Global Existence and Large Time Behavior of Solutions to the Bipolar Nonisentropic Euler-Poisson Equations

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We study the one-dimensional bipolar nonisentropic Euler-Poisson equations which can model various physical phenomena, such as the propagation of electron and hole in submicron semiconductor devices, the propagation of positive ion and negative ion in plasmas, and the biological transport of ions for channel proteins. We show the existence and large time behavior of global smooth solutions for the initial value problem, when the difference of two particles’ initial mass is nonzero, and the far field of two particles’ initial temperatures is not the ambient device temperature. This result improves that of Y.-P. Li, for the case that the difference of two particles’ initial mass is zero, and the far field of the initial temperature is the ambient device temperature.

1. Introduction

In this paper we study the following 1D bipolar nonisentropic Euler-Poisson equations:

\[\begin{align*}
n_{1t} + j_{1x} &= 0, \\
j_{1t} + \left( \frac{j_{1}^{2}}{n_{1}} \right)_{x} + (n_{1}T_{1})_{x} &= n_{1}E - j_{1}, \\
T_{1t} + \frac{j_{1}}{n_{1}}T_{1x} + \frac{2}{3}T_{1}\left( \frac{j_{1}}{n_{1}} \right)_{x} - \frac{2}{3}n_{1}T_{1xx} &= \frac{1}{3}\left( \frac{j_{1}}{n_{1}} \right)_{x}^{2} - (T_{1} - T^{*}), \\
n_{2t} + j_{2x} &= 0, \\
j_{2t} + \left( \frac{j_{2}^{2}}{n_{2}} \right)_{x} + (n_{2}T_{2})_{x} &= -n_{2}E - j_{2}, \\
T_{2t} + \frac{j_{2}}{n_{2}}T_{2x} + \frac{2}{3}T_{2}\left( \frac{j_{2}}{n_{2}} \right)_{x} - \frac{2}{3}n_{2}T_{2xx} &= \frac{1}{3}\left( \frac{j_{2}}{n_{2}} \right)_{x}^{2} - (T_{2} - T^{*}), \\
E_{x} &= n_{1} - n_{2},
\end{align*}\]

where \(n_{i} > 0, j_{i}, T_{i}, \) (\(i = 1, 2\)), and \(E\) denote the particle densities, current densities, temperatures, and the electric field, respectively, and \(T^{*} > 0\) stands for the ambient device temperature. The system (1) models various physical phenomena, such as the propagation of electron and hole in submicron semiconductor devices, the propagation of positive ion and negative ion in plasmas, and the biological transport of ions for channel proteins. When the temperature \(T_{i}(i = 1, 2)\) is the function of the density \(n_{i}(i = 1, 2)\), the system (1) reduces to the isentropic bipolar Euler-Poisson equations. For more details on the bipolar isentropic and nonisentropic Euler-Poisson equations (hydrodynamic models), we can see [1–3] and so forth.

Due to their physical importance, mathematical complexity, and wide range of applications, many results concerning the existence and uniqueness of (weak, strong, or smooth) solutions for the bipolar Euler-Poisson equations can be found in [4–14] and the references cited therein. However, the study of the corresponding nonisentropic bipolar Euler-Poisson equation is very limited in the literature. In [15] Li investigated the global existence and nonlinear diffusive waves of smooth solutions for the initial value problem of the one-dimensional nonisentropic bipolar hydrodynamic model when the difference of two particles’ initial mass is...
zero, and the far field of two particles’ initial temperatures is the ambient device temperature. We also mention that there are some results about the relaxation limit and quasineutral limit of the bipolar Euler-Poisson system see [16–19]. In this paper, we will show the existence and large time behavior of global smooth solutions for the initial value problem of (1), when the difference of two particles’ initial mass is nonzero and the far field of the initial temperatures is not the ambient device temperature. We now prescribe the following initial data:

\[ (n_i, j_i, T_i) (x, t = 0) = (n_{i0}, j_{i0}, T_{i0}) (x), \]

\[ n_{i0} > 0, \quad i = 1, 2, \]

\[ \lim_{x \to \pm \infty} (n_i, j_i, T_i) (x) = (n_i, j_i, T_i), \]

\[ n_i > 0, T_{i\pm} > 0, \quad i = 1, 2, \]

and \((n_i, j_i, T_i)\) are the state constants. We also give the electric field as \(x = -\infty\); that is,

\[ E (-\infty, t) = 0. \]

The nonlinear diffusive phenomena both in smooth and weak senses were also observed for the bipolar isotropic and nonisentropic by Gasser et al. [4], Huang and Li [5], and Li [15], respectively. Namely, according to the Darcy’s law, it is expected that the solutions \((n_i, j_i, T_1, n_{j2}, j_{j2}, T_{j2}, E)(x, t)\) converge in \(L^\infty\)-sense to \((\hat{n}, \hat{j}, T^*, \hat{T}_1, \hat{T}_2, 0)(x, t)\); here \((\hat{n}, \hat{j}) = (\hat{n}(\hat{j}))((x + x_0)/\sqrt{1 + t}) (x_0 \text{ is a shift constants})\) is the nonlinear diffusion waves, which is self-similar solutions to the following equations:

\[ \hat{n} - (nT^*)_{xx} = 0, \]

\[ \hat{j} := -(nT^*)_x, \]

\[ \hat{n}, \hat{j} \to (n_i, 0), \quad \text{as } x \to \pm \infty. \]

Note that in [15], the author assumed that

\[ j_{i+} = j_{i-}, \quad T_{i\pm} = T^*, \quad i = 1, 2 \]

which lead to the difference of two particles’ initial mass to be zero; that is,

\[ \int_{\mathbb{R}} [n_{i0} (x) - n_{20} (x)] dx = 0, \quad i = 1, 2. \]

This implies, from the last equation of (1), that

\[ E (+\infty, t) - E (-\infty, t) = 0. \]

