Research Article

An Adaptive Nonconforming Finite Element Algorithm for Laplace Eigenvalue Problem

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We establish Crouzeix-Raviart element adaptive algorithm based on Rayleigh quotient iteration and give its a priori/a posteriori error estimates. Our algorithm is performed under the package of Chen, and satisfactory numerical results are obtained.

1. Introduction

A posteriori error estimates and adaptive methods of finite element approximation for eigenvalue problems are topics attracting more attention from mathematical and physical fields; see, for example, [1–8]. Basically, there are the following three ways of combining adaptivity and eigenvalue problems in which the a posteriori error estimators are more or less the same but different in the problem solved in each iteration: (1) solving the original eigenvalue problem \( a(u, v) = \lambda b(u, v) \) (see Algorithm 10). The convergence and optimality of this adaptive procedure were proved in [2]; (2) inverse iteration type (with or without correction). The convergence has been studied in [1, 6, 7]; (3) Shifted-inverse iteration type (see [8–11]).

The triangular Crouzeix-Raviart element (C-R element) was first introduced by Crouzeix and Raviart [12] in 1973 to solve the stationary Stokes equation. After that, many scholars developed and applied it to eigenvalue problems, for instance, [13–16] discussed a posteriori error estimates and the adaptive methods of the C-R element. C-R element has important properties; for example, Armentano and Durán [17] discovered and proved that the C-R element eigenvalues approximate the exact ones of the Laplace operator from below, which is a very important property in engineering and mechanics computing.

Based on the above work, this paper further discusses the third kind of adaptive methods of the C-R finite element method for eigenvalue problems and obtains the following new results:

(1) we establish a multiscale discretization scheme of the C-R element based on Rayleigh quotient iteration and prove its convergence and a priori error estimates;
(2) we give residual type a posteriori error estimator for our adaptive algorithm, as well as its reliability and efficiency;
(3) we establish an adaptive algorithm (Algorithm 11), which is performed under the package of Chen (see [18]), and satisfactory numerical results are obtained.

As for the fundamental theory of finite elements and spectral approximation, we refer to [19–22].

Throughout this paper, \( C \) denotes a positive constant independent of mesh parameter, which may not be the same constant in different places. For simplicity, we use the notation \( a \leq b \) to mean that \( a \leq Cb, a = O(b) \) and to mean that \( a \leq b \) and \( b \leq a \).

2. Preliminaries

Consider Laplace eigenvalue problem

\[-\Delta u = \lambda u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega, (1)\]

where \( \Omega \subset R^2 \) is a polygonal domain with the maximum interior angle \( \omega \).
We denote the real order Sobolev spaces with norm \( \| \cdot \|_k \) by \( H^k(\Omega) \), \( H_0^k(\Omega) = \{ v \in H^k(\Omega), v|_{\partial \Omega} = 0 \} \). Let \( b(\cdot, \cdot) \) and \( \| \cdot \|_{0, \Omega} \) be the inner product and the norm in the space \( L^2(\Omega) \), respectively.

The weak form of (1) is as follows: find \( \lambda \in \mathbb{R} \), \( u \in H_0^1(\Omega) \), \( u \neq 0 \) such that
\[
a(u, v) = \lambda b(u, v), \quad \forall v \in H_0^1(\Omega),
\]
where
\[
a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v, \quad b(u, v) = \int_{\Omega} uv.
\]

As we know, \( a(\cdot, \cdot) \) is a symmetric, continuous, and \( H_0^1(\Omega) \)-elliptic bilinear form on \( H_1^0(\Omega) \times H_1^0(\Omega) \), and \( b(\cdot, \cdot) \) is a symmetric, continuous, and positive definite bilinear form on \( L^2(\Omega) \times L^2(\Omega) \).

The following condition results from [10, 24].

\[
\lim_{\lambda \to \lambda_k} \frac{a_h(\lambda_k, \cdot)}{b(\cdot, \cdot)},
\]
where \( a_h(\lambda_k, \cdot) := \lambda_k b(\lambda_k, \cdot), \quad \forall v \in V_h, \)
\[
\| \lambda_k - \lambda \|_2^2 = \| \lambda_k - \lambda \|_{0, \Omega}^2 + \| (T - T_h) \|_{M(\lambda_k)}^2,
\]
where the operators \( T : L^2(\Omega) \to H^1_0(\Omega) \) and \( T_h : L^2(\Omega) \to L^2(\Omega) \) are self-adjoint and completely continuous.

Define the operator \( T : L^2(\Omega) \to H_0^1(\Omega) \), satisfying
\[
a(Tg, v) = b(g, v), \quad \forall v \in H_0^1(\Omega),
\]
where \( a(Tg, v) := \int_{\Omega} \nabla (Tg) \cdot \nabla v, \quad b(g, v) := \int_{\Omega} g v \).

Then, (2) has the equivalent operator form \( Tu = \lambda^{-1} u \), where the operators \( T : H_0^1(\Omega) \to H_0^1(\Omega) \) and \( T : L^2(\Omega) \to L^2(\Omega) \) are self-adjoint and completely continuous.

\[
\frac{a_h(u, v)}{b(v, v)} - \lambda = \frac{\| w - u \|_0^2}{\| w \|_0^2} - \lambda \frac{\| w \|_0^2}{\| w \|_0^2} + 2 E_h(u, w) / b(w, w),
\]
where \( a_h(u, v) := \lambda b(u, v), \quad \forall v \in V_h, \)
\[
\| \lambda_k - \lambda \|_2^2 = \| (T - T_h) \|_{M(\lambda_k)}^2 + \| (T - T_h) \|_{M(\lambda_k)}^2,
\]
where the operators \( T : L^2(\Omega) \to H^1_0(\Omega) \) and \( T_h : L^2(\Omega) \to L^2(\Omega) \) are self-adjoint and completely continuous.

The weak form of (1) is as follows: find \( \lambda \in \mathbb{R} \), \( u \in H_0^1(\Omega) \), \( u \neq 0 \) such that
\[
a(u, v) = \lambda b(u, v), \quad \forall v \in H_0^1(\Omega),
\]
where
\[
a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v, \quad b(u, v) = \int_{\Omega} uv.
\]

As we know, \( a(\cdot, \cdot) \) is a symmetric, continuous, and \( H_0^1(\Omega) \)-elliptic bilinear form on \( H_1^0(\Omega) \times H_1^0(\Omega) \), and \( b(\cdot, \cdot) \) is a symmetric, continuous, and positive definite bilinear form on \( L^2(\Omega) \times L^2(\Omega) \).

