)\right)ds \geqslant 0, \\ &\qquad j=0,1,2,\ldots,p.\end{aligned} \tag{38}$$

Therefore, (37) and (38) indicate that $Tu \in P$.

For convenience, we introduce the following notation:

$$\Lambda = \left(\frac{1}{1-\Theta}\int_0^1 (1-s)^{n-p-1}q(s)ds\right)^{-1}. \tag{39}$$

The conditions (H1) and (H2) indicate that Λ is well defined and $\Lambda > 0$.

Theorem 3. *Suppose (H1), (H2), and (H3) hold. Assume that $f(t,0,0,\ldots,0) \not\equiv 0$ and there exists constant $a > 0$, such that*

(H4) $f(t,x_0,x_1,\ldots,x_p) \leqslant f(t,y_0,y_1,\ldots,y_p)$, *for* $0 \leqslant x_j \leqslant y_j \leqslant a$, $t\in[0,1]$, $j=0,1,2,\ldots,p$;

(H5) $\max_{0\leqslant t\leqslant 1} f(t,a,a,\ldots,a) \leqslant \Lambda a$.

Then, problems (1) *and* (2) *possess at least two positive solutions w^* and v^*, such that*

(i) $0 < \|w^*\| \leqslant a$ *and* $\lim_{k\to\infty} T^k w_0 = w^*$, *where* $w_0(t) = at^{n-1}/(n-1)!$, $t\in[0,1]$;

(ii) $0 < \|v^*\| \leqslant a$ *and* $\lim_{k\to\infty} T^k v_0 = v^*$, *where* $v_0(t) = 0$, $t\in[0,1]$.

Proof. We define $P_a = \{u\in P : \|u\| \leqslant a\}$. In what follows, we first prove $T : P_a \to P_a$. In fact, if $u\in P_a$, then $\|u\| \leqslant a$; thus

$$0 \leqslant u^{(j)}(s) \leqslant \|u\| \leqslant a, \quad s\in[0,1],\ j=0,1,2,\ldots,p. \tag{40}$$

By assumptions (H4) and (H5), we have

$$\begin{aligned}0 &\leqslant f\left(s,u(s),u'(s),\ldots,u^{(p)}(s)\right) \leqslant f(s,a,\ldots,a) \\ &\leqslant \max_{0\leqslant s\leqslant 1} f(s,a,\ldots,a) \leqslant \Lambda a, \quad s\in[0,1].\end{aligned} \tag{41}$$

Thus, by the definition of T and Lemma 2 (ii), for $j = 0,1,2,\ldots,p$, we get

$$\begin{aligned}0 &\leqslant (Tu)^{(j)}(t) \\ &= \int_0^1 \frac{\partial^j G(t,s)}{\partial t^j} q(s)\,f\left(s,u(s),u'(s),\ldots,u^{(p)}(s)\right)ds \\ &\leqslant \frac{t^{n-j-1}}{(n-j-1)!\,(1-\Theta)}\int_0^1 (1-s)^{n-p-1}q(s)\,f(s,a,\ldots,a)\,ds \\ &\leqslant \frac{1}{1-\Theta}\Lambda a\int_0^1 (1-s)^{n-p-1}q(s)\,ds = a, \quad t\in[0,1].\end{aligned} \tag{42}$$

Then, (42) shows that $\|Tu\| \leqslant a$; thus we get $T : P_a \to P_a$.

Let $v_0 = 0$; then it is evident that $v_0 \in P_a$. Let $v_{k+1} = Tv_k$ $(k = 0,1,2,\ldots)$. The fact that $T : P_a \to P_a$ implies that $v_k \in T(P_a) \subseteq P_a$ $(k = 1,2,\ldots)$. Since T is completely continuous, we assert that the sequence $\{v_k\}_{k=1}^\infty$ has a convergent subsequence $\{v_{k_i}\}_{i=1}^\infty$ such that $\lim_{i\to\infty} v_{k_i} = v^* \in P_a$.

Since $v_1 = Tv_0 = T0 \in P_a$, we have

$$\begin{aligned}a &\geqslant v_1^{(j)}(t) = \left(Tv_0\right)^{(j)}(t) = (T0)^{(j)}(t) \\ &\geqslant 0 = v_0^{(j)}(t), \quad t\in[0,1],\ j=0,1,2,\ldots,p.\end{aligned} \tag{43}$$

So, by (H4), one has

$$\begin{aligned}v_2^{(j)}(t) &= \left(Tv_1\right)^{(j)}(t) \geqslant \left(Tv_0\right)^{(j)}(t) = v_1^{(j)}(t), \quad t\in[0,1], \\ &\qquad j=0,1,2,\ldots,p.\end{aligned} \tag{44}$$

Thus, by the induction, we have

$$\begin{aligned}v_{k+1}^{(j)}(t) &\geqslant v_k^{(j)}(t), \quad t\in[0,1],\ j=0,1,2,\ldots,p, \\ &\qquad k=0,1,2,\ldots.\end{aligned} \tag{45}$$

Hence, $\lim_{k\to\infty} v_k = v^*$. Applying the continuity of T and taking the limit $k \to \infty$ in $v_{k+1} = Tv_k$, we get $Tv^* = v^*$.