In this paper, we try to drop off these too stiff conditions. That is, \(j_{i+} \neq j_{i-}, T_{i\pm} \neq T^*\) \((i = 1, 2)\). Moreover, for stating our results, set for \(i = 1, 2, \)

\[ (\varphi_{i0}, \psi_{i0}, \theta_{i0}) (x) \]

\[ = \left( \left( \int_{-\infty}^{\infty} [n_{i0} (\xi) - \hat{n}_1 (\xi, t) - \hat{n}_2 (\xi + x_0, t = 0)] d\xi, \right. \right. \]

\[ j_{i0} (x) - j_{i0} (x) - \hat{j}(x + x_0, t = 0), T_{i0} (x) - \hat{T}_{i0} (x) - T^*, \]

\[ \left. \left. \left. + \sum_{k=0}^{2} (1 + t)^{k+1} \left| \frac{d^k}{dx^k} \left( n_1 - n_2 - \hat{n}_1 - \hat{T}_1 - \hat{T}_2 - T^* \right) (t) \right| \right|^2 \right) \]

\[ + \left( \sum_{k=0}^{2} (1 + t)^{k+1} \left| \frac{d^k}{dx^k} \left( j_1 - j_2 - \hat{j} \right) (t) \right| \right|^2 \]

\[ + (1 + t)^3 \left( \frac{d^3}{dx^3} \left( T_1 - \hat{T}_1 - T^* \right) (t) \right)^2 \]

\[ \leq C (\delta + \Phi_0), \]
\[\begin{align*}
&\left\| (n_1 - \bar{n}_1 + n_2 - \bar{n}_2) (t) \right\|^2_1 + \left\| (j_1 - \bar{j}_1 - j_2 + \bar{j}_2) (t) \right\|^2_1 + \left\| (T_1 - \bar{T}_1 - T_2 + \bar{T}_2) (t) \right\|^2_1 \\
&+ \left\| \left( E - \bar{E} \right) (t) \right\|^2_1 \\
&\leq C (\Phi_0 + \delta) e^{-\alpha t},
\end{align*}\]  

for some constant \(\alpha > 0\).

**Remark 2.** It is more important and interesting that we should discuss the existence and large time behavior of global smooth solution for the bipolar nonisentropic Euler-Poisson system with the general ambient device temperature functions, instead of the constant ambient device temperature, as in [20]. Moreover, we also should consider the similar problem for the corresponding multi-dimensional bipolar non-isentropic Euler-Poisson systems. These are left for the forthcoming future.

The rest of this paper is arranged as follows. In Section 2, we make some necessary preliminaries. That is, we first give some well-known results on the diffusion waves and one key inequality will be used later; then we trickly construct the correction functions to delete the gaps between the solutions and the diffusion waves at the far field. We reformulate the original problem in terms of a perturbed variable and state local-in-time existence of classical solutions in Section 3. Section 4 is used to establish the uniformly a priori estimate and to show the global existence of smooth solutions, while we prove the algebraic convergence rate of smooth solutions in Section 5.

**2. Some Preliminaries**

In this section, we state the nonlinear diffusive wave and then construct the correction functions. First of all, we list some known results concerning the self-similar solution of the nonlinear parabolic equation (4). Let us recall that the nonlinear parabolic equation

\[\bar{n}_t - (\bar{n} T^*)_{xx} = 0,\]

possesses a unique self-similar solution \(\bar{n}(x, t) \equiv \bar{n}(\xi)\), \(\xi = x/\sqrt{1 + t}\), which is increasing if \(n_- < n_+\) and decreasing if \(n_- > n_+\). The corresponding Darcy law is \(\bar{j} := -(\bar{n} T^*)_x\) satisfying \(\bar{j} \to 0\) as \(x \to \pm \infty\).

**Lemma 3** (see [4, 15, 21]). *For the self-similar solution of* (11), it holds

\[\left| \bar{n}(\xi) - n_+ \right|_{\xi > 0} + \left| \bar{n}(\xi) - n_- \right|_{\xi < 0}, \]

\[\leq C \left| n_+ - n_- \right| e^{-\gamma \xi^2},\]

where \(\gamma > 0\) is a constant.

Next, we construct the gap function, which will be used in Sections 3 and 4. First of all, motivated by [6, 22], let us look into the behaviors of the solutions to (1)–(3) at the far fields \(x = \pm \infty\). Then we may understand how big the gaps are between the solution and the diffusion waves at the far fields. Let \((n_i^+(t), j_i^+(t), T_i^+(t)) := (n_i(\pm \infty, t), j_i(\pm \infty, t), T_i(\pm \infty, t)), i = 1, 2\) and \(E^\pm(t) := E(\pm \infty, t).\) From (1)_1 and (1)_4, since \(\partial_x j_i|_{x=\pm \infty} = 0\) for \(i = 1, 2\), it can be easily seen that

\[n_i^+(t) = n_i(\pm \infty, t) \equiv n_\pm.\]  

Differentiating (1)_7 with respect to \(t\) and applying (1)_1 and (1)_4, we have \(E_{xx} = (n_1 - n_2) = -(j_1 - j_2)_x\), which implies

\[\frac{d}{dt} E^+ (t) - \frac{d}{dt} E^- (t) \]

\[= - \left[ j_1^+ (t) - j_1^- (t) \right] + \left[ j_2^+ (t) - j_2^- (t) \right].\]  

Taking \(x = \pm \infty\) to (1)_2,3 and (1)_5,6, we also formally have

\[\frac{d}{dt} j_1^+ (t) = n_1 E^+ (t) - j_1^+ (t),\]  

\[\frac{d}{dt} j_2^+ (t) = -n_2 E^+ (t) - j_2^+ (t),\]  

\[\frac{d}{dt} T_i^+ (t) = \frac{1}{3} \left( \frac{j_i^+}{n_i^+} \right)^2 - (T_i^+ - T^*), \quad i = 1, 2.\]  

