Research Article

Complete Self-Shrinking Solutions for Lagrangian Mean Curvature Flow in Pseudo-Euclidean Space

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Let $f(x)$ be a smooth strictly convex solution of $\det(\partial^2 f / \partial x_i \partial x_j) = \exp \{ (1/2) \sum_{i=1}^n x_i (\partial f / \partial x_i) - f \}$ defined on a domain $\Omega \subset \mathbb{R}^n$; then the graph $M_{\nabla f}$ of ∇f is a space-like self-shrinker of mean curvature flow in Pseudo-Euclidean space $\mathbb{R}_n^{2n}$ with the indefinite metric $\sum dx_i dy_i$. In this paper, we prove a Bernstein theorem for complete self-shrinkers. As a corollary, we obtain if the Lagrangian graph $M_{\nabla f}$ is complete in $\mathbb{R}_n^{2n}$ and passes through the origin then it is flat.

1. Introduction

Let M be an n -dimensional submanifold immersed into the Euclidean space $\mathbb{R}^{n+m}$. Mean curvature flow is a one-parameter family $X_t = X(\cdot, t)$ of immersions $X_t : M \rightarrow \mathbb{R}^{n+m}$ with corresponding images $M_t = X_t(M)$ such that

$$\begin{aligned} \frac{d}{dt} X(x, t) &= H(x, t), \quad x \in M, \\ X(x, 0) &= X(x) \end{aligned} \quad (1)$$

is satisfied, where $H(x, t)$ is the mean curvature vector of M_t at $X(x, t)$ in $\mathbb{R}^{n+m}$. Self-similar solutions to the mean curvature flow play an important role in understanding the behavior of the flow and the types of singularities. They satisfy a system of quasilinear elliptic PDE of the second order as follows:

$$H = -\frac{X^\perp}{2}, \quad (2)$$

where $(\cdots)^\perp$ stands for the orthogonal projection into the normal bundle NM .

Self-shrinkers in the ambient Euclidean space have been studied by many authors; for example, see [1–6] and so forth. For recent progress and related results, see the introduction in [7]. When the ambient space is a pseudo-Euclidean space, there are many classification works about self-shrinkers; for

example, see [8–13] and so forth. But very little is known when self-shrinkers are complete not compact with respect to induced metric from pseudo-Euclidean space. In this paper, we will characterize self-shrinkers for Lagrangian mean curvature flow in the pseudo-Euclidean space from this aspect.

Let $(x_1, \dots, x_n; y_1, \dots, y_n)$ be null coordinates in $2n$ -dimensional pseudo-Euclidean space $\mathbb{R}_n^{2n}$. Then, the indefinite metric (cf. [14]) is defined by $ds^2 = \sum_{i=1}^n dx_i dy_i$. Suppose $f(x)$ is a smooth strictly convex function defined on domain $\Omega \subset \mathbb{R}^n$. The graph $M_{\nabla f}$ of ∇f can be written as $(x_1, \dots, x_n; \partial f / \partial x_1, \dots, \partial f / \partial x_n)$. Then, the induced Riemannian metric on $M_{\nabla f}$ is given by

$$G = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j. \quad (3)$$

In particular, if function f satisfies

$$\det \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right) = \exp \left\{ \frac{1}{2} \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i} - f \right\}, \quad (4)$$

then the graph $M_{\nabla f}$ of ∇f is a space-like self-shrinking solution for mean curvature flow in $\mathbb{R}_n^{2n}$.

Huang and Wang [12] and Chau et al. [8] have used different methods to investigate the entire solutions to the

above equation and showed that an entire smooth strictly convex solution to (4) in $\mathbb{R}^n$ is the quadratic polynomial under the decay condition on Hessian of f . Later Ding and Xin in [10] improve the previous ones in [8, 12] by removing the additional assumption and prove the following.

Theorem 1. *Any space-like entire graphic self-shrinking solution to Lagrangian mean curvature flow in $\mathbb{R}_n^{2n}$ with the indefinite metric $\sum_i dx_i dy_i$ is flat.*

These rigidity results assume that the self-shrinker graphs are entire. Namely, they are Euclidean complete. Here, we will characterize the rigidity of self-shrinker graphs from another completeness and pose the following problem.

If a graphic self-shrinker is complete with respect to induced metric from ambient space $\mathbb{R}_n^{2n}$, then is it flat?

In this paper, we will use *affine technique* (see [15–18]) to prove the following Bernstein theorem. As a corollary, it gives a partial affirmative answer to the above problem.

Theorem 2. *Let $f(x)$ be a C^∞ strictly convex function defined on a convex domain $\Omega \subseteq \mathbb{R}^n$ satisfying the PDE (4). If there is a positive constant α depending only on n such that the hypersurface $M = \{(x, f(x))\}$ in $\mathbb{R}^{n+1}$ is complete with respect to the metric*

$$\tilde{G} = \exp \left\{ \alpha \sum x_i \frac{\partial f}{\partial x_i} \right\} \sum \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j, \quad (5)$$

then f is the quadratic polynomial.

Remark 3. If $f(x)$ is a strictly convex solution to (4), then the graph $\{(x, \nabla f/2n\alpha)\}$ is a minimal manifold in $\mathbb{R}_n^{2n}$ endowed with the conformal metric $ds^2 = \exp\{-\alpha x \cdot y\} dx \cdot dy$.

As a direct application of Theorem 2, we have the following.

Corollary 4. *Let f be a strictly convex C^∞ -function defined on a convex domain $\Omega \subset \mathbb{R}^n$. If the graph $M_{\nabla f} = \{(x, \nabla f(x))\}$ in $\mathbb{R}_n^{2n}$ is a complete space-like self-shrinker for mean curvature flow and the sum $\sum x_i (\partial f / \partial x_i)$ has a lower bound, then $M_{\nabla f}$ is flat.*

When the shrinker passes through the origin especially, we have the following corollary.

