Research Article
Second-Order Multiplier Iteration Based on a Class of Nonlinear Lagrangians

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Nonlinear Lagrangian algorithm plays an important role in solving constrained optimization problems. It is known that, under appropriate conditions, the sequence generated by the first-order multiplier iteration converges superlinearly. This paper aims at analyzing the second-order multiplier iteration based on a class of nonlinear Lagrangians for solving nonlinear programming problems with inequality constraints. It is suggested that the sequence generated by the second-order multiplier iteration converges superlinearly with order at least two if in addition the Hessians of functions involved in problem are Lipschitz continuous.

1. Introduction


Consider the following inequality constrained optimization problem:

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \geq 0, \quad i = 1, \ldots, m,
\end{align*}
\]

where \( f_i : \mathbb{R}^2 \rightarrow \mathbb{R}, \ i = 0, \ldots, m \) are continuous differentiable functions.

As nonlinear Lagrangians can be used to develop dual algorithms for nonlinear programming, requiring no restrictions on primal feasibility, important contributions on this topic have been done by many authors.

Polyak and Teboulle [9] discussed a class of Lagrange functions of the form

\[
H(x, u, c) = f_0(x) - c \sum_{i=1}^{m} u_i \psi \left( c^{-1} f_i(x) \right)
\]

for solving (INP), where \( c > 0 \) is penalty parameter and \( \psi \) is twice continuous differentiable function. Furthermore, Polyak and Griva [10] proposed a general primal-dual nonlinear rescaling (PDNR) method for convex optimization with inequality constraints, and Griva and Polyak [11] developed a general primal-dual nonlinear rescaling method with dynamic scaling parameter update. Besides the works by Polyak and his coauthors, Auslender et al. [12] and Ben-Tal and Zibulevsky [13] studied other nonlinear Lagrangians and obtained interesting convergence results for convex programming problems, too. Under appropriate conditions, the sequence generated by the first-order multiplier iteration converges superlinearly.
Ren and Zhang [14] analyse the following nonlinear Lagrangians:

\[ H(x, u, c) = f_0(x) - k^{-1}\sum_{i=1}^{m} u_i \psi(kf_i(x)) \]  

and constructed the dual algorithm based on minimizing \( H(x, u, k) \) as follows.

\[ \nabla^2 x_H(x^*, u^*, k) = \nabla^2 x_L(x^*, u^*) - k\psi(0)\nabla f(\epsilon)(x^*)^T U^* \nabla f(\epsilon)(x^*), \] 

where \( U^* = \text{diag}_{1 \leq i \leq r}(u_i^*) \).

**2. Preliminaries**

Consider the inequality constrained optimization problem (INP). Let

\[ L(x, u) = f_0(x) - \sum_{i=1}^{m} u_i f_i(x) \]  

denote the Lagrange function for problem (INP) and \( I(x) = \{ i \mid f_i(x) = 0, \ i = 1, \ldots, m \} \).

For the convenience of description in the sequel, we list the following assumptions, some of which will be used somewhere.

(a) Functions \( f_i(x) (i = 0, \ldots, m) \) are twice continuously differentiable.

(b) For convenience of statement, we assume \( I(x^*) = \{ i \mid f_i(x^*) = 0, i = 1, \ldots, r \} \).

(c) Let \( (x^*, u^*) \in \mathbb{R}^n \times \mathbb{R}^m \) satisfy the Kuhn-Tucker conditions

\[ \nabla x_L(x^*, u^*) = 0, \ u^* \geq 0, \ u_i^* f_i(x^*) = 0, \ i = 1, \ldots, m. \]  

(d) Strict complementary condition holds; that is,

\[ u_i^* > 0 \quad \text{for } i \in I(x^*). \]  

(e) The set of vectors \( \{ \nabla f_i(x^*) \mid i \in I(x^*) \} \) are linearly independent.