It can be easily seen that (14)–(17) can uniquely determine the unknown state functions \(j_i^\pm(t), T_i^\pm(t)(i = 1, 2),\) and \(E^\pm(t)\) since we have known \(E^\pm(t) = E(\pm \infty, t) = 0.\) Solving these
O.D.E and noticing (13), there exists some constant $0 < \beta_0 < 1/2$ such that

$$n_i(\pm \infty, t) = n_\pm, \quad i = 1, 2,$$

$$|j_i(\pm \infty, t)| = O (1) e^{-\beta_0 t}, \quad i = 1, 2,$$

$$j_i(-\infty, 0) = O (1) e^{-t}, \quad i = 1, 2,$$

$$T_i(\pm \infty, t) = T^* + (T_{i+} - T^*) e^{-t} + O (1) e^{-\beta_0 t}, \quad i = 1, 2,$$

$$|E(\pm \infty, t)| = O (1) e^{-\beta_0 t},$$

$$E(\pm \infty, 0) = 0.$$  \hfill (18)

Obviously, there are some gaps between $j_i(\pm \infty, t)$ and $\tilde{j}_i(\pm \infty, t)$, $T_i(\pm \infty, t)$ and $T^*$, and $E(\pm \infty, t)$ and $\tilde{E}$ $\equiv 0$, which lead to $j_i(x, t) - \tilde{j}_i(x, t)$, $T_i(x, t) - T^*$, $E(x, t) \notin L^2(\mathbb{R})$. To delete these gaps, we need to introduce the correction functions $(\tilde{n}_1, \tilde{n}_2, \tilde{j}_1, \tilde{j}_2, \tilde{T}_1, \tilde{T}_2, \tilde{E})(x, t)$. As those done in [6, 22], we can construct these gap functions. That is, we can choose $(\tilde{n}_1, \tilde{n}_2, \tilde{j}_1, \tilde{j}_2, \tilde{T}_1, \tilde{T}_2, \tilde{E})(x, t)$, which solve the system

$$\tilde{n}_{1t} + \tilde{j}_{1x} = 0,$$

$$\tilde{j}_{1t} = \tilde{n}\tilde{E} - \tilde{j}_1,$$

$$\tilde{n}_{2t} + \tilde{j}_{2x} = 0,$$

$$\tilde{j}_{2t} = -\tilde{n}\tilde{E} - \tilde{j}_2,$$

$$\tilde{E}_x = \tilde{n}_1 - \tilde{n}_2,$$ \hfill (19)

with $\tilde{j}_i(x, t) \rightarrow j_i^+(t)$ as $x \rightarrow \pm \infty$, $\tilde{E}(x, t) \rightarrow 0$ as $x \rightarrow -\infty$, and $\tilde{E}(x, t) \rightarrow E^*(t)$ as $x \rightarrow +\infty$. Here, $\tilde{n}(x) = n_+ + (n_- - n_+) \int_{-\infty}^{x} m_0(y)dy$ with $m_0(x) \geq 0$, $m_0 \in C^0_\infty(\mathbb{R})$, $\supp m_0 \subseteq [-L_0, L_0]$, and $\int_{-\infty}^{+\infty} m_0(y)dy = 1$. Moreover, we take $T_1(x, t) = \tilde{T}_1(t)(1 - g(x)) + \int_{-\infty}^{x} m_0(y)dy$, which together with (17) implies

$$\frac{\partial}{\partial t} \tilde{T}_i(x, t) = -\tilde{T}_i(x, t) + \frac{1}{3} \left( \frac{\tilde{j}_i(t)}{n_-} \right)^2 (1 - g(x))$$

$$+ \frac{1}{3} \left( \frac{\tilde{j}_i(t)}{n_+} \right)^2 g(x)$$

$$= -\tilde{T}_i(x, t) + S_i(x, t), \quad i = 1, 2.$$ \hfill (20)

In conclusion, we have constructed the required correction functions $(\tilde{n}_1, \tilde{n}_2, \tilde{j}_1, \tilde{j}_2, \tilde{T}_1, \tilde{T}_2, \tilde{E})$ which satisfy

$$\tilde{n}_{1t} + \tilde{j}_{1x} = 0,$$

$$\tilde{j}_{1t} = \tilde{n}\tilde{E} - \tilde{j}_1,$$

$$\tilde{n}_{2t} + \tilde{j}_{2x} = 0,$$

$$\tilde{j}_{2t} = -\tilde{n}\tilde{E} - \tilde{j}_2,$$

$$\tilde{E}_x = \tilde{n}_1 - \tilde{n}_2,$$ \hfill (21)

with

$$\tilde{j}_i(x, t) \rightarrow j_i^+(t), \quad \text{as } x \rightarrow \pm \infty,$$

$$\tilde{T}_i(x, t) \rightarrow T_i^+(t) - T^*, \quad \text{as } x \rightarrow \pm \infty,$$

$$\tilde{E}(x, t) \rightarrow 0, \quad \text{as } x \rightarrow -\infty,$$

$$\tilde{E}(x, t) \rightarrow E^*(t), \quad \text{as } x \rightarrow +\infty.$$ \hfill (22)

Since these details can be found in [6, 22], we only give the following decay time-exponentially of $(\tilde{n}_1, \tilde{n}_2, \tilde{j}_1, \tilde{j}_2, \tilde{T}_1, \tilde{T}_2, \tilde{E})(x, t)$.

**Lemma 4.** There exist positive constants $C$ and $\gamma < 1/2$ independent of $t$, such that

$$\left\| (\tilde{n}_1, \tilde{j}_1, \tilde{T}_1, \tilde{E})(t) \right\| \leq C \delta e^{-\gamma t}, \quad i = 1, 2.$$ \hfill (22)

and $\sup \tilde{n}_i = \sup m_0 \subseteq [-L_0, L_0]$, $i = 1, 2$.