Corollary 5. *If the graph $M_{\nabla f} = \{(x, \nabla f(x))\}$ in $\mathbb{R}_n^{2n}$ is a complete space-like self-shrinker for mean curvature flow and passes through the origin, then $M_{\nabla f}$ is flat.*

2. Preliminaries

Let $f(x_1, \dots, x_n)$ be a strictly convex C^∞ -function defined on a domain $\Omega \subset \mathbb{R}^n$. Consider the graph hypersurface

$$M := \{(x, f(x)) \mid x_{n+1} = f(x_1, \dots, x_n), (x_1, \dots, x_n) \in \Omega\}. \quad (6)$$

For M , we choose the canonical relative normalization $Y = (0, 0, \dots, 1)$. Then, in terms of the language of the relative affine differential geometry, the *Calabi metric*

$$G = \sum f_{ij} dx_i dx_j \quad (7)$$

is the relative metric with respect to the normalization Y . For the position vector $y = (x_1, \dots, x_n, f(x_1, \dots, x_n))$, we have

$$y_{,ij} = \sum A_{ij}^k y_k + f_{ij} Y, \quad (8)$$

where “ $_{,}$ ” denotes the covariant derivative with respect to the Calabi metric G . We recall some fundamental formulas for the graph M ; for details, see [19]. The Levi-Civita connection with respect to the metric G has the *Christoffel symbols*

$$\Gamma_{ij}^k = \frac{1}{2} \sum f^{kl} f_{ijl}. \quad (9)$$

The *Fubini-Pick tensor* A_{ijk} satisfies

$$A_{ijk} = -\frac{1}{2} f_{ijk}. \quad (10)$$

Consequently, for the *relative Pick invariant*, we have

$$J = \frac{1}{4n(n-1)} \sum f^{il} f^{jm} f^{kn} f_{ijk} f_{lmn}. \quad (11)$$

The *Gauss integrability conditions* and the *Codazzi equations* read

$$R_{ijkl} = \sum f^{mh} (A_{jkm} A_{hil} - A_{ikm} A_{hjl}), \quad (12)$$

$$A_{ijk,l} = A_{ijl,k}. \quad (13)$$

From (12), we get the Ricci tensor

$$R_{ik} = \sum f^{mh} f^{lj} (A_{iml} A_{hjk} - A_{imk} A_{hlj}). \quad (14)$$

Introduce the Legendre transformation of f

$$\xi_i = \frac{\partial f}{\partial x_i}, \quad i = 1, 2, \dots, n, \quad (15)$$

$$u(\xi_1, \dots, \xi_n) = \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i} - f(x).$$

Define the functions

$$\rho := \left[\det \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right) \right]^{-1/(n+2)} = \left[\det \left(\frac{\partial^2 u}{\partial \xi_i \partial \xi_j} \right) \right]^{1/(n+2)},$$

$$\Phi := \sum f^{ij} (\ln \rho)_i (\ln \rho)_j = \frac{\|\nabla \rho\|^2}{\rho^2}, \quad (16)$$

here and later the norm $\|\cdot\|$ is defined with respect to the Calabi metric. From the PDE (4), we obtain

$$\frac{\partial \ln \rho}{\partial x_i} = \frac{1}{2(n+2)} (f - u)_i = \frac{1}{2(n+2)} \{f_i - x_k f_{ki}\}. \quad (17)$$

That is,

$$x_i = \sum f^{ik} (f_k - 2(n+2)(\ln \rho)_k). \quad (18)$$

Using (17) and (18), we can get

$$\rho_{ij} = \frac{\rho_i \rho_j}{\rho} + f^{kl} f_{ijk} \rho_l - \frac{f^{kl} f_{ijk} f_l \rho}{2(n+2)}. \quad (19)$$

Put $\tau := (1/2) \sum f^{ij} (\rho_i / \rho) f_j$. From (19), we have

$$\Delta \rho = -\frac{n \|\nabla \rho\|^2}{2\rho} + \tau \rho. \quad (20)$$

By (17), we get

$$4(n+2)^2 \Phi = \|\nabla u\|^2 + \|\nabla f\|^2 - 2(f+u), \quad (21)$$

and then

$$\|\nabla(f+u)\|^2 = 4(n+2)^2 \Phi + 4(f+u). \quad (22)$$

Using (17) yields

$$\Delta(f+u) = 2n + (n+2)^2 \Phi. \quad (23)$$

Define a conformal Riemannian metric $\tilde{G} := \exp\{\alpha(f+u)\}G$, where α is a constant.

Conformal Ricci Curvature. Denote by $\tilde{R}_{ij}$ the Ricci curvature with respect to the metric $\tilde{G}$; then

$$\begin{aligned} \tilde{R}_{ij} = R_{ij} - \frac{(n-2)\alpha}{2}(f+u)_{,ij} + \frac{(n-2)\alpha^2}{4}(f+u)_{,i}(f+u)_{,j} \\ - \frac{1}{2} \left(\alpha \Delta(f+u) + \frac{(n-2)\alpha^2}{2} \|\nabla(f+u)\|^2 \right) G_{ij}, \end{aligned} \quad (24)$$

where ∇ again denotes the covariant derivation with respect to the Calabi metric.

Using the above formulas, we can get the following crucial estimates.

