Research Article

On New p -Valent Meromorphic Function Involving Certain Differential and Integral Operators

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We define new subclasses of meromorphic p -valent functions by using certain differential operator. Combining the differential operator and certain integral operator, we introduce a general p -valent meromorphic function. Then we prove the sufficient conditions for the function in order to be in the new subclasses.

1. Introduction

Let Σ_p denote the class of meromorphic functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, \dots\}), \quad (1)$$

which are analytic and p -valent in the punctured unit disc:

$$\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U} - \{0\}. \quad (2)$$

A function $f \in \Sigma_p$ is said to be in the class $\Sigma_p^*(\delta)$ of meromorphic p -valent starlike of order δ ($0 \leq \delta < p$) if it satisfies the following inequality:

$$-\Re \left(\frac{zf'(z)}{f(z)} \right) > \delta. \quad (3)$$

For $f \in \Sigma_p$, Saif and Kılıçman [1] introduced the linear operator $\mathcal{D}_\lambda^k$, as follows:

$$\begin{aligned} \mathcal{D}_\lambda f(z) &= (1 + p\lambda) f(z) + \lambda z f'(z), \quad \lambda \geq 0, \\ \mathcal{D}_\lambda^0 f(z) &= f(z), \\ \mathcal{D}_\lambda^1 f(z) &= \mathcal{D}_\lambda f(z), \\ \mathcal{D}_\lambda^2 f(z) &= \mathcal{D}_\lambda (\mathcal{D}_\lambda^1 f(z)), \end{aligned} \quad (4)$$

and in general, for $k = 0, 1, 2, \dots$, we can write

$$\begin{aligned} \mathcal{D}_\lambda^k f(z) &= \frac{1}{z^p} + \sum_{n=p+1}^{\infty} (1 + p\lambda + n\lambda)^k a_n z^n, \\ (k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, p \in \mathbb{N}). \end{aligned} \quad (5)$$

It is easy to see that, for $f \in \Sigma_p$, we have

$$\begin{aligned} \lambda z (\mathcal{D}_\lambda^k f(z))' &= \mathcal{D}_\lambda^{k+1} f(z) - (1 + p\lambda) \mathcal{D}_\lambda^k f(z), \\ (k \in \mathbb{N}_0, p \in \mathbb{N}). \end{aligned} \quad (6)$$

Meromorphically multivalent functions have been extensively studied by several authors; see, for example, Uralegaddi and Somanatha [2, 3], Liu and Srivastava [4, 5], Mogra [6, 7], Srivastava et al. [8], Aouf et al. [9, 10], Joshi and Srivastava [11], Owa et al. [12], and Kulkarni et al. [13].

Now, for $f \in \Sigma_p$, we define the following new subclasses.

Definition 1. Let a function $f \in \Sigma_p$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma_p S_k(\delta, b, \lambda)$ if, and only if, f satisfies

$$\Re \left\{ p - \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{k+1} f(z)}{\mathcal{D}_\lambda^k f(z)} - 1 \right) \right\} > \delta, \quad (7)$$

where $\delta \in [0, p)$, $b \in \mathbb{C} \setminus \{0\}$, $\lambda \geq 0$, $k \in \mathbb{N}_0$.

From (6), one can see that (7) is equivalent to

$$\Re \left\{ p - \frac{\lambda}{b} \left(\frac{z(\mathcal{D}_\lambda^k f(z))'}{\mathcal{D}_\lambda^k f(z)} + p \right) \right\} > \delta. \quad (8)$$

Remark 2. In Definition 1, if we set

- (i) $k = 0$ and $p = \lambda = 1$, then we have [14, Definition 1.1];
- (ii) $k = 0$ and $p = \lambda = b = 1$, then we have $\Sigma_p^*(\delta)$, the class of meromorphic p -valent starlike of order δ ;
- (iii) $k = 1$ and $p = \lambda = 1$, then we have [14, Definition 1.7].