### 3. Reformulation of Original Problem

In this section, we first reformulate the original problem in terms of the perturbed variables. Setting for $i = 1, 2$,

$$(\varphi_i, \psi_i, \theta, \mathcal{H})(x, t)$$

$$:= \left( \int_{-\infty}^{x} \left[ n_\pm(\xi, t) - \tilde{n}_\pm(\xi, t) - \tilde{n}(\xi + x_0, t) \right] d\xi, \right.$$ \hfill (23)

$$j_i(x, t) - \tilde{j}_i(x, t) - \tilde{j}(x + x_0, t), T_i(x, t)$$

$$- \tilde{T}_i(x, t) - T^*, E(x, t) - \tilde{E}(x, t) \bigg),$$

then from (1), (11), and (21), we have for $i = 1, 2$,

$$\varphi_\pm + \psi_i = 0,$$
ψ_i + \left( \frac{(-\varphi_i + \tilde{j}_i + \tilde{j})^2}{\varphi_{ix} + \tilde{n}_i + \bar{n}} \right) + (\varphi_{ix} + \tilde{n}_i + \bar{n}) \\
\times \left( \theta_i + \tilde{T}_i + T^* - \overline{\theta_i}^{+} \right)_x \\
= (-1)^{i-1} \left( \varphi_{ix} + \tilde{n}_i + \bar{n} \right) \mathcal{H} \\
+ (-1)^{i-1} \left( \varphi_{ix} + \tilde{n}_i + \bar{n} - \bar{n} \right) \tilde{E} \\
- \psi_i + (\overline{\theta_i}^{+})_{ix},

\quad \theta_i + \tilde{T}_i + T^* \\
= \frac{2}{3} \left( \frac{(-\varphi_i + \tilde{j}_i + \tilde{j})}{\varphi_{ix} + \tilde{n}_i + \bar{n}} \right) \left( \theta_i + \tilde{T}_i + T^* \right)_x \\
+ \frac{2}{3} \left( \frac{(-\varphi_i + \tilde{j}_i + \tilde{j})}{\varphi_{ix} + \tilde{n}_i + \bar{n}} \right) \left( \theta_i + \tilde{T}_i + T^* \right)

with the initial data \((\varphi, \psi, \theta)(x,0) = (\varphi_{i0}, \psi_{i0}, \theta_{i0})(x), i = 1, 2\). Further, we have

\begin{align*}
\varphi_{1tt} + \varphi_{1t} - \left( \left( \theta_1 + \tilde{T}_1 + T^* \right) \varphi_{1ix} + \bar{\mathcal{H}}_1 \right)_x &+ (\varphi_{ix} + \tilde{n}_1 + \bar{n}) \mathcal{H} \\
= -f_1 + g_1 - (\overline{\theta_1}^{+})_{ix}, \\
\varphi_{2tt} + \varphi_{2t} - \left( \left( \theta_2 + \tilde{T}_2 + T^* \right) \varphi_{2ix} + \bar{\mathcal{H}}_2 \right)_x &- (\varphi_{ix} + \tilde{n}_2 + \bar{n}) \mathcal{H} \\
= f_2 + g_2 - (\overline{\theta_2}^{+})_{ix},
\end{align*}

(25)

with the initial data

\begin{align*}
\varphi_i (x,0) &= \varphi_{i0} (x), \\
\varphi_{ix} (x,0) &= -\psi_{i0} (x), \\
\theta_i (x,0) &= \theta_{i0} (x), \\
i &= 1, 2.
\end{align*}

(26)

By the standard iteration methods (see [23]), we can prove the local existence of classical solutions of the IVP (25) and (26). Here for the sake of clarity, we only state result and omit the proof.

Lemma 5. Suppose that \((\varphi_{i0}, -\psi_{i0}, \theta_{i0}) \in \mathcal{H}^3(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R})\) for \(i = 1, 2\). Then there is a \(C_1 > 0\) such that if

\begin{align*}
\| (\varphi_{10}, \theta_{10}, \varphi_{20}, \psi_{20}, \varphi_{20}) \|_3^2 + \| (\psi_{10}, \psi_{20}) \|_2^2 &\leq C_1, \quad (28)
\end{align*}

then there is a positive number \(T_0\) such that the initial value problems (25) and (26) have a unique solution \((\varphi_i, \theta_i, \varphi_{ix}, \psi_{ix}, \theta_{ix})\) satisfying \(\varphi_i \in C([0,T_0]; \mathcal{H}^3(\mathbb{R})) \cap \ldots\)
for some positive constant $C$.

To end this section, we also derive

$$
\mathcal{K}_{tt} + \mathcal{K}_t + 2\kappa \mathcal{K} - (\tilde{n}(\theta_1 - \theta_2) + (\theta_1 + \tilde{T} + T^*) \mathcal{K}_x)_x = h_1 - h_2 + h_3 + h_4,
$$

where

$$
h_1 := (\theta_1 + \tilde{T} + T^*) (\hat{n}_1 - \hat{n}_2) + (\theta_2 - \theta_2),$$

$$+ (\varphi_{xx} + \hat{n}_1 + \hat{n}_2) \left( \tilde{T}_1 - \tilde{T}_2 \right),$$

$$h_2 := (\varphi_{xx} + \hat{n}_1 + \hat{n}_2) \mathcal{K},$$

$$h_3 := (\varphi_{xx} + \hat{n}_1 + \hat{n}_2 + 2 (\tilde{n} - \tilde{n})) \mathcal{K},$$

$$h_4 := \frac{(-\varphi_{tt} + \tilde{T} + T^*)^2}{\varphi_{xx} + \hat{n}_1 + \hat{n}_2} - \frac{(-\varphi_{tt} + \tilde{T} + T^*)^2}{\varphi_{xx} + \hat{n}_1 + \hat{n}_2},$$