Proposition 6. *Let $f(x_1, \dots, x_n)$ be a C^∞ strictly convex function satisfying PDE (4). Then, the following estimate holds:*

$$\begin{aligned} \Delta \Phi \geq A_1 \langle \nabla \Phi, \nabla \ln \rho \rangle + \frac{1}{4} \langle \nabla(f+u), \nabla \Phi \rangle - A_2 \frac{\|\nabla \Phi\|^2}{\Phi} \\ + A_3 \Phi^2 + \Phi, \end{aligned} \quad (25)$$

where

$$\begin{aligned} A_1 = \frac{6n^2 - n + 16}{(n-1)(3n+4)}, \quad A_2 = \frac{3n^2 + 32n}{8(n-1)(3n+4)}, \\ A_3 = \frac{64n^3 - 72n^2 - 46n - 72}{5n(n-1)(3n+4)}. \end{aligned} \quad (26)$$

Because its calculation is standard as in [16], we will give its proof in the appendix.

For affine hyperspheres, Calabi in [20] calculated the Laplacian of the Pick invariant J . Later, for a general convex function, Li and Xu proved the following lemma in [17].

Lemma 7. *The Laplacian of the relative Pick invariant J satisfies*

$$\begin{aligned} \Delta J \geq \frac{n+2}{n(n-1)} \sum f^{il} f^{jm} f^{kn} A_{ijk} (\ln \rho)_{,lmn} \\ + \frac{2}{n(n-1)} \|\nabla A\|^2 + 2J^2 - \frac{(n+2)^4}{4} \Phi^2, \end{aligned} \quad (27)$$

where ∇ denotes the covariant derivative with respect to the Calabi metric.

Using Lemma 7, we get the following corollary. For the proof, see the appendix.

Corollary 8. *Let $f(x_1, \dots, x_n)$ be a C^∞ strictly convex function satisfying PDE (4); then*

$$\begin{aligned} \Delta J \geq J^2 - 20(n+2)^8 \Phi^2 + \frac{1}{4} \langle \nabla J, \nabla(f+u) \rangle \\ + J - \sqrt{n(n-1)} \|\nabla(f+u)\| J^{3/2}. \end{aligned} \quad (28)$$

3. Proof of Theorem 2

It is our aim to prove $\Phi \equiv 0$; thus, from definition of ρ ,

$$\det(f_{ij}) = \text{const}. \quad (29)$$

everywhere on M . As in [8], by Euler homogeneous theorem, we get Theorem 2.

Denote by $s(p_0, p)$ the geodesic distance function from $p_0 \in M$ with respect to the metric $\tilde{G}$. For any positive number a , let $B_a(p_0, \tilde{G}) := \{p \in M \mid s(p_0, p) \leq a\}$. Denote

$$\begin{aligned} \mathcal{A} &:= \max_{B_a(p_0, \tilde{G})} \left\{ (a^2 - s^2)^2 \exp\{-\alpha(f+u)\} \Phi \right\}, \\ \mathcal{B} &:= \max_{B_a(p_0, \tilde{G})} \left\{ (a^2 - s^2)^2 \exp\{-\alpha(f+u)\} J \right\}. \end{aligned} \quad (30)$$

Lemma 9. *Let f be a strictly convex C^∞ -function satisfying the PDE (4). Then, there exist positive constants α and C , depending only on n , such that*

$$\mathcal{A} \leq C(a^2 + a^3). \quad (31)$$

Proof. Step 1. We will prove that there exists a constant C depending only on n such that

$$\mathcal{A} \leq C(\mathcal{B}^{1/2} a + a^2 + a^3). \quad (32)$$

To this end, consider the function

$$F := (a^2 - s^2)^2 \exp\{-\alpha(f+u)\} \Phi \quad (33)$$

defined on $B_a(p_0, \tilde{G})$, where α is a positive constant to be determined later. Obviously, F attains its supremum at some interior point p^* . We may assume that s^2 is a C^2 -function in a neighborhood of p^* . Choose an orthonormal frame field on M around p^* with respect to the Calabi metric G . Then, at p^* ,

$$\frac{\Phi_{,i}}{\Phi} - \alpha(f+u)_{,i} - \frac{4ss_{,i}}{a^2 - s^2} = 0, \quad (34)$$

$$\frac{\Delta\Phi}{\Phi} - \frac{\sum(\Phi_{,i})^2}{\Phi^2} - \alpha\Delta(f+u) - \frac{12a^2 \exp\{\alpha(f+u)\}}{(a^2 - s^2)^2} \quad (35)$$

$$- \frac{4s\Delta s}{a^2 - s^2} \leq 0,$$

where “ ∇ ” denotes the covariant derivative with respect to the Calabi metric G as before, and we used the fact $\|\nabla s\|_G^2 = \exp\{\alpha(f+u)\}$. Inserting Proposition 6 into (35), we get

$$\begin{aligned} & - (1 + A_2) \frac{\sum(\Phi_{,i})^2}{\Phi^2} + A_3\Phi + \frac{1}{4}(f+u)_{,i} \frac{\Phi_{,i}}{\Phi} \\ & + A_1 \frac{\Phi_{,i}}{\Phi} \frac{\rho_{,i}}{\rho} + 1 - \alpha(2n + (n+2)^2\Phi) \\ & - \frac{12a^2 \exp\{\alpha(f+u)\}}{(a^2 - s^2)^2} - \frac{4s\Delta s}{a^2 - s^2} \leq 0. \end{aligned} \quad (36)$$

Combining (34) with (36) and using the Schwarz inequality, we have

$$\begin{aligned} & \frac{1}{4} \sum (f+u)_{,i} \frac{\Phi_{,i}}{\Phi} \\ & \geq \frac{1}{8} \alpha \sum [(f+u)_{,i}]^2 - \frac{2}{\alpha} \frac{a^2 \exp\{\alpha(f+u)\}}{(a^2 - s^2)^2}, \\ & A_1 \sum \frac{\Phi_{,i}}{\Phi} \frac{\rho_{,i}}{\rho} \geq -\frac{A_3}{4}\Phi - \frac{A_1^2}{A_3} \frac{\Phi^2}{\Phi^2}, \\ & \frac{\sum(\Phi_{,i})^2}{\Phi^2} \leq 2 \left(\alpha^2 \sum [(f+u)_{,i}]^2 + 16 \frac{a^2 \exp\{\alpha(f+u)\}}{(a^2 - s^2)^2} \right). \end{aligned} \quad (37)$$

