Research Article

Best Proximity Point for α - ψ -Proximal Contractive Multimaps

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We extend the notions of α - ψ -proximal contraction and α -proximal admissibility to multivalued maps and then using these notions we obtain some best proximity point theorems for multivalued mappings. Our results extend some recent results by Jleli and those contained therein. Some examples are constructed to show the generality of our results.

1. Introduction and Preliminaries

Samet et al. [1] introduced the notion of α - ψ -contractive type mappings and proved some fixed point theorems for such mappings in the frame work of complete metric spaces. Karapınar and Samet [2] generalized α - ψ -contractive type mappings and obtained some fixed point theorems for generalized α - ψ -contractive type mapping. Some interesting multivalued generalizations of α - ψ -contractive type mappings are available in [3–12]. Recently, Jleli and Samet [13] introduced the notion of α - ψ -proximal contractive type mappings and proved some best proximity point theorems. Many authors obtained best proximity point theorems in different setting; see, for example, [13–35]. Abkar and Gbeleh [16] and Al-Thagafi and Shahzad [18, 20] investigated best proximity points for multivalued mappings. The purpose of this paper is to extend the results of Jleli and Samet [13] for nonself multivalued mappings. To demonstrate generality of our main result we have constructed some examples.

Let (X, d) be a metric space. For $A, B \subset X$, we use the following notations: $\text{dist}(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$, $D(x, B) = \inf\{d(x, b) : b \in B\}$, $A_0 = \{a \in A : d(a, b) = \text{dist}(A, B) \text{ for some } b \in B\}$, $B_0 = \{b \in B : d(a, b) = \text{dist}(A, B) \text{ for some } a \in A\}$, $2^X \setminus \emptyset$ is the set of all nonempty subsets of X , $CL(X)$ is the set of all nonempty closed subsets

of X , and $K(X)$ is the set of all nonempty compact subsets of X . For every $A, B \in CL(X)$, let

$$H(A, B) = \begin{cases} \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\} & \text{if the maximum exists;} \\ \infty & \text{otherwise.} \end{cases} \quad (1)$$

Such a map H is called the generalized Hausdorff metric induced by d . A point $x^* \in X$ is said to be the best proximity point of a mapping $T : A \rightarrow B$ if $d(x^*, Tx^*) = \text{dist}(A, B)$. When $A = B$, the best proximity point reduces to fixed point of the mapping T .

Definition 1 (see [28]). Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the weak P -property if and only if, for any $x_1, x_2 \in A$ and $y_1, y_2 \in B$,

$$\begin{aligned} d(x_1, y_1) = \text{dist}(A, B) \\ d(x_2, y_2) = \text{dist}(A, B) \end{aligned} \implies d(x_1, x_2) \leq d(y_1, y_2). \quad (2)$$

Example 2. Let $X = \{(0, 1), (1, 0), (0, 3), (3, 0)\}$, endowed with the usual metric d . Let $A = \{(0, 1), (1, 0)\}$ and $B = \{(0, 3), (3, 0)\}$. Then for

$$\begin{aligned} d((0, 1), (0, 3)) &= \text{dist}(A, B), \\ d((1, 0), (3, 0)) &= \text{dist}(A, B), \end{aligned} \quad (3)$$

we have

$$d((0, 1), (1, 0)) < d((0, 3), (3, 0)). \quad (4)$$

Also, $A_0 \neq \emptyset$. Thus, the pair (A, B) satisfies weak P -property.

Definition 3 (see [13]). Let $T : A \rightarrow B$ and $\alpha : A \times A \rightarrow [0, \infty)$. We say that T is an α -proximal admissible if

$$\left. \begin{aligned} \alpha(x_1, x_2) &\geq 1 \\ d(u_1, Tx_1) &= \text{dist}(A, B) \\ d(u_2, Tx_2) &= \text{dist}(A, B) \end{aligned} \right\} \implies \alpha(u_1, u_2) \geq 1, \quad (5)$$

where $x_1, x_2, u_1, u_2 \in A$.