$$G_3 = G_1 - G_2 + \left[ \frac{2}{3 (\varphi_{xx} + \hat{n}_1 + \hat{n}_2)} - \frac{2}{3 \varphi_{xx} + \hat{n}_1 + \hat{n}_2} \right] \varphi_{2xx}$$

$$+ \left[ \frac{2 (\theta_1 + \tilde{T} + T^*)}{3 \varphi_{xx} + \hat{n}_1 + \hat{n}_2} - \frac{2 (\theta_2 + \tilde{T} + T^*)}{3 \varphi_{xx} + \hat{n}_1 + \hat{n}_2} \right] \varphi_{2tx}$$

$$- \frac{2 (\theta_1 + \tilde{T} + T^*)}{3 (\varphi_{xx} + \hat{n}_1 + \hat{n}_2)} \varphi_{1xx} - \frac{2 (\theta_1 + \tilde{T} + T^*)}{3 (\varphi_{xx} + \hat{n}_1 + \hat{n}_2)} \varphi_{1tx},$$

$$\leq C,$$

4. Global Existence of Smooth Solutions

In this section we mainly prove global existence of smooth solutions for the initial value problems (25) and (26). To begin with, we focus on the a priori estimates of $(\varphi, \theta, \varphi_2, \varphi_3)$. For this purpose, let $T \in (0, +\infty)$, we define

$$X(T) = \left\{ (\varphi, \varphi_2, \theta, \varphi_3) : \partial_i^j \varphi \in C \left( [0, T]; H^{3-j} (\mathbb{R}) \right), \right. $$

$$\theta \in C \left( [0, T]; H^3 (\mathbb{R}) \right), \varphi_2 \in C \left( [0, T]; H^1 (\mathbb{R}) \right),$$

$$i = 1, 2, j = 0, 1 \right\},$$

with the norm

$$N(T)^2 = \max_{0 \leq t \leq T} \left\{ \left\| (\varphi, \varphi_2, \theta, \varphi_3) (t) \right\|_3^2 + \left\| (\varphi_{tt}, \varphi_{ttt}) (t) \right\|_2^2 \right. $$

$$+ \left. \left\| (\theta_{xx}, \theta_{2xx}) (t) \right\|_1 \right\}.$$

Let $N(T)^2 \leq \varepsilon^2$, where $\varepsilon$ is sufficiently small and will be determined later. Then, by Sobolev inequality, we have for $i = 1, 2,$

$$\left\| (\varphi, \varphi_{xx}, \varphi_{xxx}, \theta, \theta_{xx}, \theta_{xxx}, \varphi_{tt}, \varphi_{ttt}) (t) \right\|_{L^\infty} \leq C^i.$$  

Clearly, there exists a positive constant $c_1, c_2$ such that

$$0 < \frac{1}{c_1} \leq n_i = \varphi_{xx} + \hat{n}_i + \tilde{n} \leq c_1,$$

$$0 < \frac{1}{c_2} \leq T_i = \theta + \tilde{T} + T^* \leq c_2,$$

$$i = 1, 2.$$  

Further, from (24), we also have $\partial_i^j \mathcal{K} \in C(0, T; H^{3-j} (\mathbb{R}))$ and

$$\left\| (\mathcal{K}, \mathcal{K}_x, \mathcal{K}_x) (t) \right\|_{L^\infty} \leq C^i.$$  

Now we first establish the following basic energy estimate.
Lemma 6. Let \((\varphi_1, \theta_1, \varphi_2, \theta_2)(x, t) \in X(T)\) be the solution of the initial value problem (25) and (26). If \(\delta + \varepsilon < 1\), then it holds that for \(0 < t < T\),

\[
\begin{align*}
&\frac{2}{\varepsilon} \left\| (\varphi_i, \varphi_{ix}, \varphi_{i\varepsilon}, \theta_i)(., t) \right\|^2 + \left\| \mathcal{H}(., t) \right\|^2 + \int_0^t \left\| \mathcal{H}(., \tau) \right\|^2 d\tau \\
&\quad + \frac{2}{\varepsilon} \int_0^t \left( \left\| (\varphi_{ix}, \varphi_{i\varepsilon}, \theta_i, \theta_{i\varepsilon})(., \tau) \right\|^2 \right) d\tau \leq C \left( \Phi_0 + \delta \right).
\end{align*}
\]

(38)

Proof. Multiplying (25) by \(\varphi_1\) and \(\varphi_2\), respectively, and integrating them over \(R\) by parts, we have for \(i = 1, 2\),

\[
\begin{align*}
&\frac{d}{dt} \int_R \left( \varphi_i \varphi_i + \frac{1}{2} \varphi_i^2 \right) dx + \int_R \left( \theta_i + \bar{T}_i + T^* \right) \varphi_i^2 dx \\
&\quad + (-1)^{i-1} \int_R \left( \varphi_i \varphi_{i\varepsilon} + \frac{\partial \varphi_i}{\partial x} + \varphi_{i\varepsilon} \right) \mathcal{H} \varphi_i dx - \int_R \varphi_i^2 dx \\
&= - \int_R \overline{n_\theta} \varphi_i dx + \int_R \left( \varphi_i \varphi_{i\varepsilon} + \frac{\partial \varphi_i}{\partial x} + \varphi_{i\varepsilon} \right) \mathcal{H} \varphi_i dx \\
&\quad + (-1)^i \int_R f_i \varphi_i dx - \int_R g_i \varphi_i dx.
\end{align*}
\]

Using Cauchy-Schwarz’s inequality, and Lemmas 3 and 4, we have

\[
\begin{align*}
&\left( \varphi_i \varphi_{i\varepsilon} + \frac{\partial \varphi_i}{\partial x} + \varphi_{i\varepsilon} \right) \mathcal{H} \varphi_i dx \\
&\leq \kappa \int_R \varphi_i^2 dx + C \int_R \left( \theta_i^2 + n_i^2 \right) dx,
\end{align*}
\]