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where \( a_h(\lambda_k, \cdot) := \lambda_k b(\lambda_k, \cdot), \quad \forall v \in V_h, \)
\[
\| \lambda_k - \lambda \|_2^2 = \| \lambda_k - \lambda \|_{0, \Omega}^2 + \| (T - T_h) \|_{M(\lambda_k)}^2,
\]
where the operators \( T : L^2(\Omega) \to H^1_0(\Omega) \) and \( T_h : L^2(\Omega) \to L^2(\Omega) \) are self-adjoint and completely continuous.


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Condition 1. There exists a properly small positive number \( \varepsilon \), \( t_i \in (1, 3 - \varepsilon) \), \( i = 1, 2, \ldots, s \) such that \( \delta_{h_1}(\lambda) = O(\delta_{h_{i-1}}(\lambda)^j) \), \( \delta_{h_1}(\lambda) \to 0 \) (\( i \to \infty \)).

The following scheme is proposed by Yang and Bi (see [11]).

Scheme 3 (multiscale discretization scheme). Consider the following steps.
Step 1. Solve (5) on \( V_H \): find \( (\lambda_{H}, u_{H}) \in R \times V_H \) such that \( \|u_{H}\|_{H} = 1 \) and

\[
a_{H}(u_{H}, \psi) = \lambda_{H} b(u_{H}, \psi), \quad \forall \psi \in V_H. \tag{16}
\]

Step 2. Execute \( u_{H} = u_{H}, \lambda_{H} = \lambda_{H}, i = 1 \).

Step 3. Solve a linear system on \( V_{H_{i}} \): find \( u' \in V_{H_{i}} \) such that

\[
a_{H_{i}}(u', \psi) - \lambda_{H_{i}} b(u', \psi) = b(u_{H_{i-1}}, \psi), \quad \forall \psi \in V_{H_{i}}. \tag{17}
\]

Set \( u^{H} = u' / \|u'\|_{H} \).

Step 4. Compute the Rayleigh quotient

\[
\lambda_{H} = \frac{a_{H}(u^{H}, u^{H})}{b(u^{H}, u^{H})}. \tag{18}
\]

Step 5. If \( i = 1 \), then output \( (\lambda^{H}, u^{H}) \), stop. Else, \( i \equiv i + 1 \), and return to Step 3.

Let \( (\lambda_{H}, u_{H}) \) be the \( k \)th eigenpair of (16), and then \( (\lambda^{H}, u^{H}) \) derived from Scheme 3 is the \( k \)th eigenpair approximation of (5).

In the sequel, we also denote \( (\lambda_{H}, u_{H}) = (\lambda_{kH}, u_{kH}), (\lambda^{H}, u^{H}) = (\lambda_{kH}^{H}, u_{kH}^{H}) \).

Lemma 4 (see [11, Lemma 3.1]). For any nonzero \( u, v \in V_{H} + H^{1}_{0}(\Omega) \),

\[
\|u\|_{H} - \|v\|_{H} \leq 2 \|u - v\|_{H}, \tag{19}
\]

Denote \( \text{dist}(u, S) = \inf_{v \in S} \|u - v\|_{H} \).

Our analysis is based on the following crucial property of the shifted-inverse iteration in finite element method (see Lemma 4.2 of [24]), which is a development of Theorem 3.2 in [11]. Let \( M = (1/\lambda_{k}) = M(\lambda_{k}), M_{k} = (1/\lambda_{k}) = M_{k}(\lambda_{k}) \).

Lemma 5 (see [24, Lemma 4.2]). Let \( \mu_{k} = 1/\lambda_{k} \) and \( \mu_{kH} = 1/(\lambda_{k} H) \) be the \( k \)th eigenvalue of \( T \) and \( T_{H} \), respectively, \( (\mu_{k}, u_{k}) \) be an approximation for the eigenpair \( (\mu_{k}, u_{k}) \), where \( \mu_{k} \) is not an eigenvalue of \( T_{H} \), and \( u_{k} \in V_{H} \) with \( \|u_{k}\|_{H} = 1 \). Suppose that

(C1) \( \text{dist}(u_{0}, M_{k}(\mu_{k})) \leq 1/2; \)

(C2) \( |\mu_{0} - \mu_{jH}| \geq \rho/2 \) for \( j \neq k, k + 1, \ldots, k + q - 1 \), where \( \rho = \min_{|\mu_{j} - \mu_{k}|}(\mu_{j} - \mu_{k}) \) is the separation constant of the eigenvalue \( \mu_{k} \);

(C3) \( u' \in V_{H}, u_{k}^{H} \in V_{H} \) satisfy

\[
(\mu_{0} - T_{H})u' = u_{0}, \quad u_{k}^{H} = \frac{u'}{\|u'\|_{H}}. \tag{20}
\]

Then,

\[
\text{dist}(u_{k}^{H}, M_{k}(\mu_{k})) \leq \frac{4}{\rho} \max_{k \leq j \leq k + q - 1} |\mu_{0} - \mu_{jH}| \text{dist}(u_{0}, M_{k}(\mu_{k})). \tag{21}
\]

Let us construct the interpolation postprocessing operator \( I_{k} : V_{H} \to V_{H} \cap H^{1}_{0}(\Omega) \) (see [25]): on the vertex \( z \) of elements,

\[
(I_{k} u_{k})(z) = \begin{cases} 0, & z \in \partial \Omega, \\ \frac{1}{J_{z}} \sum_{x \in \omega_{z}} u_{k}|_{x}(z), & z \notin \partial \Omega, \end{cases} \tag{22}
\]

where \( J_{z} \) is the number of elements containing the vertex \( z \) and \( \omega_{z} \) is the union of elements containing the vertex \( z \).

Lemma 6. Suppose that Condition 1 holds and \( H \) is properly small. Let \( (\lambda_{kH}^{H}, u_{kH}^{H}) \) be obtained by Scheme 3 for \( l = 1 \), and then there exists \( u_{k} \in M(\lambda_{k}) \) such that

\[
\|u_{k}^{H} - u_{k}\|_{H} \leq C \left( \delta_{H}^{3}(\lambda_{k}) + \delta_{kH} \right), \tag{23}
\]

\[
|\lambda_{k}^{H} - \lambda_{k}| \leq C \left( \delta_{H}^{3}(\lambda_{k}) + \delta_{kH}^{2} \right). \tag{24}
\]