Example 4. Let $X = \mathbb{R} \times \mathbb{R}$, endowed with the usual metric d . Let a be any fixed positive real number, $A = \{(a, y) : y \in \mathbb{R}\}$ and $B = \{(0, y) : y \in \mathbb{R}\}$. Define $T : A \rightarrow B$ by

$$T(a, y) = \begin{cases} (0, \frac{y}{4}) & \text{if } y \geq 0 \\ (0, 4y) & \text{if } y < 0. \end{cases} \quad (6)$$

Define $\alpha : A \times A \rightarrow [0, \infty)$ by

$$\alpha((a, x), (a, y)) = \begin{cases} 2 & \text{if } x, y \geq 0 \\ 0 & \text{otherwise.} \end{cases} \quad (7)$$

Let $w_1 = (a, y_1)$, $w_2 = (a, y_2)$, $w_3 = (a, y_3)$, and $w_4 = (a, y_4)$ be arbitrary points from A satisfying

$$\alpha(w_1, w_2) = 2, \quad (8)$$

$$\begin{aligned} d(w_3, Tw_1) &= a = \text{dist}(A, B), \\ d(w_4, Tw_2) &= a = \text{dist}(A, B). \end{aligned} \quad (9)$$

It follows from (8) that $y_1, y_2 \geq 0$. Further, from (9), $y_3 = y_1/4$ and $y_4 = y_2/4$, which implies that $y_3, y_4 \geq 0$. Hence, $\alpha(w_3, w_4) = 2$. Therefore, T is an α -proximal admissible map.

Let Ψ denote the set of all functions $\psi : [0, \infty) \rightarrow [0, \infty)$ satisfying the following properties:

- (a) ψ is monotone nondecreasing;
- (b) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$.

Definition 5 (see [13]). A nonself mapping $T : A \rightarrow B$ is said to be an α - ψ -proximal contraction, if

$$\alpha(x, y) d(Tx, Ty) \leq \psi(d(x, y)) \quad \forall x, y \in A, \quad (10)$$

where $\alpha : A \times A \rightarrow [0, \infty)$ and $\psi \in \Psi$.

Example 6. Let us consider Example 4 again with $\psi(t) = t/2$ for each $t \geq 0$. Then it is easy to see that, for each $w_1, w_2 \in A$, we have

$$\alpha(w_1, w_2) d(Tw_1, Tw_2) \leq \frac{1}{2} |w_1 - w_2| = \psi(d(w_1, w_2)). \quad (11)$$

Thus, T is an α - ψ -proximal contraction.

The following are main results of [13].

Theorem 7 (see [13], Theorem 3.1). *Let A and B be two nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $\alpha : A \times A \rightarrow [0, \infty)$ and $\psi \in \Psi$. Suppose that $T : A \rightarrow B$ be a mappings satisfying the following conditions:*

- (i) $T(A_0) \subseteq B_0$ and (A, B) satisfies the P -property;
- (ii) T is an α -proximal admissible;
- (iii) there exist elements $x_0, x_1 \in A_0$ such that

$$d(x_1, Tx_0) = \text{dist}(A, B), \quad \alpha(x_0, x_1) \geq 1; \quad (12)$$

- (iv) T is a continuous α - ψ -proximal contraction.

Then there exists an element $x^* \in A_0$ such that $d(x^*, Tx^*) = \text{dist}(A, B)$.

(C) If $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \rightarrow x \in A$ as $n \rightarrow \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all k .

Theorem 8 (see [13], Theorem 3.2). *Let A and B be two nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $\alpha : A \times A \rightarrow [0, \infty)$ and $\psi \in \Psi$. Suppose that $T : A \rightarrow B$ is a mapping satisfying the following conditions:*

- (i) $T(A_0) \subseteq B_0$ and (A, B) satisfies the P -property;
- (ii) T is an α -proximal admissible;
- (iii) there exist elements x_0 and $x_1 \in A_0$ such that

$$d(x_1, Tx_0) = \text{dist}(A, B), \quad \alpha(x_0, x_1) \geq 1; \quad (13)$$

- (iv) property (C) holds and T is an α - ψ -proximal contraction.