Definition 3. Let a function $f \in \Sigma_p$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma_p US_k(\alpha, \delta, b, \lambda)$ if, and only if, f satisfies

$$\Re \left\{ p - \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{k+1} f(z)}{\mathcal{D}_\lambda^k f(z)} - 1 \right) \right\} > \alpha \left| \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{k+1} f(z)}{\mathcal{D}_\lambda^k f(z)} - 1 \right) \right| + \delta, \quad (9)$$

where $\alpha \geq 0$, $\delta \in [-1, p)$, $b \in \mathbb{C} \setminus \{0\}$, $\lambda \geq 0$, $k \in \mathbb{N}_0$.

Inequality (9) is equivalent to

$$\Re \left\{ p - \frac{\lambda}{b} \left(\frac{z(\mathcal{D}_\lambda^k f(z))'}{\mathcal{D}_\lambda^k f(z)} + p \right) \right\} > \alpha \left| \frac{\lambda}{b} \left(\frac{z(\mathcal{D}_\lambda^k f(z))'}{\mathcal{D}_\lambda^k f(z)} + p \right) \right| + \delta. \quad (10)$$

Remark 4. In Definition 3, if we set

- (i) $k = 0$ and $p = \lambda = 1$, then we have [14, Definition 1.3];
- (ii) for $k = 1$ and $p = \lambda = 1$, then we have [14, Definition 1.8].

Definition 5. Let a function $f \in \Sigma_p$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma_p SH_k(\alpha, b, \lambda)$, if, and only if, f satisfies

$$\left| p - \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{k+1} f(z)}{\mathcal{D}_\lambda^k f(z)} - 1 \right) - 2\alpha(\sqrt{2} - 1) \right| < \sqrt{2} \Re \left\{ p - \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{k+1} f(z)}{\mathcal{D}_\lambda^k f(z)} - 1 \right) \right\} + 2\alpha(\sqrt{2} - 1), \quad (11)$$

where $\alpha > 0$, $b \in \mathbb{C} \setminus \{0\}$, $\lambda \geq 0$, $k \in \mathbb{N}_0$.

Inequality (11) is equivalent to

$$\left| p - \frac{\lambda}{b} \left(\frac{z(\mathcal{D}_\lambda^k f(z))'}{\mathcal{D}_\lambda^k f(z)} + p \right) - 2\alpha(\sqrt{2} - 1) \right| < \sqrt{2} \Re \left\{ p - \frac{\lambda}{b} \left(\frac{z(\mathcal{D}_\lambda^k f(z))'}{\mathcal{D}_\lambda^k f(z)} + p \right) \right\} + 2\alpha(\sqrt{2} - 1). \quad (12)$$

Remark 6. In Definition 5, if we set

- (i) $k = 0$ and $p = \lambda = 1$, then we have [14, Definition 1.5];
- (ii) for $k = 1$ and $p = \lambda = 1$, then we have [14, Definition 1.9].

Recently, Mohammed and Darus [15] introduced the following p -valent meromorphic function:

$$G(z) = z \mathcal{F}'_{p, \gamma_1, \dots, \gamma_n}(z) + (p+1) \mathcal{F}_{p, \gamma_1, \dots, \gamma_n}(z), \quad (13)$$

where $\mathcal{F}_{p, \gamma_1, \dots, \gamma_n}$ is the integral operator introduced and studied by the authors [15, 16] and defined by

$$\mathcal{F}_{p, \gamma_1, \dots, \gamma_n}(z) = \frac{1}{z^{p+1}} \int_0^z (u^p f_1(u))^{\gamma_1} \cdots (u^p f_n(u))^{\gamma_n} du, \quad (14)$$

where

$$n, p \in \mathbb{N}, \quad j \in \{1, 2, 3, \dots, n\}, \quad \gamma_j > 0. \quad (15)$$

For $p = 1$ we obtain [17]. It is clear that

$$G(z) = \frac{1}{z^p} (z^p f_1(z))^{\gamma_1} \cdots (z^p f_n(z))^{\gamma_n}. \quad (16)$$

By using the differential operator given by (4), we introduce the following p -valent meromorphic function.