Proof. Based on the proof of Theorem 5.1 in [11] and Lemma 5, we deduce that

\[
\|u_{k}^{H} - u_{k}\|_{H} \leq C \left( (T - T_{H})_{M(\lambda_{k})} + (T - T_{H})_{M(\lambda_{k})} \right), \tag{25}
\]

and thus (23) holds. Using Strang Lemma and Lemma 3.1 of [25], we deduce that

\[
E_{h} \left( u_{k}, u_{k}^{H} \right) = E_{h} \left( u_{k}, u_{k}^{H} - I_{k}^{H} u_{k}^{H} \right) \leq \|u_{k} - T_{H}(\lambda_{k} u_{k})\|_{H} \|u_{k}^{H} - I_{k}^{H} u_{k}^{H}\|_{H} \leq \|u_{k} - T_{H}(\lambda_{k} u_{k})\|_{H} \|u_{k}^{H} - u_{k}\|_{H} \leq \delta_{kH}^{2} \tag{26}
\]
From the above formula and (5.2) in [11], we get
\[ \left| \lambda_k - \lambda \right| \leq C \left( \left\| (T - T_H)\right\|_{M(\lambda_x)} \right)^2 + \left\| (T - T_{h_1})\right\|_{M(\lambda_x)}^2 \]
\[ + 2 E_{h_1}(u_k, u_{k+1}) \bigg/ b(u_k, u_{k+1}) \]
\[ \leq \delta_H^2(\lambda_k) + \delta_H^2(\lambda_k), \]
and thus (24) holds. □

Based on [10, 11, 24], we will prove the following Theorems 7 and 8 for Scheme 3.

**Theorem 7.** Let \((\lambda^{h}, u^{h})\) be an approximate eigenpair obtained by Scheme 3, and \(u^{h-1}\) and \(\lambda^{h-1}\) approximate \(\bar{u} \in \bar{M}(\lambda)\) and \(\lambda\), respectively, and \(\left\| u^{h-1} - \bar{u} \right\|_{h_{i-1}} \leq \delta_{h_{i-1}}(\lambda), \left| \lambda^{h-1} - \lambda \right| \leq \delta_{h_{i-1}}^2(\lambda)\). Suppose that \(H\) is properly small and Condition 1 holds. Then, there exists \(u \in M(\lambda)\) such that
\[ \left\| u - u^{h} \right\|_{h,D} = \left\| (T - T_{h_1}) \left( \lambda^{h} u^{h} \right) \right\|_{h,D} + \left\| R \right\|_{h,D}, \]
\[ \left| \lambda^{h} - \lambda \right| \leq \delta_{h_{i-1}}^3(\lambda), \]
\[ \left\| u^{h} - T_{h_1} \left( \lambda^{h} u^{h} \right) \right\|_{h,D} \leq \delta_{h_{i-1}}^3(\lambda) + \delta_{h_{i-1}}^2(\lambda), \]
where \(\left\| R \right\|_{h,D} \leq \delta_{h_{i-1}}^3(\lambda) + \delta_{h_{i-1}}^2(\lambda)\).

**Proof.** Let \(\mu_0 = 1/\lambda^{h_1}, u_0 = \lambda^{h_1} \cdot (T_{h_1} u^{h_1}) / \lambda^{h_1} \cdot (T_{h_1} u^{h_1})_{h_1} \). Since \(\bar{u} \in \bar{M}(\lambda)\), by calculation, we get
\[ \left\| \lambda^{h_1} \cdot T_{h_1} u^{h_1} - \bar{u} \right\|_{h_1} \]
\[ = \left\| \lambda^{h_1} \cdot T_{h_1} u^{h_1} - \lambda T \bar{u} \right\|_{h_1} \]
\[ \leq \left| \lambda^{h_1} - \lambda \right| \left\| T_{h_1} u^{h_1} \right\|_{h_1} + \lambda \left\| T_{h_1} \left( (u^{h_1} - \bar{u}) \right) \right\|_{h_1} \]
\[ + \lambda \left\| (T_{h_1} - T) \bar{u} \right\|_{h_1}. \]
From the definition of \(T_{h_1}\), it is easy to know that
\[ \left\| T_{h_1} \bar{v} \right\|_{h_1} \leq C \left\| \bar{v} \right\|_{0} \quad \forall \bar{v} \in L^2(\Omega). \]

From \(a_{h}(T_{h_1} v_{h_1}, v_{h_1}) = b(v_{h_1}, v_{h_1})\), we get
\[ \left\| v_{h_1} \right\|_{h_1}^2 \leq \left\| T_{h_1} v_{h_1} \right\|_{h_1} \left\| v_{h_1} \right\|_{h_1} \leq \left\| v_{h_1} \right\|_{0} \left\| v_{h_1} \right\|_{h_1}, \]
and thus
\[ \left\| v_{h_1} \right\|_{h_1} \leq \left\| v_{h_1} \right\|_{h_1}, \quad \forall v_{h_1} \in V_{h_1}. \]
By Lemma 3.1 in [25], we get that
\[ \left\| u^{h_1} - T_{h_1} u^{h_1} \right\|_{h_1} \leq \left\| u^{h_1} - T_{h_1} u^{h_1} \right\|_{h_1}, \]
\[ \left\| u^{h_1} - T_{h_1} u^{h_1} \right\|_{h_1} \leq \left\| u^{h_1} - \bar{u} \right\|_{h_1}. \]

Thus,
\[ \left\| u^{h_1} - \bar{u} \right\|_{h_1} \]
\[ \leq \left\| u^{h_1} - T_{h_1} u^{h_1} \right\|_{h_1} + \left\| T_{h_1} u^{h_1} - \bar{u} \right\|_{h_1} \]
\[ \leq \left\| u^{h_1} - T_{h_1} u^{h_1} \right\|_{h_1} + \left\| T_{h_1} u^{h_1} - \bar{u} \right\|_{h_1} \]
\[ \leq \left\| u^{h_1} - \bar{u} \right\|_{h_1}. \]
Using the above formula and (31), we can deduce that
\[ \left\| \lambda^{h_1} \cdot T_{h_1} u^{h_1} - \bar{u} \right\|_{h_1} \leq \left| \lambda^{h_1} - \lambda \right| + \left\| \lambda^{h_1} \cdot T_{h_1} u^{h_1} - \bar{u} \right\|_{h_1} \]
\[ \leq \delta_{h_{i-1}}(\lambda) \]
Using Lemma 4, we get
\[ \text{dist} (u_0, \bar{M}(\lambda)) \leq \left\| u_0 - \bar{u} \right\|_{h_1} \leq 2 \left\| \lambda^{h_1} \cdot T_{h_1} u^{h_1} - \bar{u} \right\|_{h_1} \]
\[ \leq \delta_{h_{i-1}}(\lambda). \]
Using triangle inequality and (15), we have
\[ \text{dist} (u_0, \bar{M}(\lambda)) \leq \text{dist} (u_0, \bar{M}(\lambda)) + \delta_{h_{i-1}}(\lambda). \]
From (12), for \(j = k, k + 1, \ldots, k + q - 1\), we have
\[ \left| \mu_0 - \mu_{j,h} \right| = \left| \lambda^{h_1} - \lambda + \lambda - \lambda \mu_j \right| \leq \left| \lambda^{h_1} - \lambda \right| + \delta_{h_{i-1}}^2(\lambda). \]
Noticing that \(H\) is small enough and Condition 1 holds, then by (38) and (39), we can obtain
\[ \text{dist} (u_0, M_{h_{i}}(\lambda)) \leq \frac{1}{2}. \]
Since \(\rho\) is the separation constant, \(H\) is small enough, and Condition 1 holds, we have
\[ \left| \mu_0 - \mu_{j,h} \right| \geq \frac{\rho}{2}, \quad j \neq k, k + 1, \ldots, k + q - 1. \]
From the definition of \(T_{h_1}\), we can see that Step 3 in Scheme 3 \((i = 1)\) is equivalent to
\[ a_{h} (u', \psi) - \lambda^{h_1 - 1} a_{h} (T_{h_1} u', \psi) \]
\[ = a_{h} (T_{h_1} u^{h_1}, \psi), \quad \forall \psi \in V_{h_1}. \]
where \( u^h = u'/\|u'\|_{h,l} \); that is,
\[
\left( \frac{1}{\lambda_{h,l}} - T_{h,l} \right) u' = \frac{1}{\lambda_{h,l}} T_{h,l} u^h_{h,l}, \quad u^h = \frac{u'}{\|u'\|_{h,l}}.
\]
(44)