Then there exists an element $x^* \in A_0$ such that $d(x^*, Tx^*) = \text{dist}(A, B)$.

Definition 9 (see [16]). An element $x^* \in A$ is said to be the best proximity point of a multivalued nonself mapping T , if $D(x^*, Tx^*) = \text{dist}(A, B)$.

2. Main Result

We start this section by introducing following definition.

Definition 10. Let A and B be two nonempty subsets of a metric space (X, d) . A mapping $T : A \rightarrow 2^B \setminus \emptyset$ is called α -proximal admissible if there exists a mapping $\alpha : A \times A \rightarrow [0, \infty)$ such that

$$\left. \begin{array}{l} \alpha(x_1, x_2) \geq 1 \\ d(u_1, y_1) = \text{dist}(A, B) \\ d(u_2, y_2) = \text{dist}(A, B) \end{array} \right\} \implies \alpha(u_1, u_2) \geq 1, \quad (14)$$

where $x_1, x_2, u_1, u_2 \in A$, $y_1 \in Tx_1$, and $y_2 \in Tx_2$.

Definition 11. Let A and B be two nonempty subsets of a metric space (X, d) . A mapping $T : A \rightarrow CL(B)$ is said to be an α - ψ -proximal contraction, if there exist two functions $\psi \in \Psi$ and $\alpha : A \times A \rightarrow [0, \infty)$ such that

$$\alpha(x, y) H(Tx, Ty) \leq \psi(d(x, y)), \quad \forall x, y \in A. \quad (15)$$

Lemma 12 (see [5]). Let (X, d) be a metric space and $B \in CL(X)$. Then for each $x \in X$ with $d(x, B) > 0$ and $q > 1$, there exists an element $b \in B$ such that

$$d(x, b) < qd(x, B). \quad (16)$$

Now we are in position to state and prove our first result.

Theorem 13. Let A and B be two nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $\alpha : A \times A \rightarrow [0, \infty)$ and $\psi \in \Psi$ be a strictly increasing map. Suppose that $T : A \rightarrow CL(B)$ is a mapping satisfying the following conditions:

- (i) $Tx \subseteq B_0$ for each $x \in A_0$ and (A, B) satisfies the weak P -property;
- (ii) T is an α -proximal admissible;
- (iii) there exist elements $x_0, x_1 \in A_0$ and $y_1 \in Tx_0$ such that

$$d(x_1, y_1) = \text{dist}(A, B), \quad \alpha(x_0, x_1) \geq 1; \quad (17)$$

- (iv) T is a continuous α - ψ -proximal contraction.

Then there exists an element $x^* \in A_0$ such that $D(x^*, Tx^*) = \text{dist}(A, B)$.

Proof. From condition (iii), there exist elements $x_0, x_1 \in A_0$ and $y_1 \in Tx_0$ such that

$$d(x_1, y_1) = \text{dist}(A, B), \quad \alpha(x_0, x_1) \geq 1. \quad (18)$$

Assume that $y_1 \notin Tx_1$; for otherwise x_1 is the best proximity point. From condition (iv), we have

$$\begin{aligned} 0 < d(y_1, Tx_1) &\leq H(Tx_0, Tx_1) \\ &\leq \alpha(x_0, x_1) H(Tx_0, Tx_1) \leq \psi(d(x_0, x_1)). \end{aligned} \quad (19)$$