Definition 7. Let $k \in \mathbb{N}_0$, $l = (l_1, \dots, l_n) \in \mathbb{N}_0^n$ and $\gamma_j > 0$, $1 \leq j \leq n$. One defines the p -valent meromorphic function $I_{k, n, l, \gamma}: \Sigma_p^n \rightarrow \Sigma_p$,

$$I_{k, n, l, \gamma}(f_1, \dots, f_n) = \Phi, \quad (17)$$

$$\mathcal{D}_\lambda^k \Phi(z) = \frac{1}{z^p} \left[(z^p \mathcal{D}_\lambda^{l_1} f_1(z))^{\gamma_1} \cdots (z^p \mathcal{D}_\lambda^{l_n} f_n(z))^{\gamma_n} \right], \quad (18)$$

where $f_1, \dots, f_n \in \Sigma_p$, and $\mathcal{D}_\lambda$ is the differential operator given by (4).

Remark 8. If we set $\lambda = 1$, $k = 0$, and $l_1 = \dots = l_n = 0$, then we have the p -valent meromorphic function given by (13).

2. Main Results

To prove our main results, we need the following lemma.

Lemma 9. For the p -valent meromorphic function $I_{k, n, l, \gamma}(f_1, \dots, f_n) = \Phi$ given by (18), one has

$$-\frac{\lambda z(\mathcal{D}_\lambda^k \Phi(z))'}{\mathcal{D}_\lambda^k \Phi(z)} = -\sum_{j=1}^n \gamma_j \frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} + p\lambda + \sum_{j=1}^n \gamma_j. \quad (19)$$

Proof. From (18), we have

$$z^p \mathcal{D}_\lambda^k \Phi(z) = \left[(z^p \mathcal{D}_\lambda^{l_1} f_1(z))^{\gamma_1} \cdots (z^p \mathcal{D}_\lambda^{l_n} f_n(z))^{\gamma_n} \right]. \quad (20)$$

Differentiating (20) logarithmically and then by simple computation, we get

$$\frac{z(\mathcal{D}_\lambda^k \Phi(z))'}{\mathcal{D}_\lambda^k \Phi(z)} = \sum_{j=1}^n \gamma_j \left(\frac{z(\mathcal{D}_\lambda^{l_j} f_j(z))' + p \mathcal{D}_\lambda^{l_j} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} \right) - p. \quad (21)$$

From (6), we obtain

$$(\mathcal{D}_\lambda^{l_j} f_j(z))' = \frac{\mathcal{D}_\lambda^{l_j+1} f_j(z) - (1 + p\lambda) \mathcal{D}_\lambda^{l_j} f_j(z)}{\lambda z}. \quad (22)$$

Then using (22) on the right-hand side of (21), one gets

$$\frac{z(\mathcal{D}_\lambda^k \Phi(z))'}{\mathcal{D}_\lambda^k \Phi(z)} = \sum_{j=1}^n \gamma_j \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\lambda \mathcal{D}_\lambda^{l_j} f_j(z)} - \frac{1}{\lambda} \right) - p. \quad (23)$$

Multiplying (23) by λ yields that

$$\frac{\lambda z(\mathcal{D}_\lambda^k \Phi(z))'}{\mathcal{D}_\lambda^k \Phi(z)} = \sum_{j=1}^n \gamma_j \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} - 1 \right) - p\lambda, \quad (24)$$

or, equivalently, we can write that

$$-\frac{\lambda z(\mathcal{D}_\lambda^k \Phi(z))'}{\mathcal{D}_\lambda^k \Phi(z)} = -\sum_{j=1}^n \gamma_j \frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} + p\lambda + \sum_{j=1}^n \gamma_j, \quad (25)$$

which is the desired result. $\square$

Our first theorem is as follows.