(f) For all \( y \neq 0 \) satisfying \( \nabla f_i(x^*)^T y = 0, i \in I(x^*) \), the following inequality holds:

\[ y^T \nabla^2_x L(x^*, u^*) y > 0. \]  

Let function \( \psi \) in \( H(x, u, k) \) defined in (2) and its derivatives satisfy the following conditions:

(H1) \( \psi(0) = 0; \)

(H2) \( \psi(t) > 0, \) for all \( t \in (b, +\infty), \) with \(-\infty \leq b < 0, \) and \( \psi(0) = 1; \)

(H3) \( \psi''(t) < 0, \) for all \( t \in (b, +\infty), \) with \(-\infty \leq b < 0; \)

(H4) \( k\psi'(kt) \) is bounded, where \( t \in (b, +\infty), \) with \(-\infty \leq b < 0, \) and for \( k > 0 \) large enough.

The following proposition concerns properties of \( H(x, u, k) \) at a Kuhn-Tucker point \((x^*, u^*)\).

**Proposition 1** (see [14]). Assume that (a)–(f) and (H1)–(H3) hold. For any \( k > 0 \) and any Kuhn-Tucker point \((x^*, u^*)\) the following properties are valid:

(i) \( H(x^*, u^*, k) = L(x^*, u^*) = f(x^*); \)

(ii) \( \nabla_H H(x^*, u^*, k) = \nabla_L L(x^*, u^*) = \nabla_f(x^*) - \sum_{i=1}^{m} u_i^* \nabla f_i(x^*) = 0; \)

(iii) \( \nabla_H^2 H(x^*, u^*, k) = \nabla^2_L L(x^*, u^*) - k\psi''(0)\nabla f_i^2(x^*) U^* \nabla f_i^2(x^*) U^* \nabla f_i^2(x^*) \), where \( U^* = \text{diag}_{1 \leq i \leq r}(u_i^*); \)
(iv) there exist \( c_0 > 0 \) and \( \mu > 0 \) such that, for any \( c > c_0 \),
\[
\left\langle \nabla^2_x H(x^*, u^*, k) y, y \right\rangle \geq \mu \langle y, y \rangle, \quad \forall y \in \mathbb{R}^n
\]
satisfying \( \nabla f(x^*)^T y = 0 \).

Let \( \delta > 0 \) be small enough, \( 0 < \varepsilon < \min\{u_i^* \mid i = 1, \ldots, r\} \), and \( k_0 \) large enough satisfying (iv) of Proposition 1. For any fixed \( k > k_0 \), define
\[
U_k^1(\varepsilon, \delta) = \left\{ u_1 \mid u_1 \leq u_i^* + \delta k \right\}, \quad i = 1, \ldots, r,
\]
\[
U_k^r \left( \varepsilon, \delta \right) = \left\{ u_i \mid 0 \leq u_i \leq \delta k \right\}, \quad i = r + 1, \ldots, m,
\]
\[
U_k(\varepsilon, \delta) = U_k^1(\varepsilon, \delta) \times \ldots \times U_k^r(\varepsilon, \delta) \times U_m^m(\varepsilon, \delta).
\]
For any \( k_1 > k_0 \), we denote
\[
D(\varepsilon, \delta) = \{(u, k) \mid u \in U_k(\varepsilon, \delta), k \in [k_0, k_1]\}. \tag{12}
\]
Let \( \sigma = \min\{f_i(x) \mid r + 1 \leq i \leq m\} > 0 \), \( I_r \) is the \( r \times r \) identity matrix, and \( 0_k \) is the \( r \times r \) zero matrix.

**Theorem 2** (see [14]). Assume that (a)–(f) and (H1)–(H4) hold. Then there exists \( k_0 > 0 \) large enough such that, for any \( k_1 > k_0 \), there exist \( \varepsilon_1 > 0, \delta > 0 \), satisfying that for any \( (u, k) \in D(\varepsilon, \delta) \), the following statements hold.