Choose α small enough such that

$$2 \left(1 + A_2 + \frac{A_1^2}{A_3} \right) \alpha \leq \frac{1}{16}, \quad \alpha(n+2)^2 \leq \frac{A_3}{4}, \quad (38)$$

$$100n\alpha \leq 1.$$

Then, by substituting the three estimates above, we get

$$\begin{aligned} & \frac{A_3}{2}\Phi + \frac{1}{16}\alpha(f+u)_{,i}^2 - \exp\{\alpha(f+u)\} \frac{Ca^2}{(a^2 - s^2)^2} \\ & - \frac{4s\Delta s}{a^2 - s^2} \leq 0, \end{aligned} \quad (39)$$

here and later C denotes positive constant depending only on n .

Denote $a^* = s(p_0, p^*)$. If $a^* = 0$, from (39), it is easy to complete the proof of the lemma. In the following, we assume that $a^* > 0$. Now, we calculate the term $4s\Delta s/(a^2 - s^2)$. Firstly, we will give a lower bound of the Ricci curvature $\text{Ric}(M, \tilde{G})$. Assume that

$$\begin{aligned} & \max_{B_{a^*}(p_0, \tilde{G})} \{\exp\{-\alpha(f+u)\}\Phi\} = \exp\{-\alpha(f+u)\}\Phi(\tilde{p}), \\ & \max_{B_{a^*}(p_0, \tilde{G})} \{\exp\{-\alpha(f+u)\}J\} = \exp\{-\alpha(f+u)\}J(\tilde{q}). \end{aligned} \quad (40)$$

For any $p \in B_{a^*}(p_0, \tilde{G})$, by a coordinate transformation, $f_{ij}(p) = \delta_{ij}$ and $R_{ij}(p) = 0$ hold for $i \neq j$. Then, at p ,

$$\begin{aligned} R_{ii} & \geq \frac{1}{4} \left(\sum_m f_{mii}^2 + (n+2) \sum_m f_{mii} \frac{\partial}{\partial x_m} \ln \rho \right) \geq -\frac{(n+2)^2}{16} \Phi, \\ & \frac{(n-2)\alpha}{2} (f+u)_{,ii} \\ & = \frac{(n-2)\alpha}{2} \left(2 - \frac{1}{2} f_{iik} (f+u)_{,k} \right) \\ & \leq (n-2)\alpha + \frac{(n-2)\alpha^2}{4} \|\nabla(f+u)\|^2 + CJ. \end{aligned} \quad (41)$$

Then, using the Schwarz inequality and (22)–(24), we know that at the point p

$$\begin{aligned} & \text{Ric}(M, \tilde{G}) \\ & \geq -\exp\{-\alpha(f+u)\} \\ & \quad \times \{C\Phi + CJ + \alpha[3(n-2)\alpha(f+u) + 2(n-1)]\} \tilde{G}. \end{aligned} \quad (42)$$

If $3(n-2)\alpha(f+u) + 2(n-1) \leq 0$, then

$$-\exp\{-\alpha(f+u)\} \alpha[3(n-2)\alpha(f+u) + 2(n-1)] \geq 0. \quad (43)$$

Otherwise,

$$\exp\{-\alpha(f+u)\} \alpha[3(n-2)\alpha(f+u) + 2(n-1)] \leq C. \quad (44)$$

Then, the Ricci curvature $\text{Ric}(M, \tilde{G})$ on $B_{a^*}(p_0, \tilde{G})$ is bounded from below by

$$\begin{aligned} & \text{Ric}(M, \tilde{G}) \\ & \geq -C \left(\frac{\Phi}{\exp\{\alpha(f+u)\}}(\tilde{p}) + \frac{J}{\exp\{\alpha(f+u)\}}(\tilde{q}) + 1 \right) \tilde{G}. \end{aligned} \quad (45)$$

By the Laplacian comparison theorem, we get

$$\begin{aligned} & \frac{s\Delta s}{a^2 - s^2} \\ &= \exp\{\alpha(f+u)\} \frac{s\tilde{\Delta}s}{a^2 - s^2} - \frac{(n-2)\alpha}{2} \frac{s(f+u)_{,i}s_{,i}}{a^2 - s^2} \\ &\leq C_3 \frac{\exp\{\alpha(f+u)\}(p^*)}{a^2 - s^2} \\ &\quad \times \left(\sqrt{\exp\{-\alpha(f+u)\}\Phi(\tilde{p})} \right. \\ &\quad \left. + \sqrt{\exp\{-\alpha(f+u)\}J(\tilde{q})+1} \right) s \\ &\quad + \frac{\alpha}{16}(f+u)_{,i}^2 + C \frac{a^2 \exp\{\alpha(f+u)\}}{(a^2 - s^2)^2}, \end{aligned} \quad (46)$$

where $\tilde{\Delta}$ denotes the Laplacian with respect to the metric $\tilde{G}$. Substituting (46) into (39) yields

$$\begin{aligned} & \exp\{-\alpha(f+u)\}\Phi \\ &\leq aC \left(\sqrt{\frac{\Phi}{\exp\{\alpha(f+u)\}}}(\tilde{p}) \right. \\ &\quad \left. + \sqrt{\frac{J}{\exp\{\alpha(f+u)\}}}(\tilde{q})+1 \right) \times (a^2 - s^2)^{-1} \\ &\quad + \frac{Ca^2}{(a^2 - s^2)^2}. \end{aligned} \quad (47)$$