Theorem 10. Let $\alpha_j \geq 0$, $\delta_j \in [-1, p)$, $\alpha_j + \delta_j \geq 0$, $(1 \leq j \leq n)$ and $b \in \mathbb{C} \setminus \{0\}$, $\lambda \geq 0$. Suppose that

$$\sum_{j=1}^n \gamma_j \left(\frac{p - \delta_j}{\alpha_j + 1} \right) \leq p. \quad (26)$$

If $f_j \in \Sigma_p US_{l_j}(\alpha_j, \delta_j, b, \lambda)$ ($1 \leq j \leq n$), then the function $\mathcal{D}_\lambda^k \Phi(z)$ defined by (18) is in the class $\Sigma_p S_k(\mu, b, \lambda)$, where

$$\mu = p - \sum_{j=1}^n \gamma_j \left(\frac{p - \delta_j}{\alpha_j + 1} \right). \quad (27)$$

Proof. Since $f_j \in \Sigma_p US_{l_j}(\alpha_j, \delta_j, b, \lambda)$ ($1 \leq j \leq n$), by (9), we have

$$\Re \left\{ p - \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} - 1 \right) \right\} > \frac{p\alpha_j + \delta_j}{1 + \alpha_j}. \quad (28)$$

By (19), we get

$$-\frac{\lambda z(\mathcal{D}_\lambda^k \Phi(z))'}{\mathcal{D}_\lambda^k \Phi(z)} - p\lambda = -\sum_{j=1}^n \gamma_j \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} - 1 \right). \quad (29)$$

This is equivalent to

$$\begin{aligned} & p - \frac{\lambda}{b} \left(\frac{z(\mathcal{D}_\lambda^k \Phi(z))'}{\mathcal{D}_\lambda^k \Phi(z)} + p \right) \\ &= p - \frac{1}{b} \sum_{j=1}^n \gamma_j \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} - 1 \right) \\ &= \sum_{j=1}^n \gamma_j \left[p - \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} - 1 \right) \right] + p - p \sum_{j=1}^n \gamma_j. \end{aligned} \quad (30)$$

From (28) together with (30), we can get

$$\begin{aligned} & \Re \left\{ p - \frac{\lambda}{b} \left(\frac{z(\mathcal{D}_\lambda^k \Phi(z))'}{\mathcal{D}_\lambda^k \Phi(z)} + p \right) \right\} \\ &= \sum_{j=1}^n \gamma_j \Re \left[p - \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} - 1 \right) \right] \\ &+ p - p \sum_{j=1}^n \gamma_j \\ &> \sum_{j=1}^n \gamma_j \left(\frac{p\alpha_j + \delta_j}{1 + \alpha_j} \right) - p \sum_{j=1}^n \gamma_j + p \\ &= p - \sum_{j=1}^n \gamma_j \left(\frac{p - \delta_j}{1 + \alpha_j} \right). \end{aligned} \quad (31)$$

Hence, we obtain $\mathcal{D}_\lambda^k \Phi(z) \in \Sigma_p S_k(\mu, b, \lambda)$, where $\mu = p - \sum_{j=1}^n \gamma_j ((p - \delta_j)/(\alpha_j + 1))$. $\square$

Corollary 11. Let $\alpha_j \geq 0$, $\delta_j \in [-1, p)$, $\alpha_j + \delta_j \geq 0$, $(1 \leq j \leq n)$, and $b \in \mathbb{C} \setminus \{0\}$, $\lambda \geq 0$. Suppose that

$$\sum_{j=1}^n \gamma_j \left(\frac{p - \delta_j}{\alpha_j + 1} \right) \leq p. \quad (32)$$

If $f_j \in \Sigma_p US_{l_j}(\alpha_j, \delta_j, b, 1)$ ($1 \leq j \leq n$), then the function $\mathcal{D}_\lambda^k \Phi(z)$, defined by (18), is in the class $\Sigma_p S_{k+1}(\mu, b, 1)$, where μ is defined as in (27).