(i) There exists a vector
\[
\hat{x} = \hat{x}(u, k) \in \text{arg min}\{H(x, u, k) \mid x \in \mathcal{S}(x^*, \varepsilon_1)\}. \tag{13}
\]

(ii) For \( \hat{x} \) in (i) and \( \hat{u} = \hat{u}(u, k) = \text{diag}_{i=1}^{m}(\psi'(k f_i(\hat{x})))u \), the following estimate is valid:
\[
\max \left\{ \|\hat{x} - x^*\|, \|\hat{u} - u^*\| \right\} \leq c k^{-1} \|u - u^*\|, \tag{14}
\]
where \( c > 0 \) is a scalar independent of \( k_0 \) and \( k_1 \).

(iii) Function \( H(x, u, k) \) is strongly convex in a neighborhood of \( \hat{x} \).

### 3. The Second-Order Multiplier Iteration

Based on the nonlinear Lagrange function \( H(x, u, k) \), we consider the dual function defined on \( \mathcal{S}(x^*, \varepsilon_1) \times \mathbb{R}^m_+ \) as follows:
\[
d_k(u) = \inf \{H(x, u, k) \mid x \in \mathcal{S}(x^*, \varepsilon_1)\} - \delta(u | U_k(\varepsilon, \delta)), \tag{15}
\]
where \( \delta(u | U_k(\varepsilon, \delta)) = \left\{ \begin{array}{ll} 0 & \text{if } u \in U_k(\varepsilon, \delta) \text{ is the indicator function of } U_k(\varepsilon, \delta) \end{array} \right\} \).

**Lemma 3.** Assume that conditions (a)–(f) and (H1)–(H4) hold; then for any fixed \( k \geq k_0 \) function \( d_k(u) \) is twice continuously differentiable and concave on \( U_k(\varepsilon, \delta) \).

**Proof.** Obviously, for \( k > 0 \), function \( d_k(u) \) is concave. In view of Theorem 2, for any \( (u, k) \in D(\varepsilon, \delta) \), function \( H(x, u, k) \) is strongly convex in the neighborhood of \( \hat{x} = \hat{x}(u, k) \). So \( \hat{x}(u, k) \) is unique minimizer of function \( H(x, u, k) \) with respect to \( x \) in the neighborhood of point \( \hat{x} \), and \( d_k(u) = H(\hat{x}(u, k), u, k) \) is smooth in \( U_k(\varepsilon, \delta) \); that is, the Jacobian of \( d_k(u) \) exists, and
\[
\nabla d_k(u) = \nabla u \hat{x}(u, k) \nabla_x H(\hat{x}(u, k), u, k), \quad \forall (u, k) \in D(\varepsilon, \delta). \tag{16}
\]
In view of \( \nabla_x H(\hat{x}(u, k), u, k) = 0_k \), we have
\[
\nabla u d_k(u) = \nabla_x H(\hat{x}(u, k), u, k) \left( \nabla^2_x H(\hat{x}(u, k), u, k) \right)^{-1} \psi', \quad \forall (u, k) \in D(\varepsilon, \delta). \tag{17}
\]
It follows from (18) that
\[
\nabla^2_x H(\hat{x}(u, k), u, k) = -\nabla^2_x H(\hat{x}(u, k), u, k) \left( \nabla^2_x H(\hat{x}(u, k), u, k) \right)^{-1} \psi', \quad \forall (u, k) \in D(\varepsilon, \delta). \tag{19}
\]
Thus,
\[
\nabla^2_x d_k(u) = \nabla^2_x(\nabla^2_x H(\hat{x}(u, k), u, k)) \nabla^2_x H(\hat{x}(u, k), u, k) \nabla^2_x H(\hat{x}(u, k), u, k)^{-1} \psi', \quad \forall (u, k) \in D(\varepsilon, \delta). \tag{20}
\]
So,
\[
\nabla^2_u u_d(k^* u^*) = -\psi'(k f(x^*)) (\nabla f(x^*))^T (\nabla^2_x H(x^*, u^*, k)^{-1} x (\nabla f(x^*)) \psi'(k f(x^*)).
\]
\[
(22)
\]