For $q > 1$, it follows from Lemma 12 that there exists $y_2 \in Tx_1$ such that

$$0 < d(y_1, y_2) < qd(y_1, Tx_1). \quad (20)$$

From (19) and (20), we have

$$0 < d(y_1, y_2) < qd(y_1, Tx_1) \leq q\psi(d(x_0, x_1)). \quad (21)$$

As $y_2 \in Tx_1 \subseteq B_0$, there exists $x_2 \neq x_1 \in A_0$ such that

$$d(x_2, y_2) = \text{dist}(A, B); \quad (22)$$

for otherwise x_1 is the best proximity point. As (A, B) satisfies the weak P -property, from (18) and (22), we have

$$0 < d(x_1, x_2) \leq d(y_1, y_2). \quad (23)$$

From (21) and (23), we have

$$0 < d(x_1, x_2) < qd(y_1, Tx_1) \leq q\psi(d(x_0, x_1)). \quad (24)$$

Since ψ is strictly increasing, we have $\psi(d(x_1, x_2)) < \psi(q\psi(d(x_0, x_1)))$. Put $q_1 = \psi(q\psi(d(x_0, x_1))) / \psi(d(x_1, x_2))$. Also, we have $\alpha(x_0, x_1) \geq 1$, $d(x_1, y_1) = \text{dist}(A, B)$, and $d(x_2, y_2) = \text{dist}(A, B)$. Since T is an α -proximal admissible, then $\alpha(x_1, x_2) \geq 1$. Thus we have

$$d(x_2, y_2) = \text{dist}(A, B), \quad \alpha(x_1, x_2) \geq 1. \quad (25)$$

Assume that $y_2 \notin Tx_2$; for otherwise x_2 is the best proximity point. From condition (iv), we have

$$\begin{aligned} 0 < d(y_2, Tx_2) &\leq H(Tx_1, Tx_2) \\ &\leq \alpha(x_1, x_2) H(Tx_1, Tx_2) \leq \psi(d(x_1, x_2)). \end{aligned} \quad (26)$$

For $q_1 > 1$, it follows from Lemma 12 that there exists $y_3 \in Tx_2$ such that

$$0 < d(y_2, y_3) < q_1 d(y_2, Tx_2). \quad (27)$$

From (26) and (27), we have

$$\begin{aligned} 0 < d(y_2, y_3) &< q_1 d(y_2, Tx_2) \leq q_1 \psi(d(x_1, x_2)) \\ &= \psi(q\psi(d(x_0, x_1))). \end{aligned} \quad (28)$$

As $y_3 \in Tx_2 \subseteq B_0$, there exists $x_3 \neq x_2 \in A_0$ such that

$$d(x_3, y_3) = \text{dist}(A, B); \quad (29)$$

for otherwise x_2 is the best proximity point. As (A, B) satisfies the weak P -property, from (25) and (29), we have

$$0 < d(x_2, x_3) \leq d(y_2, y_3). \quad (30)$$

From (28) and (30), we have

$$\begin{aligned} 0 < d(x_2, x_3) &< q_1 d(y_2, Tx_2) \leq q_1 \psi(d(x_1, x_2)) \\ &= \psi(q\psi(d(x_0, x_1))). \end{aligned} \quad (31)$$

Since ψ is strictly increasing, we have $\psi(d(x_2, x_3)) < \psi^2(q\psi(d(x_0, x_1)))$. Put $q_2 = \psi^2(q\psi(d(x_0, x_1)))/\psi(d(x_2, x_3))$. Also, we have $\alpha(x_1, x_2) \geq 1$, $d(x_2, y_2) = \text{dist}(A, B)$, and $d(x_3, y_3) = \text{dist}(A, B)$. Since T is an α -proximal admissible then $\alpha(x_2, x_3) \geq 1$. Thus, we have

$$d(x_3, y_3) = \text{dist}(A, B), \quad \alpha(x_2, x_3) \geq 1. \quad (32)$$

Continuing in the same way, we get sequences $\{x_n\}$ in A_0 and $\{y_n\}$ in B_0 , where $y_n \in Tx_{n-1}$ for each $n \in \mathbb{N}$ such that

$$d(x_{n+1}, y_{n+1}) = \text{dist}(A, B), \quad \alpha(x_n, x_{n+1}) \geq 1, \quad (33)$$