Proof. In Theorem 10, we consider $\lambda = 1$. $\square$

By Corollary 11, we easily get the following.

Corollary 12. Let $\alpha_j \geq 0$, $\delta_j \in [-1, p)$, $\alpha_j + \delta_j \geq 0$, $(1 \leq j \leq n)$, and $b \in \mathbb{C} \setminus \{0\}$, $\lambda \geq 0$. Suppose that

$$\sum_{j=1}^n \gamma_j \left(\frac{p - \delta_j}{\alpha_j + 1} \right) \leq p. \quad (33)$$

If $f_j \in \Sigma_p US_{l_j}(\alpha_j, \delta_j, b, 1)$ ($1 \leq j \leq n$), then the function $\mathcal{D}_\lambda^k \Phi(z)$, defined by (18), is in the class $\Sigma_p S_{k+1}(0, b, 1)$.

Now, we prove a sufficient condition for the function $\mathcal{D}_\lambda^k \Phi(z)$ defined by (18) to belong to the class $\Sigma_p US_k(\alpha, \delta, b, \lambda)$.

Theorem 13. Let $\alpha \geq 0$, $\delta \in [-1, p)$, $\alpha + \delta \geq 0$ ($1 \leq j \leq n$), and $b \in \mathbb{C} \setminus \{0\}$, $\lambda \geq 0$. Suppose that

$$\sum_{j=1}^n \gamma_j \leq 1. \quad (34)$$

If $f_j \in \Sigma_p US_{l_j}(\alpha, \delta, b, \lambda)$ ($1 \leq j \leq n$), then the function $\mathcal{D}_\lambda^k \Phi(z)$ defined by (18) is in the class $\Sigma_p US_k(\alpha, \delta, b, \lambda)$.

Proof. Since $f_j \in \Sigma_p US_{l_j}(\alpha, \delta, b, \lambda)$ ($1 \leq j \leq n$), by (9), we have

$$\begin{aligned} & \Re \left\{ p - \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} - 1 \right) \right\} \\ & > \alpha \left| \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} - 1 \right) \right| + \delta. \end{aligned} \quad (35)$$

On the other hand, from (19), we obtain the following:

$$\begin{aligned} & p - \frac{\lambda}{b} \left(\frac{z(\mathcal{D}_\lambda^k \Phi(z))'}{\mathcal{D}_\lambda^k \Phi(z)} + p \right) \\ & = \sum_{j=1}^n \gamma_j \left[p - \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} - 1 \right) \right] + p - p \sum_{j=1}^n \gamma_j. \end{aligned} \quad (36)$$

Considering (10) with the above equality, we find

$$\begin{aligned} & \Re \left\{ p - \frac{\lambda}{b} \left(\frac{z(\mathcal{D}_\lambda^k \Phi(z))'}{\mathcal{D}_\lambda^k \Phi(z)} + p \right) \right\} \\ & - \alpha \left| \frac{\lambda}{b} \left(\frac{z(\mathcal{D}_\lambda^k \Phi(z))'}{\mathcal{D}_\lambda^k \Phi(z)} + p \right) \right| - \delta \\ & = p - p \sum_{j=1}^n \gamma_j + \sum_{j=1}^n \gamma_j \Re \left[p - \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} - 1 \right) \right] \end{aligned}$$

$$\begin{aligned} & - \alpha \left| \sum_{j=1}^n \gamma_j \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} - 1 \right) \right| - \delta \\ & \geq p - p \sum_{j=1}^n \gamma_j + \sum_{j=1}^n \gamma_j \Re \left[p - \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} - 1 \right) \right] \\ & - \alpha \sum_{j=1}^n \gamma_j \left| \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} - 1 \right) \right| - \delta \\ & > p - p \sum_{j=1}^n \gamma_j + \sum_{j=1}^n \gamma_j \left[\alpha \left| \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} - 1 \right) \right| + \delta \right] \\ & - \alpha \sum_{j=1}^n \gamma_j \left| \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} - 1 \right) \right| - \delta \\ & = (p - \delta) \left(1 - \sum_{j=1}^n \gamma_j \right) \geq 0. \end{aligned} \quad (37)$$