Noticing that \((1/\lambda_{h,l})T_{h,l}u^h_{h,l} = \|(1/\lambda_{h,l})T_{h,l}u^h_{h,l}\|_{h,l} \), \( u_0 \) differs from \( u_0 \) by only a constant, then Step 3 is equivalent to
\[
\left( \frac{1}{\lambda_{h,l}} - T_{h,l} \right) u = u_0, \quad u^h = \frac{u}{\|u\|_{h,l}}.
\]
(45)

From the above formulae, (41), (42), and (45), we can see that the conditions in Lemma 5 hold; therefore, substituting (39) and (40) into (21), we derive
\[
\text{dist}(u^h, \tilde{M}_h(\lambda)) \\
\leq \left( |\lambda_{h,l} - \lambda| + \delta_{h,l}^2(\lambda) \right) \left( \text{dist}(u_0, \tilde{M}(\lambda)) + \delta_h(\lambda) \right) \leq \delta_{h,l}^3(\lambda) + \delta_h^2(\lambda) \leq \delta_{h,l}^3(\lambda).
\]
(46)

Let eigenfunctions \( \{u_j(\lambda)\}^{k+q-1}_{j=k} \) be an orthonormal basis of \( M_h(\lambda) \) in the sense of inner product \( a_h(u,v) \) and then
\[
\text{dist}(u^h, M_h(\lambda)) \leq \left\| u^h - \frac{1}{k+q-1} \sum_{j=k}^{k+q-1} a_h(u^h, u_j(\lambda)) u_j(\lambda) \right\|_{h,l}.
\]
(47)

Let
\[
u^* = \sum_{j=k}^{k+q-1} a_h(u^h, u_j(\lambda)) u_j(\lambda),
\]
(48)

and then it follows directly from (46) that
\[
\|u^h - u^*\|_{h,l} \leq \text{dist}(u^h, \tilde{M}_h(\lambda)) \leq \delta_{h,l}^3(\lambda).
\]
(49)

By Lemma 2, there exists \( \{u_j^0\}^{k+q-1}_{j=k} \subset \tilde{M}(\lambda) \) so that \( u_{j,\lambda} - u_j^0 \) satisfies (14).

Let
\[
u = \sum_{j=k}^{k+q-1} a_h(u^h, u_j(\lambda)) u_j^0.
\]
(50)

Then, \( u \in M(\lambda) \), and
\[
u - u^* = \sum_{j=k}^{k+q-1} a_h(u^h, u_j(\lambda)) (u_j^0 - u_{j,\lambda}).
\]
(51)

By calculation,
\[
u_j^0 - u_{j,\lambda} = \lambda_j T u_j^0 - \lambda_{j,\lambda} u_{j,\lambda} = \lambda_j T u_j^0 - \lambda_{j,\lambda} T h_j u_{j,\lambda}
\]
\[
= \lambda_j T u_j^0 - \lambda_{j,\lambda} T h_j u_j^0 + \lambda_{j,\lambda} T u_{j,\lambda}
\]
\[
= (\lambda_j - \lambda_{j,\lambda}) T u_j^0 + \lambda_{j,\lambda} T (u_j^0 - u_{j,\lambda})
\]
\[
= \lambda_{j,\lambda} (T - T_{h,j}) u_{j,\lambda} + R_j',
\]
(52)

where \( R_j' = (\lambda_j - \lambda_{j,\lambda}) T u_j^0 + \lambda_{j,\lambda} T (u_j^0 - u_{j,\lambda}) \).

From (12) and (13), we deduce that
\[
\|R_j'\|_{h,D} = \|\lambda_j - \lambda_{j,\lambda} T u_j^0\|_{h,D} + \lambda_{j,\lambda} T (u_j^0 - u_{j,\lambda})\|_{h,D} \leq \delta_{h,l}^3(\lambda).
\]
(53)

Substituting (52) into (51), we have
\[
u - u^* = \sum_{j=k}^{k+q-1} a_h(u^h, u_j(\lambda)) (\lambda_{j,\lambda} (T - T_{h,j}) u_{j,\lambda} + R_j')
\]
\[
= \lambda_{j,\lambda} (T - T_{h,j}) \sum_{j=k}^{k+q-1} a_h(u^h, u_j(\lambda)) u_{j,\lambda}
\]
\[
+ \sum_{j=k}^{k+q-1} a_h(u^h, u_j(\lambda)) R_j'
\]
\[
= \lambda_{j,\lambda} (T - T_{h,j}) \sum_{j=k}^{k+q-1} a_h(u^h, u_j(\lambda)) u_{j,\lambda}
\]
\[
+ \sum_{j=k}^{k+q-1} a_h(u^h, u_j(\lambda)) R_j'.
\]
(54)

Let
\[
R' = \lambda_{j,\lambda} (T - T_{h,j}) (u^* - u^h) + \sum_{j=k}^{k+q-1} a_h(u^h, u_j(\lambda)) R_j'
\]
\[
+ u^* - u^h.
\]
(55)

By the above two equalities, we obtain
\[
u - u^h = u - u^* + u^* - u_h = (T - T_{h,j}) (\lambda_{j,\lambda} u^h) + R'.
\]
(56)