Let $w_0(t) = at^{n-1}/(n-1)!$, $t \in [0,1]$, and $w_{k+1} = Tw_k$ $(k = 0,1,2,\ldots)$. Then, $w_0 \in P_a$. Since $T : P_a \to P_a$, we have $w_k \in T(P_a) \subseteq P_a$ $(k = 1,2,\ldots)$. Since T is completely continuous, we assert that the sequence $\{w_k\}_{k=1}^{\infty}$ has a convergent subsequence $\{w_{k_i}\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty} w_{k_i} = w^* \in P_a$.

Since $w_1 = Tw_0 \in P_a$, by Lemma 2 (ii) and (H4) and (H5), for $j = 0,1,2,\ldots,p$, we have

$$\begin{aligned}
&(Tw_0)^{(j)}(t) \\
&\quad = \int_0^1 \frac{\partial^j G(t,s)}{\partial t^j} q(s) f\left(s, w_0(s), w_0'(s), \ldots, w_0^{(p)}(s)\right) ds \\
&\quad \leqslant \frac{t^{n-j-1}}{(n-j-1)!(1-\Theta)} \\
&\qquad \times \int_0^1 (1-s)^{n-p-1} q(s) f(s,a,\ldots,a)\, ds \\
&\quad \leqslant \frac{w_0^{(j)}(t)}{1-\Theta} \Lambda \int_0^1 (1-s)^{n-p-1} q(s)\, ds = w_0^{(j)}(t), \\
&\qquad\qquad t \in [0,1].
\end{aligned} \tag{46}$$

Thus, we obtain that

$$w_1^{(j)}(t) \leqslant w_0^{(j)}(t), \quad t \in [0,1], \ j = 0,1,2,\ldots,p. \tag{47}$$

So, by Lemma 2 (ii) and (H4), we have

$$\begin{aligned}
&w_2^{(j)}(t) \\
&\quad = (Tw_1)^{(j)}(t) \\
&\quad = \int_0^1 \frac{\partial^j G(t,s)}{\partial t^j} q(s) f\left(s, w_1(s), w_1'(s), \ldots, w_1^{(p)}(s)\right) ds \\
&\quad \leqslant \int_0^1 \frac{\partial^j G(t,s)}{\partial t^j} q(s) f\left(s, w_0(s), w_0'(s), \ldots, w_0^{(p)}(s)\right) ds \\
&\quad = (Tw_0)^{(j)}(t) = w_1^{(j)}(t), \quad t \in [0,1], \ j = 0,1,2,\ldots,p.
\end{aligned} \tag{48}$$

By the induction, we have

$$\begin{aligned}
&w_{k+1}^{(j)}(t) \leqslant w_k^{(j)}(t), \quad t \in [0,1], \ j = 0,1,2,\ldots,p, \\
&\qquad\qquad k = 0,1,2,\ldots.
\end{aligned} \tag{49}$$

Hence, $\lim_{k\to\infty} w_k = w^*$. Applying the continuity of T and $w_{k+1} = Tw_k$, we get $Tw^* = w^*$.

Furthermore, assumption $f(t,0,0,\ldots,0) \not\equiv 0$ implies that the zero function is not a solution of problems (1) and (2); thus $\|w^*\|_\infty > 0$, $\|v^*\|_\infty > 0$. The definition of the cone P follows that we have $w^*(t) \geqslant t^{n-1}\|w^*\|_\infty > 0$, $v^*(t) \geqslant t^{n-1}\|v^*\|_\infty > 0$, $t \in (0,1]$. Thus, w^* and v^* are positive solutions of problems (1) and (2). The proof is completed. □

Remark 4. The iterative sequences in Theorem 3 start off with the zero function and a known simple function, respectively.

Remark 5. We can easily get that w^* and v^* are the maximal and minimal solution of problems (1) and (2) in P_a. Of course, $w^* = v^*$ may happen and then problems (1) and (2) have only one solution in P_a.