$$d(y_{n+1}, y_{n+2}) < \psi^n(q\psi(d(x_0, x_1))). \quad (34)$$

As $y_{n+2} \in Tx_{n+1} \subseteq B_0$, there exists $x_{n+2} \neq x_{n+1} \in A_0$ such that

$$d(x_{n+2}, y_{n+2}) = \text{dist}(A, B). \quad (35)$$

Since (A, B) satisfies the weak P -property form (33) and (35), we have $d(x_{n+1}, x_{n+2}) \leq d(y_{n+1}, y_{n+2})$. Then from (34), we have

$$d(x_{n+1}, x_{n+2}) < \psi^n(q\psi(d(x_0, x_1))). \quad (36)$$

For $n > m$ we have

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) < \sum_{i=n}^{m-1} \psi^{i-1}(q\psi(d(x_0, x_1))). \quad (37)$$

Hence, $\{x_n\}$ is a Cauchy sequence in A . Similarly, we show that $\{y_n\}$ is a Cauchy sequence in B . Since A and B are closed subsets of a complete metric space, there exist x^* in A and y^* in B such that $x_n \rightarrow x^*$ and $y_n \rightarrow y^*$ as $n \rightarrow \infty$. By (35), we conclude that $d(x^*, y^*) = \text{dist}(A, B)$ as $n \rightarrow \infty$. Since T is continuous and $y_n \in Tx_{n-1}$, we have $y^* \in Tx^*$. Hence, $\text{dist}(A, B) \leq D(x^*, Tx^*) \leq d(x^*, y^*) = \text{dist}(A, B)$. Therefore, x^* is the best proximity point of the mapping T . $\square$

Theorem 14. Let A and B be two nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $\alpha : A \times A \rightarrow [0, \infty)$ and $T : A \rightarrow K(B)$ be mappings satisfying the following conditions:

- (i) $Tx \subseteq B_0$ for each $x \in A_0$ and (A, B) satisfies the weak P -property;
- (ii) T is an α -proximal admissible;
- (iii) there exist elements $x_0, x_1 \in A_0$ and $y_1 \in Tx_0$ such that

$$d(x_1, y_1) = \text{dist}(A, B), \quad \alpha(x_0, x_1) \geq 1; \quad (38)$$

- (iv) T is a continuous α - ψ -proximal contraction.

Then there exists an element $x^* \in A_0$ such that $D(x^*, Tx^*) = \text{dist}(A, B)$.

Theorem 15. Let A and B be two nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $\alpha : A \times A \rightarrow [0, \infty)$ and $\psi \in \Psi$ be a strictly increasing map. Suppose that $T : A \rightarrow CL(B)$ is a mapping satisfying the following conditions:

- (i) $Tx \subseteq B_0$ for each $x \in A_0$ and (A, B) satisfies the weak P -property;
- (ii) T is an α -proximal admissible;
- (iii) there exist elements $x_0, x_1 \in A_0$ and $y_1 \in Tx_0$ such that

$$d(x_1, y_1) = \text{dist}(A, B), \quad \alpha(x_0, x_1) \geq 1; \quad (39)$$

- (iv) property (C) holds and T is an α - ψ -proximal contraction.

Then there exists an element $x^* \in A_0$ such that $D(x^*, Tx^*) = \text{dist}(A, B)$.

Proof. Following the proof of Theorem 13, there exist Cauchy sequences $\{x_n\}$ in A and $\{y_n\}$ in B such that (33) holds and $x_n \rightarrow x^* \in A$ and $y_n \rightarrow y^* \in B$ as $n \rightarrow \infty$. From the condition (C), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x^*) \geq 1$ for all k . Since T is an α - ψ -proximal contraction, we have

$$\begin{aligned} H(Tx_{n_k}, Tx^*) &\leq \alpha(x_{n_k}, x^*) H(Tx_{n_k}, Tx^*) \\ &\leq \psi(d(x_{n_k}, x^*)), \quad \forall k. \end{aligned} \quad (40)$$