From (49) and (53), we have
\[
\|R'\|_{h,D} = \|\lambda_{j,\lambda} (T - T_{h,j}) (u^* - u^h)
\]
\[
+ \sum_{j=k}^{k+q-1} a_h(u^h, u_j(\lambda)) R_j'
\]
\[
+ u^* - u^h\|_{h,D}
\]
\[
\leq \lambda_{j,\lambda} \|T - T_{h,j}\|_{h,D} \|u^* - u^h\|_{h,D}
\]
\[
+ \sum_{j=k}^{k+q-1} a_h(u^h, u_j(\lambda)) R_j'\|_{h,D}
\]
\[
+ \|u^* - u^h\|_{h,D}
\]
\[
\leq \delta_{h,l}^3(\lambda) + \delta_{h,l}^3(\lambda).
\]
(57)
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Therefore,
\[
\|u - u_h\|_{h,D} = \|(T - T_{h_1}) (\lambda_{j,h} u^h)\|_{h,D} + \|R\|_{h,D} \leq \delta_{h_1}(\lambda) .
\]

By Lemma 1, we have
\[
ap_h(u^h, u^h) = \frac{\|u^h - u\|_0^2}{\|u^h\|_0^2} = \frac{\|u^h - u\|_0^2}{\|u^h\|_0^2} - \lambda \frac{\|u^h - u\|_0^2}{\|u^h\|_0^2}
+ 2 \frac{E_h(u, u^h)}{b(u^h, u^h)}. \tag{60}
\]

Since \(I_h^2 u^h \in C^0(\Omega)\), using Strang Lemma and Lemma 3.1 of [25], we deduce that
\[
E_h(u, u^h) = E_h(u, u^h - I_h^2 u^h)
\leq \|u - T_{h_1}(\lambda u)\|_{h_1} \|u^h - I_h^2 u^h\|_{h_1}
\leq \|u - T_{h_1}(\lambda u)\|_{h_1} \|u^h - u\|_{h_1}
\leq \delta_{h_1}^2(\lambda) .
\tag{61}
\]

Substituting (59) and (61) into (60), we get
\[
\|u - u^h\|_{h, D} = \|(T - T_{h_1}) (\lambda_{j,h} u^h)\|_{h,D} + \|R\|_{h,D}
\leq \|(T - T_{h_1}) (\lambda^h u^h)\|_{h,D}
+ \|(T - T_{h_1}) (\lambda_{j,h} - \lambda^h) u^h\|_{h,D} + \|R\|_{h,D}, \tag{62}
\]

where \(R = (T - T_{h_1})(\lambda_{j,h} - \lambda^h) u^h + R'\).

By (57) and (29), we know that \(\|R\|_{h,D} \leq \delta_{h_1}^3(\lambda) + \delta_{h_1}^2(\lambda)\); thus, (28) holds.

By calculation,
\[
\sum_{j,k} a_h(u^h, u_{j,h_1}) \lambda^h T_{h_1} u_{j,h_1}
= \lambda^h T_{h_1} \left( \sum_{j,k} a_h(u^h, u_{j,h_1}) u_{j,h_1} \right)
= \lambda^h T_{h_1} u^* . \tag{63}
\]

By the above formulae and (12), we deduce that
\[
\|u^h - T_{h_1}(\lambda^h u^h)\|_{h,D}
= \|u^h - u^* + u^* - T_{h_1}(\lambda^h u^*) + T_{h_1}(\lambda^h u^*) - T_{h_1}(\lambda^h u^h)\|_{h,D}
\leq \|u^h - u^*\|_{h,D}
+ \sum_{j,k} (\lambda_{j,h_1} - \lambda^h) a_h(u^h, u_{j,h_1}) T_{h_1} u_{j,h_1}
+ \lambda^h T_{h_1} (u^* - u^h)\|_{h,D}
\leq \|u^h - u^*\|_{h,D} + \sum_{j,k} |\lambda_{j,h_1} - \lambda^h|
\leq \|u^h - u^*\|_{h,D} + \delta_{h_1}^2(\lambda),
\]

which together with (49) leads to (30). This completes the proof. \(\Box\)

**Theorem 8.** Let \((\lambda^h, u^h)\) be the kth approximate eigenpair of (1) obtained by Scheme 3, let \(\lambda\) be the kth eigenvalue of (1), and let \(H\) be properly small. Suppose that Condition 1 holds, then there exists \(u \in M(\lambda)\) such that
\[
\|u^h - u\|_{h_1} \leq C \delta_{h_1}(\lambda), \tag{65}
|\lambda^h - \lambda| \leq C \delta_{h_1}^2(\lambda), \quad l \geq 1.
\]

**Proof.** The proof of (65) is completed by using induction. When \(l = 1\), by Lemma 6, we know that Theorem 8 holds.

Suppose that Theorem 8 holds for \(l - 1\); that is,
\[
\|u^{h_{l-1}} - u\|_{h_{l-1}} \leq C \delta_{h_{l-1}}(\lambda), \tag{66}
|\lambda^{h_{l-1}} - \lambda| \leq C \delta_{h_{l-1}}^2(\lambda),
\]

which together with the assumptions in Theorem 8, we know that Theorem 7 holds. For \(l\), by (29) and (59), we get (65). The proof is completed. \(\Box\)

**4. A Posteriori Error Estimates for Multiscale Discretization Scheme**

Based on the work of [14, 26], in this section, we will discuss a posteriori error estimates of the C-R element approximation for Laplace eigenvalue problem.

Consider the boundary value problem corresponding to (2): find \(w \in H^2_{0}(\Omega)\) such that
\[
a(w, v) = b(f, v), \quad \forall v \in H^1_{0}(\Omega), \tag{67}
\]

and its C-R element approximation: find \(w_h \in V_h\) such that
\[
a_h(w_h, v) = b(f, v), \quad \forall v \in V_h. \tag{68}
\]
Let $κ^+ ∈ π_h, κ^- ∈ π_h$ be two elements sharing one edge $e$. For any piecewise continuous function $φ$, we denote by $[(φ)]_e = (φ|_{κ^+})_e - (φ|_{κ^-})_e$ the jump of $φ$ across $e$.