The proof is complete. $\square$

Corollary 14. Let $\alpha \geq 0$, $\delta \in [-1, p)$, $\alpha + \delta \geq 0$ ($1 \leq j \leq n$), and $b \in \mathbb{C} \setminus \{0\}$. Suppose that

$$\sum_{j=1}^n \gamma_j \leq 1. \quad (38)$$

If $f_j \in \Sigma_p US_{l_j}(\alpha, \delta, b, 1)$ ($1 \leq j \leq n$), then the function $\mathcal{D}_\lambda^k \Phi(z)$ defined by (18) is in the class $\Sigma_p US_{k+1}(\alpha, \delta, b, 1)$.

Proof. In Theorem 13, we consider that $\lambda = 1$ $\square$

Next, for the function $\mathcal{D}_\lambda^k \Phi$ defined by (18) to belong to the class $\Sigma_p SH_k(\alpha, b, \lambda)$, we have the following result.

Theorem 15. Let $\alpha \geq 0$, $\lambda \geq 0$, and $b \in \mathbb{C} \setminus \{0\}$. Suppose that

$$\sum_{j=1}^n \gamma_j \leq 1. \quad (39)$$

If $f_j \in \Sigma_p SH_{l_j}(\alpha, b, \lambda)$, then the function $\mathcal{D}_\lambda^k \Phi(z) \in \Sigma_p SH_k(\alpha, b, \lambda)$.

Proof. Since $f_j \in \Sigma_p SH_{l_j}(\alpha, b, \lambda)$, by (11), we have

$$\begin{aligned} & \sqrt{2} \Re \left\{ p - \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} - 1 \right) \right\} + 2\alpha (\sqrt{2} - 1) \\ & - \left| p - \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} - 1 \right) - 2\alpha (\sqrt{2} - 1) \right| > 0. \end{aligned} \quad (40)$$

Combining (12), (30), and the above inequality, we obtain

$$\begin{aligned}
 & \sqrt{2}\Re \left\{ p - \frac{\lambda}{b} \left(\frac{z(\mathcal{D}_\lambda^k f(z))'}{\mathcal{D}_\lambda^k f(z)} + p \right) \right\} + 2\alpha(\sqrt{2}-1) \\
 & - \left| p - \frac{\lambda}{b} \left(\frac{z(\mathcal{D}_\lambda^k f(z))'}{\mathcal{D}_\lambda^k f(z)} + p \right) - 2\alpha(\sqrt{2}-1) \right| \\
 & = \sqrt{2}\Re \left\{ \sum_{j=1}^n \gamma_j \left[p - \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} - 1 \right) \right] \right. \\
 & \quad \left. + p - p \sum_{j=1}^n \gamma_j \right\} + 2\alpha(\sqrt{2}-1) \\
 & - \left| \sum_{j=1}^n \gamma_j \left[p - \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} - 1 \right) \right] \right. \\
 & \quad \left. + p - p \sum_{j=1}^n \gamma_j - 2\alpha(\sqrt{2}-1) \right| \\
 & = \sum_{j=1}^n \gamma_j \left\{ \sqrt{2}\Re \left(p - \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} - 1 \right) \right) \right. \\
 & \quad \left. + 2\alpha(\sqrt{2}-1) \right\} - 2\alpha(\sqrt{2}-1) \sum_{j=1}^n \gamma_j \\
 & + \sqrt{2} \left(p - p \sum_{j=1}^n \gamma_j \right) + 2\alpha(\sqrt{2}-1) \\
 & - \left| \sum_{j=1}^n \gamma_j \left\{ \left(p - \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} - 1 \right) \right) \right. \right. \\
 & \quad \left. \left. - 2\alpha(\sqrt{2}-1) \right\} \right. \\
 & \quad \left. + 2\alpha(\sqrt{2}-1) \sum_{j=1}^n \gamma_j - 2\alpha(\sqrt{2}-1) + p - p \sum_{j=1}^n \gamma_j \right| \\
 & = \sum_{j=1}^n \gamma_j \left\{ \sqrt{2}\Re \left(p - \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} - 1 \right) \right) \right. \\
 & \quad \left. + 2\alpha(\sqrt{2}-1) \right\} \\
 & + \left[\sqrt{2}p + 2\alpha(\sqrt{2}-1) \right] \left(1 - \sum_{j=1}^n \gamma_j \right)
 \end{aligned}$$