Let \( \bar{x}(u, k) \) be the minimizer of \( H(x, u, k) \) in a neighborhood of \( x^* \); then we obtain that
\[
\nabla_v d_v(u) = \nabla_v H(\bar{x}(u, k), u, k)
\]
\[
= -k^{-1}(\psi(k f(\bar{x}(u, k))), \ldots, \psi(k f_m(\bar{x}(u, k))))^T.
\]
\[
\nabla^2_v d_v(u) = -\psi(k f) (\nabla f(\bar{x}(u, k)))^T
\]
\[
\times \left( \nabla^2_x H(\bar{x}(u, k), u, k)^{-1}
\right)
\]
\[
\times \nabla_f (\bar{x}(u, k)) \psi'(k f(\bar{x}(u, k))).
\]
\[
(23)
\]

In view of the interpretation of the multiplier iteration as the steepest ascent method, it is natural to consider Newton's method for maximizing the dual functional \( d_k \) which is given by
\[
u^+ = u^+ - [\nabla^2 d_k(u^+)]^{-1} \nabla d_k(u^+).
\]
\[
(24)
\]
In view of (23), this iteration can be written as
\[
u^+ = u^+ - B_k^{-1} k^{-1} \psi(k f(x(u^+, k)));
\]
where
\[
B_k = \psi'(k f(x(u^+, k))) (\nabla f(x(u^+, k)) )^T
\]
\[
\times \left( \nabla^2_x H(x(u^+, k), u, k)^{-1}
\right)
\]
\[
\times (\nabla f(x(u^+, k)) \psi'(k f(x(u^+, k))))
\]
\[
(25)
\]
We will provide a convergence and rate of convergence result for iteration (23) and (26).

For \( k > 0 \) and \( (x, u) \in \mathbb{R}^m \), we define
\[
A^+(x, u) = \{ i | u_i \psi(k f_i(x)) > 0, \ i = 1, \ldots, m \}
\]
\[
A^-(x, u) = \{ i | i \notin A^+(x, u), \ i = 1, \ldots, m \}
\]
\[
(27)
\]
For a given \( (x, u) \), assume (by reordering indices if necessary) that \( A^+(x, u) \) contains the first \( r \) indices where \( r \) is an integer with \( 0 \leq r \leq m \). Define
\[
\psi_+(k f(x)) = \begin{bmatrix} \psi(k f_1(x)) \\ \vdots \\ \psi(k f_r(x)) \end{bmatrix}
\]
\[
\psi_-(k f(x)) = \begin{bmatrix} \psi(k f_{r+1}(x)) \\ \vdots \\ \psi(k f_m(x)) \end{bmatrix}
\]
\[
u_+ = (u_1, \ldots, u_r)^T, \quad \nu_- = (u_{r+1}, \ldots, u_m)^T,
\]
\[
H_+(x, u, k) = f_0(x) - k^{-1} u_+ \psi_+(k f(x)).
\]
\[
(28)
\]
We note that \( r, \psi_+, \psi_-, u_+, u_- \) and \( H_+ \) depend on \( (x, u) \), but to simplify notation we do not show this dependence explicitly.