Let $u_h$ be the solution of (68), $\bar{J}_{e,θ}$ be the jump of $\nabla u_h$ across $e$ along $θ_e$, and $\bar{J}_{e,τ}$ be the jump of $\nabla u_h$ across $e$ along $τ_e$; let $\bar{R}_κ(u_h)$ be element residual; that is,

$$
\bar{R}_κ(u_h) = f + Δu_h, \quad κ ∈ π_h,
$$

$$
\bar{J}_{e,θ}(u_h) = \begin{cases}
[\nabla u_h]_e \cdot θ_e, & e ∈ Ω, \\
0, & e ∈ (∂Ω),
\end{cases}
$$

$$
\bar{J}_{e,τ}(u_h) = \begin{cases}
[\nabla u_h]_e \cdot τ_e, & e ∈ Ω, \\
-\sqrt{2}∇u_h \cdot τ_e, & e ∈ (∂Ω).
\end{cases}
$$

For $κ ∈ π_h$, define the residual on the element $κ$ as

$$
\eta_h(u_h, κ) = \left( h^2_κ \| \bar{R}_κ(u_h) \|_{0,κ}^2 + \frac{1}{2} \sum_{e ∈ Ω_e} h_e \left( \| \bar{J}_{e,τ}(u_h) \|_{0,e}^2 + \| \bar{J}_{e,θ}(u_h) \|_{0,θ}^2 \right) \right)^{1/2},
$$

and thus, for $G ⊂ Ω$, the residual sum on $G$ is given by

$$
\eta_h(u_h, G) = \left( \sum_{κ ∈ π_h, κ ⊂ G} \eta_h^2(u_h, κ) \right)^{1/2}.
$$

For $f ∈ L^2(Ω)$, define the date oscillation by

$$
osc(f, π_h) = \left( \sum_{κ ∈ π_h} h^2_κ \| f - f_h \|_{0,κ}^2 \right)^{1/2},
$$

where $f_h$ stands for a piecewise polynomial approximation of $f$ over $π_h$.

For the boundary value problem (67), Carstensen and Hu [27] have proved the following a posteriori error estimates:

$$
\| w - w_h \|_{h, Ω} ≤ \bar{C}_1 (\eta_h(w_h, Ω) + osc(f, π_h)),
$$

where constant $\bar{C}_1$ is only dependent on minimum angle of $π_h$, and if the right-hand side $f$ of (67) is a piecewise linear polynomial over $π_h$, then

$$
\bar{C}_2 \eta_h(w_h, Ω) ≤ \| w - w_h \|_{h, Ω}.
$$

Selecting $f = λ^h u^h$ in (67) and (68), then the generalized solution and the nonconforming finite element solution are $w = T(λ^h u^h)$ and $w_h = T_h(λ^h u^h)$, respectively, and the a posteriori error indicator of $w_h$ is $\eta_h(w_h, Ω)$, which is defined by (71).

Define the element residual $R_h(u^h)$ and the jump residual $J_{e,θ}(u^h)$ and $J_{e,τ}(u^h)$ for $u^h$ as follows:

$$
R_h(u^h) = λ^h u^h + Δu^h, \quad κ ∈ π_h,
$$

$$
J_{e,θ}(u^h) = \begin{cases}
[ [u^h]_e]_e \cdot θ_e, & e ∈ Ω, \\
0, & e ∈ (∂Ω),
\end{cases}
$$

$$
J_{e,τ}(u^h) = \begin{cases}
[ [u^h]_e]_e \cdot τ_e, & e ∈ Ω, \\
-\sqrt{2}∇u^h \cdot τ_e, & e ∈ (∂Ω).
\end{cases}
$$

For $κ ∈ π_h$, define the residual on the element $κ$ as

$$
\eta_h(u^h, κ) = \left( h^2_κ \| R_h(u^h) \|_{0,κ}^2 + \frac{1}{2} \sum_{e ∈ Ω_e} h_e \left( \| J_{e,τ}(u^h) \|_{0,e}^2 + \| J_{e,θ}(u^h) \|_{0,θ}^2 \right) \right)^{1/2}.
$$

For $G ⊂ Ω$, define the residual sum on $G$ as

$$
\eta_h(u^h, G) = \left( \sum_{κ ∈ π_h, κ ⊂ G} \eta_h^2(u^h, κ) \right)^{1/2}.
$$

**Theorem 9.** Suppose that the conditions in Theorem 7 hold and $V_h$ is a finite element space consisting of piecewise linear polynomials, then there exists a positive constant $δ$ which is independent of mesh parameter, such that

$$
\| u - u_h \|_{h, Ω} ≤ \left( \bar{C}_1 + δ \right) \eta_h(u_h, Ω),
$$

$$
(\bar{C}_2 + δ) \eta_h(u^h, G) ≤ \| u - u^h \|_{h, G},
$$

$$
| λ - λ^h | ≤ \eta_h^2(u^h, κ).
$$

**Proof.** Let $u_h = T_h(λ^h u^h)$, and by calculation

$$
\eta_h(u_h, Ω) = \left( \sum_{κ ∈ π_h, κ ⊂ G} \eta_h^2(u_h, κ) \right)^{1/2} = \left( \sum_{κ ∈ π_h, κ ⊂ Ω} \eta_h^2(u^h, κ) \right)^{1/2} + \left( \sum_{κ ∈ π_h, κ ⊂ Ω} \eta_h^2(u_h, κ) \right)^{1/2} - \left( \sum_{κ ∈ π_h, κ ⊂ Ω} \eta_h^2(u^h, κ) \right)^{1/2}
≡ \eta_h(u^h, G) + R_2.
By triangle inequality, we have

\[
|R_2| = \left( \sum_{k \in \tau_i \times C} \tilde{\eta}_h (w_l, \kappa) \right)^{1/2} - \left( \sum_{k \in \tau_i \times C} \eta_h (u^h, \kappa) \right)^{1/2} \leq \left( \sum_{k \in \tau_i \times C} \left( \tilde{\eta}_h (w_l, \kappa) - \eta_h (u^h, \kappa) \right)^2 \right)^{1/2}.
\]

From triangle inequality, (69)-(70), and (75)-(76), we deduce that

\[
\left| \tilde{\eta}_h (w_l, \kappa) - \eta_h (u^h, \kappa) \right| \leq \left( h_x^2 \| R_x (w_l) \|_{0,x}^2 + \frac{1}{2} \sum_{k \in \tau_i \times C} \left( \left\| J_{e,x} (w_l) \right\|_{0,e}^2 + \left\| J_{e,\theta} (w_l) \right\|_{0,e}^2 \right) \right)^{1/2} - \left( h_x^2 \| R_x (u^h) \|_{0,x}^2 + \frac{1}{2} \sum_{k \in \tau_i \times C} \left( \left\| J_{e,x} (u^h) \right\|_{0,e}^2 + \left\| J_{e,\theta} (u^h) \right\|_{0,e}^2 \right) \right)^{1/2} \leq \left( h_x^2 \| R_x (w_l) - R_x (u^h) \|_{0,x}^2 + \frac{1}{2} \sum_{k \in \tau_i \times C} \left( \left\| J_{e,x} (w_l) - J_{e,x} (u^h) \right\|_{0,e}^2 + \left\| J_{e,\theta} (w_l) - J_{e,\theta} (u^h) \right\|_{0,e}^2 \right) \right)^{1/2} \leq \left( h_x^2 \| \Delta (w_l - u^h) \|_{0,x}^2 + \frac{1}{2} \sum_{k \in \tau_i \times C} \left( \left\| \nabla (w_l - u^h) \right\|_{0,e}^2 + \left\| \nabla (w_l - u^h) \right\|_{0,e}^2 \right) \right)^{1/2}.
\]