Note that

$$\begin{aligned} \mathcal{A} &\geq \left[(a^2 - s^2)^2 \exp\{-\alpha(f+u)\}\Phi \right](\tilde{p}) \\ &\geq (a^2 - s^2)^2 (p^*) \exp\{-\alpha(f+u)\}(\tilde{p})\Phi(\tilde{p}), \\ \mathcal{B} &\geq \left[(a^2 - s^2)^2 \exp\{-\alpha(f+u)\}J \right](\tilde{q}) \\ &\geq (a^2 - s^2)^2 (p^*) \exp\{-\alpha(f+u)\}(\tilde{q})J(\tilde{q}). \end{aligned} \quad (48)$$

Multiplying by $(a^2 - s^2)^2(p^*)$, at both sides of (47), yields

$$\mathcal{A} \leq Ca(\mathcal{A}^{1/2} + \mathcal{B}^{1/2}) + C(a^2 + a^3). \quad (49)$$

Using the Schwarz inequality, we complete Step 1.

Step 2. We will prove that there is a constant C depending only on n such that

$$\mathcal{B} \leq C(\mathcal{A} + a^2 + a^4). \quad (50)$$

Consider

$$H = (a^2 - s^2)^2 \exp\{-\alpha(f+u)\}J \quad (51)$$

defined on $B_a(p_0, \tilde{G})$, where α is the constant in (38). Obviously, H attains its supremum at some interior point q^* . Choose an orthonormal frame field on M around q^* with respect to the Calabi metric G . Then, at q^* ,

$$\frac{J_{,i}}{J} - \alpha(f+u)_{,i} - \frac{4ss_{,i}}{a^2 - s^2} = 0, \quad (52)$$

$$\begin{aligned} & \frac{\Delta J}{J} - \frac{\sum (J_{,i})^2}{J^2} - \alpha\Delta(f+u) - \frac{12a^2 \exp\{\alpha(f+u)\}}{(a^2 - s^2)^2} \\ & - \frac{4s\Delta s}{a^2 - s^2} \leq 0, \end{aligned} \quad (53)$$

where ∇ denotes the covariant derivative with respect to the Calabi metric G as before. Inserting Corollary 8 into (53), we get

$$\begin{aligned} & J - 20(n+2)^8 \frac{\Phi^2}{J} + \frac{1}{4} \sum \frac{J_{,i}}{J} (f+u)_{,i} + 1 - \frac{\sum (J_{,i})^2}{J^2} \\ & - \alpha(2n + (n+2)^2\Phi) - \sqrt{n(n-1)} \|\nabla(f+u)\| J^{1/2} \\ & - \frac{12a^2 \exp\{\alpha(f+u)\}}{(a^2 - s^2)^2} - \frac{4s\Delta s}{a^2 - s^2} \leq 0. \end{aligned} \quad (54)$$

Applying the Schwarz inequality, we have

$$\begin{aligned} & \frac{1}{4} \sum \frac{J_{,i}}{J} (f+u)_{,i} \geq \frac{\alpha}{8} \sum [(f+u)_{,i}]^2 - C \frac{a^2 \exp\{\alpha(f+u)\}}{(a^2 - s^2)^2}, \\ & \sum \frac{(J_{,i})^2}{J^2} \leq 2\alpha^2 \sum [(f+u)_{,i}]^2 + \frac{32a^2 \exp\{\alpha(f+u)\}}{(a^2 - s^2)^2}, \\ & \sqrt{n(n-1)} \|\nabla(f+u)\| J^{1/2} \\ & \leq \frac{J}{4} + 4n(n-1)((n+2)^2\Phi + (f+u)). \end{aligned} \quad (55)$$

Inserting these estimates into (54) yields

$$\begin{aligned} & \frac{3}{4}J - 20(n+2)^8 \frac{\Phi^2}{J} - C\Phi + \frac{\alpha}{16} \sum (f+u)_{,i}^2 \\ & - C \frac{a^2 \exp\{\alpha(f+u)\}}{(a^2 - s^2)^2} - C(f+u) - \frac{4s\Delta s}{a^2 - s^2} \leq 0, \end{aligned} \quad (56)$$

here and later C denotes different positive constants depending only on n .

We discuss two subcases.

Case 1. If

$$\frac{J}{\exp\{\alpha(f+u)\}}(q^*) \leq \frac{\Phi}{\exp\{\alpha(f+u)\}}(q^*), \quad (57)$$

then $\mathcal{B} \leq \mathcal{A}$. In this case, Step 2 is complete.

Case 2. Now, assume that

$$\frac{J}{\exp\{\alpha(f+u)\}}(q^*) > \frac{\Phi}{\exp\{\alpha(f+u)\}}(q^*). \quad (58)$$

Then, $1 > (\Phi/J)(q^*)$. Thus,

$$\begin{aligned} & \frac{3}{4}J - C\Phi + \frac{\alpha}{16} \sum [(f+u)_{,i}]^2 - C \frac{a^2 \exp\{\alpha(f+u)\}}{(a^2 - s^2)^2} \\ & - C(f+u) - \frac{4s\Delta s}{a^2 - s^2} \leq 0. \end{aligned} \quad (59)$$

The rest of the estimate is almost the same as in Step 1. The only difference is to deal with the term $(f+u)$. If $(f+u)(q^*) \leq 0$, then $-C(f+u)(q^*) \geq 0$. We can drop this term.

Otherwise, $\exp\{-\alpha(f+u)\}(f+u)$ has a uniform upper bound.