(40)

where and in the subsequent \(\kappa > 0\) is some proper small constant, and

\[
(-1)^i \int_R f_i \varphi_i dx \leq C \int_R \varphi_i^2 dx + C\delta^2 \left( 1 + t \right)^{1/4} e^{-\varepsilon t},
\]

(41)

where we also used the facts

\[
\int_R (\bar{n} - n)^2 dx \leq C\delta^2 \left( 1 + t \right)^{1/2}
\]

(42)

which can be proved from the construction of \(\bar{n}(x) \to n_\pm\), as \(x \to \pm \infty\), and the property of the diffusion wave \(\bar{n}(x + \chi_0)/\sqrt{1 + t}\). Similarly, we can show

\[
- \int_R g_i \varphi_i dx
\]

\[
= - \int_R \left( \varphi_i \varphi_{i\varepsilon} + \varphi_i \varphi_{i\varepsilon} \right) dx
\]

\[
\leq C(\delta + \varepsilon) \int_R \left( \varphi_i^2 + \varphi_i^2 \right) dx + C\delta \int_R \varphi_{i\varepsilon}^2 dx + C\delta^2 e^{-\varepsilon t},
\]

(43)

which together with (39)–(41) implies,

\[
\begin{align*}
&\frac{d}{dt} \int_R \left( \varphi_i \varphi_i + \frac{1}{2} \varphi_i^2 \right) dx + \int_R \left( \theta_i + \bar{T}_i + T^* \right) \varphi_i^2 dx \\
&\quad + (-1)^{i-1} \int_R \left( \varphi_i \varphi_{i\varepsilon} + \frac{\partial \varphi_i}{\partial x} + \varphi_{i\varepsilon} \right) \mathcal{H} \varphi_i dx - \int_R \varphi_i^2 dx \\
&\leq C(\delta + \varepsilon) \int_R \left( \varphi_i^2 + \varphi_i^2 \right) dx + C\delta^2 e^{-\varepsilon t},
\end{align*}
\]

(44)

where \(0 < \nu_i < \nu\). Moreover, for the coupled term with the electric field, we have

\[
\begin{align*}
&\int_R \left( \varphi_{ix} + \bar{n}_1 + \bar{n} \right) \mathcal{H} \varphi_1 - \left( \varphi_{ix} + \bar{n}_2 + \bar{n} \right) \mathcal{H} \varphi_2 dx \\
&\geq C \int_R \left( \varphi_{ix}^2 + \varphi_{ix}^2 \right) dx - C\delta^2 \cdot e^{-\varepsilon t}.
\end{align*}
\]

(45)

Next, multiplying (25) by \(\varphi_i\) and \(\varphi_{ix}\), respectively, and integrating their sum over \(R\) by parts, we have

\[
\begin{align*}
&\frac{d}{dt} \int_R \left( \frac{1}{2} \varphi_i^2 + \varphi_i \left( \theta_i + \bar{T}_i + T^* \right) \varphi_i^2 dx \\
&\quad + \int_R \left( \varphi_i^2 + \bar{n}_1 \varphi_{ix} + (-1)^{i-1} \varphi_{ix} + \bar{n}_1 \right) \mathcal{H} \varphi_i dx \\
&\leq (-1)^i \int_R f_i \varphi_i dx + \int_R g_i \varphi_i dx \\
&\quad - \int_R \left( \varphi_i \varphi_{i\varepsilon} + \frac{\partial \varphi_i}{\partial x} + \varphi_{i\varepsilon} \right) \mathcal{H} \varphi_i dx.
\end{align*}
\]

(46)

Using Schwartz’s inequality, (42), and Lemmas 3 and 4, we have

\[
\begin{align*}
&\int_R \left( \varphi_i \varphi_{i\varepsilon} + \frac{\partial \varphi_i}{\partial x} + \varphi_{i\varepsilon} \right) \mathcal{H} \varphi_i dx \\
&\leq \kappa \int_R \varphi_i^2 dx + C \int_R \left( \theta_i^2 + n_i^2 \right) dx \\
&\leq \kappa \int_R \varphi_i^2 dx + C(\delta + \varepsilon) \int_R \left( \varphi_i^2 + \varphi_i^2 \right) dx + C\delta^2 \cdot e^{-\varepsilon t}.
\end{align*}
\]

(47)

Since

\[
\begin{align*}
g_i \varphi_{ix} = &\left( \frac{-\varphi_i + \bar{j}_i + \bar{j}}{\varphi_i + \bar{n}_i + \bar{n}} \right)^2 \varphi_{ix} - \left( \frac{2(-\varphi_i + \bar{j}_i + \bar{j})}{\varphi_i + \bar{n}_i + \bar{n}} \right) \varphi_{i\varepsilon} \\
&+ O(1) \left( \left( \varphi_i + \bar{j}_i + \bar{j} \right) (\bar{n}_i + \bar{n})_x \right) + \left( \bar{n}_i + \bar{n} \right) \varphi_i^2,
\end{align*}
\]