It is obvious that \( \| \Delta (w_l - u^h) \|_{0,x}^2 = 0 \), and, by the trace theorem (see e.g., [28]) and the inverse estimates, we get

\[
\frac{1}{2} \sum_{k \in \tau_i \times C} \left( \left\| \nabla (w_l - u^h) \right\|_{0,e}^2 + \left\| \nabla (w_l - u^h) \right\|_{0,e}^2 \right) \leq h_x \left( h_x^{-1} \| \nabla (w_l - u^h) \|_{0,e}^2 + h_x \| \nabla (w_l - u^h) \|_{0,e}^2 \right) \leq \| w_l - u^h \|_{h,\omega_x}^2.
\]

Thus,

\[
\left| \tilde{\eta}_h (w_l, \kappa) - \eta_h (u^h, \kappa) \right| \leq \left\| \tilde{\eta}_h (\lambda^h u^h) - \eta_h (u^h, \kappa) \right\|_{h,\omega_x} = \left\| w_l - u^h \right\|_{h,\omega_x}.
\]

Combining (82), (85), and (85), we get

\[
|R_2| \leq \left\| w_l - u^h \right\|_{h,\omega_x} \leq \delta_{h,1} (\lambda) + \delta^2 (\lambda).
\]

Hence, from Condition 1, we know that \( R_3 \) is a small quantity of higher order than \( \tilde{\eta}_h (w_l, \kappa) \). Using (81), we obtain that \( R_2 \) is also a small quantity of higher order than \( \eta_h (u^h, \kappa) \).

Therefore, by (28), (73), (81), and (86), we have

\[
\left\| u - u^h \right\|_{h,\omega} = \left\| (T - T_h) (\lambda^h u^h) \right\|_{h,\omega} + \| R \|_{h,\omega} \leq C_1 \left\| \tilde{\eta}_h (T_h (\lambda^h u^h), \kappa) + \| R \|_{h,\omega} \leq C_1 \left\| \tilde{\eta}_h (u^h, \kappa) + \| R \|_{h,\omega} \leq C_1 \left( \tilde{\eta}_h (T_h (\lambda^h u^h), \kappa) - \eta_h (u^h, \kappa) \right) + \| R \|_{h,\omega} \leq C_1 \left( \tilde{\eta}_h (u^h, \kappa) + C_2 \| R \|_{h,\omega} + \| R \|_{h,\omega} \leq C_1 + \delta \right) \eta_h (u^h, \kappa)
\]

which is (78).
Similarly, by (28) and (74), we get
\[
\| u - u^h \|_{h, \Omega}
\geq C_2 \| (T - T_{h}) (\lambda^h \cdot u^h) \|_{h, \Omega} + \| R \|_{h, \Omega}
\geq C_2 \| \eta_h (T_{h} (\lambda^h \cdot u^h), \Omega) \|_{h, \Omega} + \| R \|_{h, \Omega}
\geq C_2 \| \eta_h (u^h, \Omega)
+ C_2 (\eta_h (T_{h} (\lambda^h \cdot u^h), \Omega) - \eta_h (u^h, \Omega))
+ \| R \|_{h, \Omega}
\geq (C_2 + \delta) \eta_h (u^h, \Omega),
\] (88)
and thus (79) holds.

By (61) and (28), we get
\[
E_{h} (u, u^h) \leq \| u - T_{h} (\lambda u) \|_{h} \| u^h - u \|_{h}
\leq \| u^h - u \|_{h}^2
\] (89)
and, by substituting the above relation into (60), we obtain
\[
\| \lambda^h - \lambda \| \leq \| u^h - u \|_{h}^2
\] (90)
which together with (78) yields (80). This completes the proof.

5. Adaptive Finite Element Algorithm Based on Multiscale Discretizations

As we know, The following Algorithm 10 is fundamental and important; see [14, 16] for its detailed theoretical results.

\textbf{Algorithm 10.} Choose parameter $0 < \theta < 1$.

\begin{enumerate}
  \item Step 1. Pick any initial mesh $\pi_{h_0}$ with mesh size $h_0$.
  \item Step 2. Solve (5) on $\pi_{h_0}$ for discrete solution $(\lambda_{h_0}, u_{h_0})$.
  \item Step 3. $l \Leftarrow 0$.
  \item Step 4. Compute the local indicators $\eta_{h_0}(u_{h_0}, \kappa)$.
  \item Step 5. Construct $\pi_{h_l} \subset \pi_{h_0}$ by Marking Strategy E and parameter $\theta$.
  \item Step 6. Refine $\pi_{h_l}$ to get a new mesh $\pi_{h_{l+1}}$.
  \item Step 7. Solve (5) on $\pi_{h_{l+1}}$ for discrete solution $(\lambda_{h_{l+1}}, u_{h_{l+1}})$.
  \item Step 8. $l \Leftarrow l + 1$, and go to Step 4.
\end{enumerate}

\textbf{Marking Strategy E.} Give parameter $0 < \theta < 1$.

\textbf{Step 1.} Construct a minimal subset $\pi_{h_l}$ of $\pi_{h_l}$ by selecting some elements in $\pi_{h_l}$ such that
\[
\sum_{\kappa \in \overline{\pi}_{h_l}} \eta_{h_l} (u_{h_l}, \kappa) \geq \theta \eta_{h_l} (u_{h_l}, \Omega).
\] (91)

\textbf{Step 2.} Mark all the elements $\pi_{h_l}$.

\textbf{Step 3.} $l \Leftarrow l + 1$, and go to Step 4.

\textbf{Algorithm II.} Choose parameter $0 < \theta < 1$.

\begin{enumerate}
  \item Step 1. Pick any initial mesh $\pi_{h_0}$ with mesh size $h_0$.
  \item Step 2. Solve (5) on $\pi_{h_0}$ for discrete solution $(\lambda_{h_0}, u_{h_0})$.
  \item Step 3. $l \Leftarrow 0$, $\lambda_0 \Leftarrow \lambda_{h_0}$.
  \item Step 4. Compute the local indicators $\eta_{h_0}(u_{h_0}, \kappa)$.
  \item Step 5. Construct $\pi_{h_l} \subset \pi_{h_0}$ by Marking Strategy E and parameter $\theta$.
  \item Step 6. Refine $\pi_{h_l}$ to get a new mesh $\pi_{h_{l+1}}$.
  \item Step 7. Find $u' \in V_{h_{l+1}}$ such that
\[
a_{h_{l+1}} (u', \psi) = b (u_{h_{l+1}}, \psi), \quad \forall \psi \in V_{h_{l+1}}.
\] (92)
  \item Step 8. $\lambda_0 \Leftarrow \lambda_{h_{l+1}}, l \Leftarrow l + 1$ and go to Step 4.
\end{enumerate}

\textbf{Marking Strategy E in Algorithm II will be the same as that in Algorithm 10, except for replacing $u_{h_0}$ with $u_{h_0}$.}

Note that when $|\lambda_0 - \lambda|$ is too small, (92) is an almost singular linear equation. Although it has no difficulty in solving (92) numerically (see Lecture 27.4 in [29]), one would like to think of selecting a proper integer $l_0 \geq 0$. When $l \geq l_0$, set $\lambda^h = \lambda_{h_{l+1}}$ in (92). So, we can establish the following algorithm (see e.g., Scheme 3.2 in [24]).