Using the same method as in Step 1, we can estimate the term $4s\Delta s/(a^2 - s^2)$ and finally get

$$\mathcal{B} \leq C(\mathcal{A} + a^2 + a^4). \quad (60)$$

Then, combining the conclusion of Step 1, we get

$$\mathcal{A} \leq C(a^2 + a^3). \quad (61)$$

This completes the proof of Lemma 9. $\square$

Proof of Theorem 2. For any point $q \in M$, choose sufficient large constant R_0 such that $q \in B_{R_0}(p_0, \bar{G})$. Then, for all $a \geq R_0$, $q \in B_a(p_0)$. Using Lemma 9, we know

$$\exp\{-\alpha(f+u)\} \Phi(q) \leq \frac{C(n)(a^2 + a^3)}{(a^2 - s^2)^2}. \quad (62)$$

Now, let $a \rightarrow +\infty$, and we have

$$0 \leq \exp\{-\alpha(f+u)\} \Phi(q) \leq 0. \quad (63)$$

Consequently,

$$\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)(q) = \text{const}. \quad (64)$$

This completes the proof of Theorem 2. $\square$

4. Appendix

Proof of Proposition 6. Let $p \in M$, and we choose a local orthonormal frame field of the metric G around p . Then,

$$\begin{aligned} \Phi &= \frac{\sum (\rho_{,j})^2}{\rho^2}, \quad \Phi_{,i} = 2 \sum \frac{\rho_{,j} \rho_{,ji}}{\rho^2} - 2\rho_{,i} \frac{\sum (\rho_{,j})^2}{\rho^3}, \\ \Delta\Phi &= 2 \frac{\sum (\rho_{,ji})^2}{\rho^2} + 2 \sum \frac{\rho_{,j} \rho_{,jii}}{\rho^2} - 8 \sum \frac{\rho_{,j} \rho_{,ji} \rho_{,ji}}{\rho^3} \\ &\quad + (n+6)\Phi^2 - 2\tau\Phi, \end{aligned} \quad (65)$$

where we used (20). In the case $\Phi(p) = 0$, it is easy to get, at p ,

$$\Delta\Phi \geq 2 \frac{\sum (\rho_{,ij})^2}{\rho^2}. \quad (66)$$

Now, we assume that $\Phi(p) \neq 0$. Choose a local orthonormal frame field of the metric G around p such that $\rho_{,1}(p) = \|\nabla\rho\|(p) > 0$, $\rho_{,i}(p) = 0$, for all $i > 1$. Then,

$$\begin{aligned} \Delta\Phi &= 2(1 - \delta + \delta) \sum \frac{(\rho_{,ij})^2}{\rho^2} + 2 \sum \frac{\rho_{,j} \rho_{,jii}}{\rho^2} \\ &\quad - 8 \frac{(\rho_{,1})^2 \rho_{,11}}{\rho^3} + (n+6)\Phi^2 - 2\tau\Phi, \end{aligned} \quad (67)$$

where $1 > \delta > 0$ is a constant to be determined later. Applying (20), we obtain

$$\begin{aligned} 2 \frac{\sum (\rho_{,ij})^2}{\rho^2} &\geq \frac{2n}{n-1} \frac{(\rho_{,11})^2}{\rho^2} + 4 \frac{\sum_{i>1} (\rho_{,1i})^2}{\rho^2} \\ &\quad + \frac{2n}{n-1} \frac{(\rho_{,1})^2 \rho_{,11}}{\rho^3} + \frac{n^2}{2(n-1)} \Phi^2 \\ &\quad - \frac{4}{n-1} \frac{\rho_{,11}}{\rho} \tau + \frac{2}{n-1} \tau^2 - \frac{2n}{n-1} \Phi \tau. \end{aligned} \quad (68)$$

An application of the Ricci identity shows that

$$\begin{aligned} \frac{2}{\rho^2} \sum \rho_{,j} \rho_{,jii} &= -2n \frac{(\rho_{,1})^2 \rho_{,11}}{\rho^3} + n \frac{(\rho_{,1})^4}{\rho^4} \\ &\quad + 2R_{11} \frac{(\rho_{,1})^2}{\rho^2} + 2 \frac{\rho_{,1}}{\rho^2} (\rho\tau)_{,1}. \end{aligned} \quad (69)$$

Substituting (68) and (69) into (67), we obtain

$$\begin{aligned} \Delta\Phi &\geq 2\delta \sum \frac{(\rho_{,ij})^2}{\rho^2} + \left(-2n - 8 + \frac{2n(1-\delta)}{n-1}\right) \\ &\quad \times \frac{(\rho_{,1})^2 \rho_{,11}}{\rho^3} + 2R_{11} \frac{(\rho_{,1})^2}{\rho^2} \\ &\quad + \left(\frac{n^2(1-\delta)}{2(n-1)} + 2(n+3)\right) \frac{(\rho_{,1})^4}{\rho^4} \\ &\quad + 2 \frac{\rho_{,1}}{\rho^2} (\rho\tau)_{,1} - \frac{4n-2-2n\delta}{n-1} \Phi\tau + (1-\delta) \\ &\quad \times \left(\frac{2n}{n-1} \frac{(\rho_{,11})^2}{\rho^2} + 4 \frac{\sum_{i>1} (\rho_{,1i})^2}{\rho^2} \right. \\ &\quad \left. - \frac{4}{n-1} \frac{\rho_{,11}}{\rho} \tau + \frac{2}{n-1} \tau^2\right). \end{aligned} \quad (70)$$

Note that

$$\frac{(\rho_{,11})^2}{\rho^2} = \frac{1}{4} \sum \frac{(\Phi_{,i})^2}{\Phi} - \frac{\sum_{i>1} (\rho_{,1i})^2}{\rho^2} + 2 \frac{(\rho_{,1})^2 \rho_{,11}}{\rho^3} - \frac{(\rho_{,1})^4}{\rho^4}. \quad (71)$$