Now, we consider Newton's method for solving the system of necessary conditions
\[
\nabla_v H_+(x, u, k) = \nabla f_0(x) - \sum_{i=1}^r u_i \psi'(k f_i(x)) \nabla f_i(x) = 0,
\]
\[
k^{-1} \psi(k f_i(x)) = 0, \quad i = 1, \ldots, r.
\]
\[
(29)
\]
Considering the extension of Newton's method, given \( (x, u) \), we denote the next iterate by \( (\bar{x}, \bar{u}) \) where \( \bar{u} = (\bar{u}_1, \ldots, \bar{u}_m)^T \). We also write
\[
\bar{u}_+ = (\bar{u}_1, \ldots, \bar{u}_r)^T, \quad \bar{u}_- = (\bar{u}_{r+1}, \ldots, \bar{u}_m)^T.
\]
\[
(30)
\]
The iteration, roughly speaking, consists of setting the multipliers of the inactive constraints \( (j \in A^-(x, u)) \) to zero and treating the remaining constraints as equalities. More precisely, we set \( \bar{u}_- = 0_{m-r} \) and obtain \( \bar{x}, \bar{u}_+ \) by solving the system
\[
\begin{bmatrix} \nabla^2_x H_+(x, u, k) & -\nabla f_0(x) \psi_+(k f(x)) \\ \psi_+(k f(x)) \nabla f_0(x)^T & 0 \end{bmatrix} \begin{bmatrix} \bar{x} - x \\ \bar{u}_+ - u_+ \end{bmatrix} = 0.
\]
\[
(31)
\]
where \( \psi_+(k f(x)) = [\text{diag} \, \psi'(k f(x))]_{i=1}^r \).

If \( \nabla^2_x H_+(x, u, k) \) is invertible and \( \nabla f_0(x) \) has rank \( r \), we can solve system (31) explicitly. It follows from (31) that
\[
\nabla^2_x H_+(x, u, k) (\bar{x} - x) = -\nabla f_0(x) \psi_+(k f(x)) (\bar{u}_+ - u_+),
\]
\[
(32)
\]
Premultiplying (32) with \( \psi_+(k f(x)) \nabla f_0(x)^T \)[\( \nabla^2_x H_+(x, u, k) \)]^{-1} and using (33), we obtain
\[
\bar{x} - x = \left[ \nabla^2_x H_+(x, u, k) \right]^{-1} \left\{ \nabla f_0(x) \psi_+(k f(x)) (\bar{u}_+ - u_+) - \nabla^2_x H_+(x, u, k) \right\},
\]
\[
(34)
\]
from which, we have
\[
\bar{u}_+ = u_+ - \left\{ \psi_+(k f(x)) \nabla f_0(x)^T \left[ \nabla^2_x H_+(x, u, k) \right]^{-1} \right. \]
\[
\times \nabla f_0(x) \psi_+(k f(x)) \left[ \nabla^2_x H_+(x, u, k) \right]^{-1} k^{-1} \psi_+(k f(x)),
\]
\[
(35)
\]
Substitution in (32) yields
\[
\bar{x} = x - \nabla^2_x H_+(x, u, k) \nabla^2_x H_+(x, u, k).
\]
\[
(36)
\]

Return to (25) and (26), and using the fact that \( \nabla_x H_\pm(x(u,k),u,k) = 0 \), we see that iteration (25) and (26) is of the form (35).

For a triple \((x, u, k)\) for which the matrix on the left-hand side of (31) is invertible, we denote by \( \bar{x}(x, u, k), \bar{u}_+(x, u, k) \) the unique solution of (31) and say that \( \bar{x}(x, u, k), \bar{u}_+(x, u, c) \) are well defined.

Define

\[
  u_+^{s+1} = \bar{u}_+(x(u^s, k), u^s, k), \quad u_-^{s+1} = 0. \quad (37)
\]

**Proposition 4.** Let \( k \) be a scalar. For every triple \((x, u, k)\), if \( \psi' \) satisfies

\[
  \psi'^2 - 2\psi' + I = 0, \quad (38)
\]

then the vectors \( \bar{x}(x, u, k), \bar{u}_+(x, u, k) \) are well defined if and only if the vectors \( \bar{x}(x, \psi'(kf(x))u, 0), \bar{u}_+(x, \psi'(kf(x))u, 0) \) are well defined.