$$\begin{aligned}
 & - \left| \sum_{j=1}^n \gamma_j \left\{ \left(p - \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} - 1 \right) \right) \right. \right. \\
 & \quad \left. \left. - 2\alpha(\sqrt{2}-1) \right\} \right. \\
 & \quad \left. + \left[p - 2\alpha(\sqrt{2}-1) \right] \left(1 - \sum_{j=1}^n \gamma_j \right) \right|,
 \end{aligned} \tag{41}$$

which is

$$\begin{aligned}
 & \geq \sum_{j=1}^n \gamma_j \left\{ \sqrt{2}\Re \left(p - \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} - 1 \right) \right) \right. \\
 & \quad \left. + 2\alpha(\sqrt{2}-1) \right\} \\
 & + \left[\sqrt{2}p + 2\alpha(\sqrt{2}-1) \right] \left(1 - \sum_{j=1}^n \gamma_j \right) \\
 & - \sum_{j=1}^n \gamma_j \left| \left\{ \left(p - \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} - 1 \right) \right) \right. \right. \\
 & \quad \left. \left. - 2\alpha(\sqrt{2}-1) \right\} \right| \\
 & - \left| p - 2\alpha(\sqrt{2}-1) \right| \left(1 - \sum_{j=1}^n \gamma_j \right) \\
 & = \sum_{j=1}^n \gamma_j \left\{ \sqrt{2}\Re \left[p - \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} - 1 \right) \right] \right. \\
 & \quad \left. + 2\alpha(\sqrt{2}-1) \right. \\
 & \quad \left. - \left| p - \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} - 1 \right) - 2\alpha(\sqrt{2}-1) \right| \right\} \\
 & + \left[\sqrt{2}p + 2\alpha(\sqrt{2}-1) - \left| p - 2\alpha(\sqrt{2}-1) \right| \right] \\
 & \times \left(1 - \sum_{j=1}^n \gamma_j \right) \\
 & > \left[\sqrt{2}p + 2\alpha(\sqrt{2}-1) - \left| p - 2\alpha(\sqrt{2}-1) \right| \right] \\
 & \times \left(1 - \sum_{j=1}^n \gamma_j \right),
 \end{aligned} \tag{42}$$

and finally

$$> \left(1 - \sum_{j=1}^n \gamma_j\right) \min \left\{ (\sqrt{2} - 1)(p + 4\alpha), p(\sqrt{2} + 1) \right\} \geq 0. \quad (43)$$

Hence, by (12), we have $\mathcal{D}_\lambda^k \Phi(z) \in \Sigma_p SH_k(\alpha, b, \lambda)$. $\square$

Corollary 16. Let $\alpha \geq 0$ and $b \in \mathbb{C} \setminus \{0\}$. Suppose that

$$\sum_{j=1}^n \gamma_j \leq 1. \quad (44)$$

If $f_j \in \Sigma_p SH_{l_j}(\alpha, b, 1)$, then the function $\mathcal{D}_\lambda^k \Phi(z)$ defined by (18) is in the class $\Sigma_p SH_{k+1}(\alpha, b, 1)$.