Letting $k \rightarrow \infty$ in the above inequality, we get $Tx_{n_k} \rightarrow Tx^*$. By continuity of the metric d , we have

$$d(x^*, y^*) = \lim_{k \rightarrow \infty} d(x_{n_k+1}, y_{n_k+1}) = \text{dist}(A, B). \quad (41)$$

Since $y_{n_k+1} \in Tx_{n_k}$, $y_{n_k} \rightarrow y^*$, and $Tx_{n_k} \rightarrow Tx^*$, then $y^* \in Tx^*$. Hence, $\text{dist}(A, B) \leq D(x^*, Tx^*) \leq d(x^*, y^*) = \text{dist}(A, B)$. Therefore, x^* is the best proximity point of the mapping T . $\square$

Theorem 16. Let A and B be two nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $\alpha : A \times A \rightarrow [0, \infty)$ and $T : A \rightarrow K(B)$ be mappings satisfying the following conditions:

- (i) $Tx \subseteq B_0$ for each $x \in A_0$ and (A, B) satisfies the weak P -property;
- (ii) T is an α -proximal admissible;
- (iii) there exist elements $x_0, x_1 \in A_0$ and $y_1 \in Tx_0$ such that

$$d(x_1, y_1) = \text{dist}(A, B), \quad \alpha(x_0, x_1) \geq 1; \quad (42)$$

- (iv) property (C) holds and T is an α - ψ -proximal contraction.

Then there exists an element $x^* \in A_0$ such that $D(x^*, Tx^*) = \text{dist}(A, B)$.

Example 17. Let $X = [0, \infty) \times [0, \infty)$ be endowed with the usual metric d . Suppose that $A = \{(1/2, x) : 0 \leq x < \infty\}$ and $B = \{(0, x) : 0 \leq x < \infty\}$. Define $T : A \rightarrow CL(B)$ by

$$T\left(\frac{1}{2}, a\right) = \begin{cases} \left\{\left(0, \frac{x}{2}\right) : 0 \leq x \leq a\right\} & \text{if } a \leq 1 \\ \left\{\left(0, x^2\right) : 0 \leq x \leq a^2\right\} & \text{if } a > 1, \end{cases} \quad (43)$$

and $\alpha : A \times A \rightarrow [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in \left\{\left(\frac{1}{2}, a\right) : 0 \leq a \leq 1\right\} \\ 0 & \text{otherwise.} \end{cases} \quad (44)$$

Let $\psi(t) = t/2$ for all $t \geq 0$. Notice that $A_0 = A$, $B_0 = B$, and $Tx \subseteq B_0$ for each $x \in A_0$. Also, the pair (A, B) satisfies the weak P -property. Let $x_0, x_1 \in \{(1/2, x) : 0 \leq x \leq 1\}$; then $Tx_0, Tx_1 \subseteq \{(0, x/2) : 0 \leq x \leq 1\}$. Consider $y_1 \in Tx_0$, $y_2 \in Tx_1$, and $u_1, u_2 \in A$ such that $d(u_1, y_1) = \text{dist}(A, B)$ and $d(u_2, y_2) = \text{dist}(A, B)$. Then we have $u_1, u_2 \in \{(1/2, x) : 0 \leq x \leq 1/2\}$. Hence, T is an α -proximal admissible map. For $x_0 = (1/2, 1) \in A_0$ and $y_1 = (0, 1/2) \in Tx_0$ in B_0 , we have $x_1 = (1/2, 1/2) \in A_0$ such that $d(x_1, y_1) = \text{dist}(A, B)$ and $\alpha(x_0, x_1) = 1$. If $x, y \in \{(1/2, a) : 0 \leq a \leq 1\}$, then we have

$$\alpha(x, y) H(Tx, Ty) = \frac{|x - y|}{2} = \frac{1}{2} d(x, y) = \psi(d(x, y)); \quad (45)$$

for otherwise

$$\alpha(x, y) H(Tx, Ty) \leq \psi(d(x, y)). \quad (46)$$

Hence, T is an α - ψ -proximal contraction. Moreover, if $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) = 1$ for all n and $x_n \rightarrow x \in A$ as $n \rightarrow \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) = 1$ for all k . Therefore, all the conditions of Theorem 15 hold and T has the best proximity point.