Furthermore,

\[
  \bar{x}(x, u, k) = \bar{x}(x, \psi'(kf(x))u, 0), \quad (39)
\]
\[
  \bar{u}_+(x, u, k) = \bar{u}_+(x, \psi'(kf(x))u, 0). \quad (40)
\]

**Proof.** By calculating, we have

\[
  \nabla_x H_+(x, u, k) = \nabla_x L(x, \psi'(kf(x))u),
\]
\[
  \nabla^2_x H_+(x, u, k) = \nabla^2_x L(x, \psi'(kf(x))u) + k \sum_{i=1}^r u_i \psi''(kf_i(x)) \nabla f_i(x) \nabla f_i(x)^T
\]
\[
  - k \sum_{i=1}^r u_i \psi''(kf_i(x)) \nabla f_i(x) \nabla f_i(x)^T.
\]

As a result, the system (31) can be written as

\[
  \begin{pmatrix}
    \nabla^2_x L(x, \psi'(kf(x))u) - k \sum_{i=1}^r u_i \psi''(kf_i(x)) \nabla f_i(x) \nabla f_i(x)^T - \nabla f_{(r)}(x) \psi'_+(kf(x)) \\
    \psi'_+(kf(x)) \nabla f_{(r)}(x)^T
  \end{pmatrix}
  \times
  \begin{pmatrix}
    \bar{x} - x \\
    \bar{u}_+ - u_+
  \end{pmatrix} = 0. \quad (42)
\]

The second equation yields

\[
  \psi'_+(kf(x)) \nabla f_{(r)}(x)^T (\bar{x} - x) = -k^{-1} \psi'_+(kf(x)). \quad (43)
\]

If we form the second-order Taylor series expansion of \( \psi' \) around \( t_k \),

\[
  \psi(t) = \psi(t_k) + \psi'(t_k)(t - t_k) + \frac{1}{2}(t - t_k)^T \psi''(t_k)(t - t_k), \quad (44)
\]

we obtain

\[
  \psi'(t) = \psi'(t_k) + \psi''(t_k)(t - t_k). \quad (45)
\]

Take \( t = kf_i(\bar{x}) \), \( t_k = kf_i(x), i = 1, \ldots, r \), and it follows that

\[
  \psi'(kf_i(x)) = 1 - k(\bar{x} - x)^T \nabla f_i(x) \psi''(kf_i(x)), \quad (i = 1, \ldots, r). \quad (46)
\]

Substituting (46) into (43), we have

\[
  \begin{pmatrix}
    \psi'_+(kf(x)) \nabla f_{(r)}(x)^T \nabla f_i(x) \\
    \psi'_+(kf(x)) \nabla f_{(r)}(x)^T \nabla f_i(x)
  \end{pmatrix}
  \times
  \begin{pmatrix}
    \bar{x} - x \\
    \bar{u}_+ - u_+
  \end{pmatrix} = 0. \quad (49)
\]

This shows (39) and (40). \( \square \)

In view of (40), we can write (37) as

\[
  u_+^{s+1} = \bar{u}_+(x(u^s, k), u^s, k), \quad u_-^{s+1} = 0. \quad (50)
\]

where

\[
  \bar{u}(u^s, k) = \psi'(kf(x(u^s, k)))u^s. \quad (51)
\]
This means that one can carry out the second-order multiplier iteration (25), (26) in two stages. First execute the first-order iteration (51) and then the second-order iteration (50), which is part of Newton’s iteration at \((x(u', k), \bar{u}(u', k))\) for solving the system of necessary conditions (29).

Now, we know that \(x(u', k), \bar{u}(u', k)\) is close to \((x^*, u^*)\) for \((u', k)\) in an appropriate region of \(\mathbb{R}^{n+s+1}\). Therefore, using known results for Newton’s method, we expect that (50) will yield a vector \(u'^{+}\) which is closer to \(u^*\) than \(u'\). This argument is the basis for the proof of the following proposition.