Then, (70) and (71) together give us

$$\begin{aligned} \Delta\Phi \geq & 2\delta \sum \frac{(\rho_{ij})^2}{\rho^2} + \frac{n(1-\delta)}{2(n-1)} \frac{\sum (\Phi_{,i})^2}{\Phi} \\ & + \left(\frac{6n(1-\delta)}{n-1} - 2(n+4) \right) \frac{(\rho_{,1})^2 \rho_{,11}}{\rho^3} + 2R_{11} \frac{(\rho_{,1})^2}{\rho^2} \\ & + \left[\frac{(n^2-4n)(1-\delta)}{2(n-1)} + 2(n+3) \right] \frac{(\rho_{,1})^4}{\rho^4} \\ & + \frac{1-\delta}{n-1} \left(2\tau^2 - 4 \frac{\rho_{,11}}{\rho} \tau \right) - \frac{4n-2-2n\delta}{n-1} \Phi\tau \\ & + 2 \frac{\rho_{,1}}{\rho^2} (\rho\tau)_{,1}. \end{aligned} \quad (72)$$

Using the Schwarz inequality gives

$$2 \frac{\rho_{,11}}{\rho} \tau \leq \frac{7}{3} \sum \frac{(\rho_{ij})^2}{\rho^2} + \frac{3}{7} \tau^2. \quad (73)$$

Using

$$\frac{(\rho_{,1})^2 \rho_{,11}}{\rho^3} = \frac{1}{2} \Phi_{,i} \frac{\rho_{,i}}{\rho} + \Phi^2, \quad (74)$$

and choosing $\delta = 7/(3n+4)$, we get

$$\begin{aligned} \Delta\Phi \geq & \frac{n(1-\delta)}{2(n-1)} \frac{\sum (\Phi_{,i})^2}{\Phi} + \left(\frac{3n(1-\delta)}{n-1} - (n+4) \right) \\ & \times \sum \Phi_{,i} \frac{\rho_{,i}}{\rho} + 2R_{11} \frac{(\rho_{,1})^2}{\rho^2} \\ & + \left[\frac{(n^2+8n)(1-\delta)}{2(n-1)} - 2 \right] \Phi^2 + \frac{8(1-\delta)}{7(n-1)} \tau^2 \\ & - \frac{4n-2-2n\delta}{n-1} \Phi\tau + 2 \frac{\rho_{,1}}{\rho^2} (\rho\tau)_{,1}. \end{aligned} \quad (75)$$

In the following, we will calculate the terms $R_{11}((\rho_{,1})^2/\rho^2)$ and $(\rho_{,1}/\rho^2)(\rho\tau)_{,1}$. Note that (17) is invariant under an affine transformation of coordinates that preserved the origin. So, we can choose the coordinates $x_1, x_2, \dots, x_n$ such that $f_{ij}(p) = \delta_{ij}$ and $\partial\rho/\partial x_1 = \|\text{grad}\rho\|(p) > 0$, $(\partial\rho/\partial x_i)(p) = 0$, for all $i > 1$. From (19), we easily obtain

$$\rho_{,ij} = \rho_{ij} + A_{ij1}\rho_{,1} = \frac{\rho_{,i}\rho_{,j}}{\rho} - A_{ij1}\rho_{,1} + \frac{A_{ijk}f_k\rho}{n+2}. \quad (76)$$

Thus, we get

$$\Phi_{,i} = \frac{2\rho_{,1}\rho_{,1i}}{\rho^2} - 2 \frac{\rho_{,i}(\rho_{,1})^2}{\rho^3} = -2A_{i11} \frac{(\rho_{,1})^2}{\rho^2} + 2 \frac{\rho_{,1}f_k A_{ki1}}{(n+2)\rho}, \quad (77)$$

$$\sum \Phi_{,i} \frac{\rho_{,i}}{\rho} = -2A_{111} \frac{(\rho_{,1})^3}{\rho^3} + 2 \frac{f_k A_{k11}}{n+2} \frac{(\rho_{,1})^2}{\rho^2}. \quad (78)$$

By the same method, as deriving (69), we have

$$\begin{aligned} \sum (A_{m11})^2 \geq & \frac{n}{n-1} \sum (A_{i11})^2 - \frac{2}{n-1} A_{111} \sum A_{i11} \\ & + \frac{1}{n-1} (\sum A_{i11})^2. \end{aligned} \quad (79)$$

Note that $\sum A_{i11} = ((n+2)/2)(\rho_{,1}/\rho)$. Therefore, by (14), (77), (78), and (79), we obtain

$$\begin{aligned} 2R_{11} \frac{(\rho_{,1})^2}{\rho^2} &= 2 \sum (A_{kj1})^2 \frac{(\rho_{,1})^2}{\rho^2} - (n+2) A_{111} \frac{(\rho_{,1})^3}{\rho^3} \\ &\geq \frac{n}{2(n-1)} \frac{\sum (\Phi_{,i} - 2(\rho_{,1}f_k A_{ki1}/(n+2)\rho))^2}{\Phi} \\ &\quad + \frac{(n+2)(n+1)}{2(n-1)} \sum \Phi_{,i} \frac{\rho_{,i}}{\rho} \\ &\quad - \frac{n+1}{n-1} f_k A_{k11} \frac{(\rho_{,1})^2}{\rho^2} + \frac{(n+2)^2}{2(n-1)} \Phi^2. \end{aligned} \quad (80)$$

On the other hand, we have

$$2 \frac{\rho_{,1}}{\rho^2} (\rho\tau)_{,1} = 2\Phi\tau + \frac{1}{n+2} \sum A_{lik} f_k f_i \frac{\rho_{,1}}{\rho} + \Phi. \quad (81)$$