Proof. In Theorem 15, we consider $\lambda = 1$. $\square$

Finally, we end this paper by the following theorem and its consequence.

Theorem 17. Let $\alpha \geq 0$, $\lambda \geq 0$, and $b \in \mathbb{C} \setminus \{0\}$. Suppose that

$$(p + \sqrt{2}\alpha(\sqrt{2} - 1)) \sum_{j=1}^n \gamma_j < p. \quad (45)$$

If $f_j \in \Sigma_p SH_{l_j}(\alpha, b, \lambda)$, then the function $\mathcal{D}_\lambda^k \Phi(z)$ defined by (18) is in the class $\Sigma_p SH_k(0, b, \lambda)$.

Proof. Since $f_j \in \Sigma_p SH_{l_j}(\alpha, b, \lambda)$, by (11), we have

$$\begin{aligned} & \sqrt{2}\Re \left\{ p - \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} - 1 \right) \right\} + 2\alpha(\sqrt{2} - 1) \\ & > \left| p - \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} - 1 \right) - 2\alpha(\sqrt{2} - 1) \right|. \end{aligned} \quad (46)$$

Considering this inequality and (30), we obtain

$$\begin{aligned} & \sqrt{2}\Re \left\{ p - \frac{\lambda}{b} \left(\frac{z(\mathcal{D}_\lambda^k f(z))'}{\mathcal{D}_\lambda^k f(z)} + p \right) \right\} \\ & = \sqrt{2}\Re \left\{ \sum_{j=1}^n \gamma_j \left[p - \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} - 1 \right) \right] \right\} \end{aligned}$$

$$\begin{aligned} & + p - p \sum_{j=1}^n \gamma_j \Big\} \\ & = \sum_{j=1}^n \gamma_j \left\{ \sqrt{2}\Re \left[p - \frac{1}{b} \left(\frac{\mathcal{D}_\lambda^{l_j+1} f_j(z)}{\mathcal{D}_\lambda^{l_j} f_j(z)} - 1 \right) \right] \right. \\ & \quad \left. + 2\alpha(\sqrt{2} - 1) \right\} \end{aligned} \quad (47)$$

$$\begin{aligned} & + \sqrt{2}p \left(1 - \sum_{j=1}^n \gamma_j \right) - 2\alpha(\sqrt{2} - 1) \sum_{j=1}^n \gamma_j \\ & > \sqrt{2}p \left(1 - \sum_{j=1}^n \gamma_j \right) - 2\alpha(\sqrt{2} - 1) \sum_{j=1}^n \gamma_j \\ & = \sqrt{2} \left(p - (p + \sqrt{2}\alpha(\sqrt{2} - 1)) \sum_{j=1}^n \gamma_j \right) > 0. \end{aligned}$$

Hence, we have $\mathcal{D}_\lambda^k \Phi(z) \in \Sigma_p SH_k(0, b, \lambda)$. $\square$

Corollary 18. Let $\alpha \geq 0$ and $b \in \mathbb{C} \setminus \{0\}$. Suppose that

$$(p + \sqrt{2}\alpha(\sqrt{2} - 1)) \sum_{j=1}^n \gamma_j < p. \quad (48)$$

If $f_j \in \Sigma_p SH_{l_j}(\alpha, b, 1)$, then the function $\mathcal{D}_\lambda^k \Phi(z)$ defined by (18) is in the class $\Sigma_p SH_{k+1}(0, b, 1)$.

Proof. In Theorem 17, we consider that $\lambda = 1$. $\square$

For other work that we can look at regarding differential and integral operators, see [14, 18–24].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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