**Proposition 5.** Assume (a)–(f) hold, and let \(k_0 > 0\), \(\delta > 0\) be as in Theorem 2. Then, given any scalar \(\gamma > 0\), there exists a scalar \(\delta_1\) with \(0 < \delta_1 \leq \delta\) such that for all \((u, k) \in D_1 = \{(u, k) : \bar{u} \in U_k(\varepsilon, \delta), k \geq k_0\} \) there holds

\[
\left\| (\tilde{x}(u, k), \tilde{u}(u, k)) - (x^*, u^*) \right\| \leq \gamma k^{-1} \left\| u - u^* \right\|, \tag{52}
\]

where

\[
u^{+1} = u^* - B_k^{-1} k^{-1} \psi (k f(x(u', k))), \tag{53}
\]

where

\[
B_k = \psi' \left( k f(x(u', k)) \right) \nabla f(x(u', k))^T \left[ \frac{\partial^2 f(x(u', k))}{\partial u^2}(x(u', k), u', k) \right]^{-1} \psi' (k f(x(u', k))) \tag{54}
\]

If, in addition, \(\nabla^2 f_i(x), i = 0, \ldots, m\) are Lipschitz continuous in a neighborhood of \(x^*\), there exists a scalar \(\gamma_1 > 0\) such that, for all \((u, k) \in D_1\), there holds

\[
\left\| (\tilde{x}(u, k), \tilde{u}(u, k)) - (x^*, u^*) \right\| \leq \gamma_1 k^{-2} \left\| u - u^* \right\|^2. \tag{55}
\]

**Proof.** In view of Theorem 2, given any \(\gamma > 0\), there exist \(\varepsilon_1 > 0, \varepsilon_2 > 0\) and \(M > 0\) such that if \(x(u, k) \in S(x^*, \varepsilon_1)\) and \(\bar{u}(u, k) \in S(u^*, \varepsilon_2)\), there holds

\[
\left\| (\tilde{x}(u, k), \tilde{u}(u, k), 0), \bar{u}(x(u, k), \tilde{u}(u, k), 0)) - (x^*, u^*) \right\|
\leq \frac{\gamma}{M} \left\| (x(u, k), \tilde{u}(u, k)) - (x^*, u^*) \right\| \tag{56}
\]

(compare with Proposition 1.17, Bertsekas [8]). Take \(\delta_1\) sufficiently small so that, for all \((u, k) \in D_1\), we have \(x(u, k) \in S(x^*, \varepsilon_1), \bar{u}(u, k) \in S(u^*, \varepsilon_2)\), and

\[
\left\| (x(u, k), \bar{u}(u, k)) - (x^*, u^*) \right\| \leq Mk^{-1} \left\| u - u^* \right\|. \tag{57}
\]

From (50), we have

\[
\left\| (\tilde{x}(u, k), \tilde{u}(u, k), 0), \bar{u}(x(u, k), \tilde{u}(u, k), 0)) - (x^*, u^*) \right\|
\leq \frac{\gamma}{M} \cdot Mk^{-1} \left\| u - u^* \right\| = \gamma k^{-1} \left\| u - u^* \right\|. \tag{58}
\]

If \(\nabla^2 f_i(x), i = 0, \ldots, m\) are Lipschitz continuous, then there exists a \(\gamma_1 > 0\) such that for \(x(u, k) \in S(x^*, \varepsilon)\) and \(\bar{u}(u, k) \in S(u^*, \varepsilon)\), we have

\[
\left\| (\tilde{x}(x(u, k), \bar{u}(u, k), 0), \bar{u}(x(u, k), \bar{u}(u, k), 0)) - (x^*, u^*) \right\|
\leq \frac{\gamma_1}{2M} \left\| (x(u, k), \bar{u}(u, k)) - (x^*, u^*) \right\|^2 \leq \frac{\gamma_1}{2M} \left( (MK^{-1})^2 \left\| u - u^* \right\|^2 + (MK^{-1})^2 \left\| u - u^* \right\|^2 \right)
\leq \gamma k^{-2} \left\| u - u^* \right\|^2. \tag{59}
\]

From the above analysis, we know that the sequence generated by the second-order multiplier iteration converges superlinearly with order at least two if the Hessians of functions involved in problem are Lipschitz continuous.

**Conflict of Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.

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**References**


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