4. Example

Example 1. Consider the fourth-order four-point boundary value problem

$$\begin{gathered}
u''''(t) + \frac{2}{9}\left[t + u^2(t) + tu'(t) + u''(t)\right] = 0, \quad t \in (0,1), \\
u(0) = u'(0) = u''(0) = 0, \\
u''(1) = u''\left(\frac{1}{4}\right) + \frac{1}{2}u''\left(\frac{1}{2}\right).
\end{gathered} \tag{50}$$

In this problem, $n = 4$, $m = 2$, $p = 2$, $k_1 = 1$, $k_2 = 1/2$, $\eta_1 = 1/4$, $\eta_2 = 1/2$, $q(t) \equiv 2/9$, and $f(t,x_0,x_1,x_2) = t + x_0^2 + tx_1 + x_2$. It is obvious that (H1)–(H3) hold. By direct calculation, we get

$$\Lambda = \left(\frac{1}{1-\Theta}\int_0^1 (1-s)^{n-p-1} q(s) ds\right)^{-1} = \frac{9}{2}. \tag{51}$$

Choose $a = 2$; then it is easy to check that (H4) and (H5) hold. Thus, all the conditions of Theorem 3 are satisfied. By Theorem 3, problem (50) has two positive solutions v^* and w^*, such that $0 < \|v^*\| \leqslant 2$, $0 < \|w^*\| \leqslant 2$, $\lim_{k\to\infty} v_k = v^*$, and $\lim_{k\to\infty} w_k = w^*$.

The two iterative sequences are as follows:

$$v_0(t) = 0, \quad t \in [0,1],$$

$$\begin{aligned}
&v_{k+1}(t) \\
&\quad = -\frac{1}{27}\int_0^t (t-s)^3 \left[s + v_k^2(s) + sv_k'(s) + v_k''(s)\right] ds \\
&\qquad + \frac{2t^3}{27}\left[\int_0^1 (1-s)\left[s + v_k^2(s) + sv_k'(s) + v_k''(s)\right] ds\right. \\
&\qquad\qquad - \int_0^{1/4}\left(\frac{1}{4} - s\right)\left[s + v_k^2(s) + sv_k'(s) + v_k''(s)\right] ds \\
&\qquad\qquad - \frac{1}{2}\int_0^{1/2}\left(\frac{1}{2} - s\right) \\
&\qquad\qquad\qquad \left.\times \left[s + v_k^2(s) + sv_k'(s) + v_k''(s)\right] ds\right], \\
&\qquad\qquad\qquad t \in [0,1], \quad k = 0,1,2,\ldots,
\end{aligned}$$

$$w_0(t) = \frac{1}{3}t^3, \quad t \in [0,1],$$

$$
\begin{aligned}
&w_{k+1}(t) \\
&= -\frac{1}{27}\int_0^t (t-s)^3\left[s + w_k^2(s) + sw_k'(s) + w_k''(s)\right]ds \\
&\quad + \frac{2t^3}{27}\left[\int_0^1 (1-s)\left[s + v_k^2(s) + sv_k'(s) + v_k''(s)\right]ds\right. \\
&\qquad - \int_0^{1/4}\left(\frac{1}{4}-s\right) \\
&\qquad\quad \times\left[s + w_k^2(s) + sw_k'(s) + w_k''(s)\right]ds \\
&\qquad - \frac{1}{2}\int_0^{1/2}\left(\frac{1}{2}-s\right) \\
&\qquad\quad \left.\times\left[s + w_k^2(s) + sw_k'(s) + w_k''(s)\right]ds\right], \\
&\qquad\qquad t\in[0,1], \quad k = 0,1,2,\ldots.
\end{aligned}
\tag{52}
$$

The second and third terms of the two schemes are as follows:

$$
\begin{aligned}
v_1(t) &= \frac{59}{5184}t^3 - \frac{1}{540}t^5, \\
v_2(t) &= \frac{59}{5184}t^3 - \frac{1}{540}t^5 + \frac{45253}{11609505792}t^9 \\
&\quad - \frac{3481}{2902376448}t^{10} - \frac{767}{604661760}t^{11} \\
&\quad + \frac{59}{151165440}t^{12} + \frac{13}{125971200}t^{13} \\
&\quad - \frac{1}{31492800}t^{14}, \\
w_1(t) &= \frac{84571643}{2229534720}t^3 - \frac{1}{180}t^5 - \frac{1}{3780}t^7 - \frac{1}{204120}t^{10}, \\
w_2(t) &= \frac{59}{5184}t^3 - \frac{1}{540}t^5 + \frac{92980716396352837}{2147396429240126668800}t^9 \\
&\quad - \frac{7152362799719449}{536849107310031667200}t^{10} \\
&\quad - \frac{1099431359}{86684309913600}t^{11} + \frac{84571643}{21671077478400}t^{12} \\
&\quad + \frac{591299137}{1820370508185600}t^{13} - \frac{45484549}{455092627046400}t^{14} \\
&\quad + \frac{13}{146966400}t^{15} - \frac{3774873023}{98300007442022400}t^{16} \\
&\quad + \frac{136328699}{24575001860505600}t^{17} + \frac{11}{11110659840}t^{18} \\
&\quad - \frac{1}{1984046400}t^{19} + \frac{13}{166659897600}t^{20} \\
&\quad - \frac{1}{41664974400}t^{21} + \frac{13}{17999268940800}t^{23} \\
&\quad - \frac{1}{4499817235200}t^{24}.
\end{aligned}
\tag{53}
$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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