(48)
we obtain, after integration by parts, that
\[
\int_{\mathbb{R}} g_{ix} \varphi_{it} \, dx \\
\leq \frac{d}{dt} \int_{\mathbb{R}} \left( -\varphi_{ix} + \frac{\tilde{j}_i + j}{2} \right)^2 \varphi_{ix} \, dx \\
+ C(\delta + \epsilon) \int_{\mathbb{R}} \left( \varphi_{i}^2 + \varphi_{i}^2 \right) \, dx \\
+C\delta \int_{\mathbb{R}} \left( \varphi_{i}^2 + \varphi_{i}^2 \right) \, dx + C\delta^2 e^{-\nu t},
\]
where we have used
\[
\left\| \left( \frac{-\varphi_{ix} + \tilde{j}_i + j}{(\varphi_{ix} + \tilde{n}_i + \tilde{n})^2} \right) \right\|_{L^\infty} \leq C(\delta + \epsilon),
\]
with the aid of \(|\varphi_{i}| < C|\varphi_{i} + \tilde{x} + \varphi_{i} + \theta_{i} + \theta_{i} + \tilde{n}_{i} + \tilde{n}| + C\delta^2 e^{-\nu t} \). Putting the above inequality into (46), we have
\[
\frac{d}{dt} \int_{\mathbb{R}} \left( \frac{1}{2} \varphi_{i}^2 + \frac{1}{2} \left( \theta_{i} + \tilde{T}_i + \tilde{T}^* \right) \varphi_{i}^2 \right) \varphi_{ix} \, dx \\
+ \left( 1 - \kappa \right) \int_{\mathbb{R}} \varphi_{it}^2 \, dx + \left( -1 \right)^{\frac{1}{2}} \int_{\mathbb{R}} \left( \varphi_{ix} + \tilde{n}_i + \tilde{n} \right) \varphi_{ix} \, dx \\
+ \int_{\mathbb{R}} \overline{\eta}_{ix} \varphi_{ix} \, dx \\
\leq C(\delta + \epsilon) \int_{\mathbb{R}} \left( \varphi_{i}^2 + \varphi_{i}^2 \right) \, dx \\
+ C \int_{\mathbb{R}} \left( \tilde{n}_{ix}^2 + \tilde{n}_{ix}^2 \right) \, dx + C\delta^2 e^{-\nu t}.
\]
On the other hand, we have
\[
\int_{\mathbb{R}} \left( \varphi_{ix} + \tilde{n}_i + \tilde{n} \right) \varphi_{ix} - \left( \varphi_{ix} + \tilde{n}_i + \tilde{n} \right) \varphi_{ix} \, dx \\
\geq \frac{d}{dt} \int_{\mathbb{R}} \frac{1}{2} \varphi_{ix} \varphi_{ix} \, dx - \int_{\mathbb{R}} \frac{1}{2} \varphi_{ix} \varphi_{ix} \, dx \\
- C\epsilon \int_{\mathbb{R}} \left( \varphi_{i}^2 + \varphi_{i}^2 + \varphi_{i}^2 + \varphi_{i}^2 \right) \, dx - C\delta e^{-\nu t}.
\]
Finally, multiplying (25) by \(3\tilde{n}(\tilde{n} + \tilde{n} + \tilde{n})/2(\theta_{i} + \tilde{T}_i + \tilde{T}^*) \theta_{i} \) and integrating the resultant equation by parts over \(\mathbb{R}\), we have
\[
\frac{d}{dt} \int_{\mathbb{R}} \frac{3\tilde{n}}{2} \left( \varphi_{ix} + \tilde{n}_i + \tilde{n} \right) \theta_{i}^2 \, dx \\
+ \int_{\mathbb{R}} \frac{3\tilde{n}}{2} \left( \varphi_{ix} + \tilde{n}_i + \tilde{n} \right) \theta_{i}^2 + \frac{\tilde{n}}{\theta_i + \tilde{T}_i + \tilde{T}^*} \theta_{ix}^2 \, dx,
\]
\[ + \int_{\mathbb{R}} \frac{3n}{2} \frac{(q_{ix} + \hat{n}_i + \overline{n})}{\theta_i + T_1 + T^*} d\theta_i dx \]
\[ + \int_{\mathbb{R}} \frac{3n}{2} \frac{(q_{ix} + \hat{n}_i + \overline{n})}{\theta_i + T_1 + T^*} \frac{2}{\theta_i + T_1 + T^*} d\theta_i dx \]
\[ \leq C(\delta + \epsilon) \int_{\mathbb{R}} \frac{(\theta_i^2 + \theta_{ix}^2 + \phi_{ix}^2 + \phi_{ix}^2)}{2} \]
\[ + C \int_{\mathbb{R}} \frac{\overline{n}^2}{\theta_i + T_1 + T^*} dx + C\delta^2(1 + t)^{1/2} e^{-\nu t}. \]
(58)

Combining (44), (45), (51), (52), and (58), we can obtain (38); this completes the proof.

Further, in the completely similar way, we can show the following.

**Lemma 7.** Let \((\varphi_1, \varphi_2, \varphi_3)(x, t) \in X(T)\) be the solution of the initial value problems (25) and (26); then it holds that for \(0 < t < T\),
\[ \sum_{i=1}^2 \left\| (\varphi_{ix}, \varphi_{ixx}, \varphi_{ixxx}, \varphi_{ixxxx}, \varphi_{ixxx}, \varphi_{ixxxx}, \varphi_{ixxxx}, \varphi_{ixxxx}) (x, t) \right\|_2^2 \]
\[ + \int_0^t \left( \sum_{i=1}^2 \left\| (\varphi_{ixx}, \varphi_{ixxx}, \varphi_{ixxxx}, \varphi_{ixxxx}, \varphi_{ixxxx}, \varphi_{ixxxx}, \varphi_{ixxxx}, \varphi_{ixxxx}) (x, \tau) \right\|_2^2 \right) d\tau \]
\[ \leq C(\Phi_0 + \delta), \]
(59)

provided that \(\epsilon + \delta \ll 1\).