Example 18. Let $X = [0, \infty) \times [0, \infty)$ be endowed with the usual metric d . Let $a > 1$ be any fixed real number, $A = \{(a, x) : 0 \leq x < \infty\}$ and $B = \{(0, x) : 0 \leq x < \infty\}$. Define $T : A \rightarrow CL(B)$ by

$$T(a, x) = \{(0, b^2) : 0 \leq b \leq x\} \quad (47)$$

and $\alpha : A \times A \rightarrow [0, \infty)$ by

$$\alpha((a, x), (a, y)) = \begin{cases} 1 & \text{if } x = y = 0 \\ \frac{1}{a(x+y)} & \text{otherwise.} \end{cases} \quad (48)$$

Let $\psi(t) = (1/a)t$ for all $t \geq 0$. Notice that $A_0 = A$, $B_0 = B$, and $Tx \subseteq B_0$ for each $x \in A_0$. If $w_1 = (a, y_1), w_2 = (a, y_2) \in A$ with either $y_1 \neq 0$ or $y_2 \neq 0$ or both are nonzero, we have

$$\begin{aligned} \alpha(w_1, w_2) H(Tw_1, Tw_2) &= \frac{1}{a(y_1 + y_2)} |(y_1)^2 - (y_2)^2| \\ &= \frac{1}{a} |y_1 - y_2| = \psi(d(w_1, w_2)); \end{aligned} \quad (49)$$

for otherwise

$$\alpha(w_1, w_2) H(Tw_1, Tw_2) = 0 = \psi(d(w_1, w_2)). \quad (50)$$

For $x_0 = (a, 1/2a) \in A_0$ and $y_1 = (0, 1/4a^2) \in Tx_0$ in B_0 , we have $x_1 = (a, 1/4a^2) \in A_0$ such that $d(x_1, y_1) = a = \text{dist}(A, B)$ and $\alpha(x_0, x_1) > 1$. Furthermore, it is easy to see that remaining conditions of Theorem 13 also hold. Thus, T has the best proximity point.

3. Consequences

From results of previous section, we immediately obtain the following results.

Corollary 19. Let A and B be two nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $\alpha : A \times A \rightarrow [0, \infty)$ and $T : A \rightarrow B$ be mappings satisfying the following conditions:

- (i) $T(A_0) \subseteq B_0$ and (A, B) satisfies the weak P -property;
- (ii) T is an α -proximal admissible;
- (iii) there exist elements x_0 and x_1 in A_0 such that

$$d(x_1, Tx_0) = \text{dist}(A, B), \quad \alpha(x_0, x_1) \geq 1; \quad (51)$$

- (iv) T is a continuous α - ψ -proximal contraction.

Then there exists an element $x^* \in A_0$ such that $d(x^*, Tx^*) = \text{dist}(A, B)$.

Corollary 20. Let A and B be two nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $\alpha : A \times A \rightarrow [0, \infty)$ and $T : A \rightarrow B$ be mappings satisfying the following conditions:

- (i) $T(A_0) \subseteq B_0$ and (A, B) satisfies the weak P -property;
- (ii) T is an α -proximal admissible;
- (iii) there exist elements x_0 and x_1 in A_0 such that

$$d(x_1, Tx_0) = \text{dist}(A, B), \quad \alpha(x_0, x_1) \geq 1; \quad (52)$$

- (iv) property (C) holds and T is an α - ψ -proximal contraction.

Then there exists an element $x^* \in A_0$ such that $d(x^*, Tx^*) = \text{dist}(A, B)$.

Remark 21. Note that Corollaries 19 and 20 generalize Theorems 7 and 8 in Section 1, respectively.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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