Then, inserting (80) and (81) into (75), we get

$$\begin{aligned} \Delta\Phi \geq & \frac{2n-n\delta}{2(n-1)} \sum \frac{(\Phi_{,i})^2}{\Phi} - \frac{(n+2)(n-5)+6n\delta}{2(n-1)} \\ & \times \sum \Phi_{,i} \frac{\rho_{,i}}{\rho} + \Phi + \frac{2(n+2)^2 - (n^2+8n)\delta}{2(n-1)} \Phi^2 \\ & + \frac{8(1-\delta)}{7(n-1)} \tau^2 - \frac{2n(1-\delta)}{n-1} \Phi\tau + \frac{1}{n+2} \\ & \times \sum A_{lik} f_k f_i \frac{\rho_{,1}}{\rho} - \frac{n+1}{n-1} \frac{(\rho_{,1})^2}{\rho^2} f_k A_{k11} \\ & - \frac{2n}{(n-1)(n+2)} \frac{\sum \Phi_{,i} f_k A_{ki1}}{\sqrt{\Phi}} + \frac{2n}{(n-1)(n+2)^2} \\ & \times \sum (f_k A_{ki1})^2. \end{aligned} \quad (82)$$

Using (77), we have

$$\begin{aligned} \frac{1}{n+2} \sum A_{lik} f_k f_i \frac{\rho_{,1}}{\rho} &- \frac{n+1}{n-1} \frac{(\rho_{,1})^2}{\rho^2} f_k A_{k11} \\ &= \frac{1}{2} f_i \Phi_{,i} - \frac{2}{n-1} \frac{(\rho_{,1})^2}{\rho^2} f_k A_{k11}. \end{aligned} \quad (83)$$

One observes that the Schwarz inequality gives

$$\begin{aligned}
 & \frac{2n}{(n-1)(n+2)} \frac{\sum \Phi_{,i} f_k A_{ki1}}{\sqrt{\Phi}} \\
 & \leq \frac{9n}{8(n-1)} \sum \frac{(\Phi_{,i})^2}{\Phi} + \frac{8n}{9(n-1)(n+2)^2} \\
 & \quad \times \sum (f_k A_{ki1})^2, \\
 & \frac{2}{n-1} \frac{(\rho_{,1})^2}{\rho^2} f_k A_{k11} \\
 & \leq \frac{9(n+2)^2}{10n(n-1)} \Phi^2 + \frac{10n}{9(n-1)(n+2)^2} \\
 & \quad \times \sum (f_k A_{ki1})^2, \\
 & 2n\Phi\tau \leq \tau^2 + n^2\Phi^2.
 \end{aligned} \tag{84}$$

Note that by (17) we have

$$\begin{aligned}
 \frac{1}{4} f^{ij} \Phi_{,j} f_{,i} &= \frac{n+2}{2} f^{ij} \Phi_{,j} (\ln \rho)_{,i} + \frac{1}{4} \Phi_{,j} x_j \\
 &= \frac{n+2}{2} f^{ij} \Phi_{,j} (\ln \rho)_{,i} + \frac{1}{4} f^{ij} \frac{\partial \Phi}{\partial x_i} \frac{\partial u}{\partial x_j}.
 \end{aligned} \tag{85}$$

Then, inserting these estimates into (82) yields Proposition 6. $\square$

Proof of Corollary 8. Now, we will calculate the term $(\ln \rho)_{,ijk}$. In particular, if f satisfies PDE (4), choose the coordinate $(x_1, x_2, \dots, x_n)$ such that $f_{ij}(p) = \delta_{ij}$; then we have

$$\begin{aligned}
 (\ln \rho)_{,ijk} &= \frac{1}{n+2} (A_{ijk} + A_{ijk,p} f_{,p}) - (\ln \rho)_{,l} A_{ijk,l} \\
 &\quad + A_{ijl} A_{klp} \left(3(\ln \rho)_{,p} - \frac{2}{n+2} f_{,p} \right).
 \end{aligned} \tag{86}$$

Using (17), we have

$$3(\ln \rho)_{,p} - \frac{2}{n+2} f_{,p} = -\frac{1}{n+2} (f+u)_{,p} + (\ln \rho)_{,p}. \tag{87}$$

By the Young inequality and the Schwarz inequality, we have

$$\begin{aligned}
 & \frac{n+2}{n(n-1)} \sum A_{ijk} A_{ijl} A_{klh} (\ln \rho)_{,h} \\
 & \leq \frac{1}{2} J^2 + 16n^2(n-1)^2(n+2)^4 \Phi^2, \\
 & \frac{1}{n(n-1)} \sum A_{ijk} A_{ijl,k} f_l \\
 & = \frac{1}{2} \sum J_{,l} f_{,l} = \frac{1}{4} \langle \nabla J, \nabla (f+u) \rangle + \frac{n+2}{2} \langle \nabla J, \nabla \ln \rho \rangle,
 \end{aligned}$$

$$\begin{aligned}
 & \frac{n+2}{n(n-1)} \sum A_{ijk} A_{ijl,k} (\ln \rho)_{,l} \\
 & = \frac{n+2}{2} \sum J_{,i} (\ln \rho)_{,i} \\
 & \leq \frac{1}{n(n-1)} \sum (A_{ijk,l})^2 + \frac{(n+2)^2}{4} J \Phi \\
 & \leq \frac{1}{n(n-1)} \sum (A_{ijk,l})^2 + \frac{1}{4} J^2 + \frac{(n+2)^4}{16} \Phi^2, \\
 & \frac{1}{n(n-1)} \sum A_{ijk} A_{jil} A_{klp} (f+u)_{,p} \\
 & \leq \sqrt{n(n-1)} \|\nabla (f+u)\| J^{3/2}.
 \end{aligned} \tag{88}$$

Thus, by inserting (88) into Lemma 7, we obtain Corollary 8. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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