\textbf{Algorithm II.} Choose parameter $0 < \theta < 1$.

\begin{enumerate}
  \item Step 1–Step 7. Execute Step 1–Step 7 of Algorithm II.
  \item Step 8. If $l < l_0$, $\lambda_0 \Leftarrow \lambda_{h_{l+1}}, l \Leftarrow l + 1$, go to Step 4; else $l \Leftarrow l + 1$, go to Step 4.
\end{enumerate}

\textbf{Marking Strategy E in Algorithm II will be the same as that in Algorithm II.}
6. Numerical Experiments

In this section, we will report two numerical examples for Algorithms 10 and 11 to illustrate the theoretical results in this paper. We use MATLAB 2012 to solve Examples 1 and 2. Our program is compiled under the package of Chen. We take $\theta = 0.5$ in two Algorithms.

For reading convenience, we use the following notations in our tables.

- $l^*$: The $l^*$th iteration of Algorithm 10
- $\lambda_{k,h_l^*}$: The $k$th approximate eigenvalue derived from the $l^*$th iteration obtained by Algorithm 10
- $\text{dof}_{k,l^*}$: The degrees of freedom of the $l^*$th iteration for computing $\lambda_{k,h_l^*}$
- $\text{CPU}_{l^*}(s)$: The total CPU time(s) for computing $\lambda_{k,h_l^*}$
- $|\lambda_{k,h_l^*} - \lambda_k|$: The error of $k$th approximate eigenvalue $\lambda_{k,h_l^*}$
- $l$: The $l$th iteration of Algorithm 11
- $\lambda_{k,h_l}^*$: The $k$th approximate eigenvalue derived from the $l$th iteration obtained by Algorithm 11
- $\text{dof}_{k,l}$: The degrees of freedom of the $l$th iteration for computing $\lambda_{k,h_l}^*$
Example 1. We use Algorithms 10 and 11 to compute the approximate eigenvalues of (1) on the \( L \)-shaped domain \( \Omega = ((0,2) \times (0,2)) \setminus ([1,2] \times [1,2]) \) (see Figure 1(a)).

The first and fifth eigenvalues of (1) are \( \lambda_1 \approx 9.639723844 \) and \( \lambda_5 \approx 31.912636 \) on this domain, respectively. The associated numerical results are presented in Table 1 and Figures 1(a), 2, 3, and 4. Figure 1(a) gives the uniform initial mesh with \( H = \frac{\sqrt{2}}{16} \). Figures 2 and 3 show the adaptive meshes of the first and fifth eigenvalues after the fifth iteration by two algorithms, respectively. It is indicated in Figure 4 that the error curves of the first and fifth approximate eigenvalues and the curves of the associated a posteriori error estimators obtained by Algorithms 10 and 11 are approximately parallel to the line with slope \(-1\), respectively; this coincides with our theory in Section 4.
Table 1: The 1st and 5th eigenvalues obtained by two algorithms on L-shaped domain with $H = \sqrt{2}/16$.

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Table 2: The 1st and 6th eigenvalues obtained by two algorithms on slit domain with $H = \sqrt{2}/16$.

<table>
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<tr>
<th>$k$</th>
<th>$l^*$</th>
<th>$\text{dof}_{l^*}$</th>
<th>$\lambda_{k,l^*}$</th>
<th>$\text{CPU}_{l^*}$ (s)</th>
<th>$l$</th>
<th>$\text{dof}_l$</th>
<th>$\lambda^h_l$</th>
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<td>1.33</td>
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<td>0.79</td>
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<td>54.9</td>
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</table>

Figure 5: The adaptive meshes of 1st eigenvalue after 5th iteration by Algorithm 10 (a) and Algorithm 11 (b).
Figure 6: The adaptive meshes of 6th eigenvalue after 5th iteration by Algorithm 10 (a) and Algorithm II (b).

Figure 7: The error curves of two algorithms on slit domain.

But from Table 1, using Algorithm II, we will spend much less time in the case of the same number of degrees of freedom but get the same accuracy to Algorithm 10. In addition, Algorithm 10, due to not having enough memory, can not proceed, while Algorithm II can have one more iteration; thus, more accurate numerical results will be obtained.

Example 2. We use Algorithms 10 and II to compute the approximate eigenvalues of (1) on \( \Omega = ((0, 2) \times (0, 2)) \backslash ([1, 2] \times \{1\}) \) with a slit (see Figure 1(b)).

The first and sixth eigenvalues of (1) are \( \lambda_1 \approx 8.3713297112 \) and \( \lambda_6 \approx 30.536 \) on this domain, respectively. The associated numerical results are presented in Table 2 and Figures 5, 6, and 7. Figure 7 show that the error curves of the first and sixth approximate eigenvalues and the curves of the associated a posteriori error estimators obtained by Algorithms 10 and II are approximately parallel to the line with slope –1, respectively, which suffices to support our theory.

From Table 2, using Algorithm II, compared with Algorithm 10, we can get the same accurate results in the case
of the almost same degrees of freedom, but the CPU time is significantly decreased.

Remark 13. Based on the work of [30], we would like to believe that \( \lambda^h \) and Rayleigh quotient \( a(I_h^h u^h, I_h^h v^h)/b(I_h^h v^h, I_h^h v^h) \) of \( I_h^h u^h \) are the lower and upper bounds of the exact eigenvalue \( \lambda \), respectively. To see this point, the numerical results of Tables 1 and 2 also illustrate that the C-R element eigenvalues approximate the exact ones of the Laplace operator from below. Thus, we can establish iterative control condition by computing \( \lambda^h \) and \( a(I_h^h u^h, I_h^h v^h)/b(I_h^h v^h, I_h^h v^h) \) for the two algorithms.

Remark 14. For Algorithm 12, by calculating, in the case of the almost same number of degrees of freedom, we can get the same accurate results to Algorithm 11, and CPU time is almost the same; thus, we do not list the associated numerical results in this paper.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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