Based on the local existence given in Lemma 5 and the a priori estimates given in Lemmas 6 and 7, by the standard continuity argument, we can prove the global existence of the unique solutions of the IVP (25) and (26).
\[ +\|\theta_1, \theta_2 \|^2 + \|\theta_{1t}, \theta_{2t} \|^2 \, dx \\
\leq C \left( \|(\psi_{10}, \psi_{10}, \psi_{20}, \psi_{20})\|^2 + \|(\psi_{10}, \psi_{20})\|^2 + \delta \right), \quad t > 0, \\
\|\theta_1, \theta_2 \|_{L^2} + \|\theta_{1t}, \theta_{2t} \|_{L^2} + \|\theta_{1t}, \theta_{2t} \|_{H^1} \xrightarrow{t \to \infty} 0, \quad t \to \infty. \quad (60) \]

5. The Algebraic Decay Rates

In this section, we prove the time-decay rate of smooth solutions \((\varphi_1, \theta_1, \varphi_2, \theta_2)\) of (25) with the initial data \((\varphi_{10}, -\psi_{10}, \theta_{10}, \psi_{20}, \theta_{20})\). For this aim, using the idea of [4, 15, 24], we first prove the exponential decay of \(H\) and \(\theta_1 - \theta_2\) to zero then obtain the algebraic convergence of \((\varphi_1, \theta_1, \varphi_2, \theta_2)\). Due to Theorem 8, we know that the global smooth solutions \((\varphi_1, \theta_1, \varphi_2, \theta_2)\) satisfy

\[ \|\varphi_{1t}, \varphi_{2t}, \varphi_{1x}, \varphi_{2x}, \varphi_{1xx}, \varphi_{2xx}, H_x, H_t, H_x, H_t\|_{L^\infty(R)} \leq C(\Phi_0 + \delta). \quad (62) \]

Further, by (25), we also have

\[ \|\varphi_{1t}, \varphi_{2t}, \theta_{1t}, \theta_{2t}\|_{L^\infty(R)} \leq C(\Phi_0 + \delta). \quad (63) \]

Lemma 9. Let \((\varphi_1, \theta_1, \varphi_2, \theta_2)\) be the global classical solutions of IVP (25) and (26) satisfying \(\Phi_0 + \delta \ll 1\). Then it holds for \(H\) and \(\theta_1 - \theta_2\) that for \(t > 0,\)

\[ \|H_x, H_t, H_{xx}, H_{xt}, \theta_1 - \theta_2, \theta_{1x}, \theta_{1xx}, \theta_{2x}, \theta_{2xx}, H_x, H_t, H_x, H_t\|_{L^\infty(R)} \leq C(\Phi_0 + \delta) e^{-\gamma t}. \quad (64) \]

Proof. Multiplying (30) by \(H\) and integrating it by parts over \(R\), we obtain

\[ \frac{d}{dt} \int_R \left( H H_t + \frac{1}{2} H^2 \right) dx - \int_R H_x^2 dx + 2 \int_R \tilde{\rho} H^2 dx \\
+ \int_R (\theta_1 + \tilde{T}_1 + T^*) H_x^2 dx \\
= - \int_R \tilde{\rho} (\theta_1 - \theta_2) H_x dx \\
+ \int_R (h_1^2 - h_2 - h_3 + h_{1x}) H \, dx. \quad (65) \]

Using Cauchy-Schwartz’s inequality, Lemmas 3 and 4, (62), and (63), we have

\[ - \int_R \tilde{\rho} (\theta_1 - \theta_2) H_x \, dx + \int_R h_{1x} H \, dx \\
\leq \kappa \int_R H_x^2 \, dx + C \int_R (\theta_1 - \theta_2)^2 \, dx + C \delta e^{-\gamma t}, \quad (66) \]

Moreover, noticing that

\[ h_{4x} = -\frac{j_0^2}{n_1^2} + \frac{2j_1}{n_1} \frac{H_{1x}}{H_x} + O(1) \left( \frac{n_{1x} + \frac{2j_1}{n_1} + \tilde{j}_{1x} + \tilde{j}_{2x}}{n_{1x} + \tilde{j}_{1x} + \tilde{j}_{2x}} \right) \\
+ O(1) \left( \frac{H_{2xx} + \frac{2j_1}{n_1} + \tilde{j}_{1x} + \tilde{j}_{2x}}{n_{1x} + \tilde{j}_{1x} + \tilde{j}_{2x}} \right) \times \left( H_x + H_{1x} + \tilde{j}_{1x} + \tilde{j}_{2x} \right), \quad (67) \]

then

\[ \int_R h_{4x} \, dx \leq C(\Phi_0 + \delta) \int_R (H_x^2 + H_{1x}^2 + H_{1x}^2) \, dx + C \delta e^{-\gamma t}. \quad (68) \]

Therefore, we have

\[ \frac{d}{dt} \int_R \left( H H_t + \frac{1}{2} H^2 \right) dx - \int_R H_t^2 dx + 2 \int_R \tilde{\rho} H^2 dx \\
+ \int_R \left( (\theta_1 + \tilde{T}_1 + T^*) - \kappa \right) H_x^2 dx \\
\leq C(\Phi_0 + \delta) \int_R (H_x^2 + H_{1x}^2 + H_{1x}^2) \, dx \\
+ C \int_R (\theta_1 - \theta_2)^2 \, dx + C \delta e^{-\gamma t}. \quad (69) \]

While multiplying (30) by \(H_t\) and integrating the resultant equation by parts over \(R\), similarly, we can show

\[ \frac{d}{dt} \int_R \left( \frac{1}{2} H_t^2 + \frac{1}{2} \tilde{\rho} H^2 + \left( \frac{1}{2} (\theta_1 + \tilde{T}_1 + T^*) - \frac{j_0^2}{n_1} \right) H_x^2 \right) dx \\
+ \int_R H_t^2 dx + \int_R \tilde{\rho} (\theta_1 - \theta_2) H_{1x} \, dx \\
\leq C(\Phi_0 + \delta) \times \int_R (\theta_1 - \theta_2)^2 + (\theta_1 - \theta_2)^2 + H^2 + H_x^2 + H_{1x} \, dx \\
+ C \delta e^{-\gamma t}. \quad (70) \]
Next, multiplying (31) by $(3\pi(\phi_{1x} + \bar{n}_1 + \bar{n})/2(\theta_1 + \bar{T}_1 + T^*))((\theta_1 - \theta_2))$ and integrating the resultant equation by parts over $\mathbb{R}$, we get

$$\frac{d}{dt} \int_{\mathbb{R}} \frac{3\pi(\phi_{1x} + \bar{n}_1 + \bar{n})}{4(\theta_1 + \bar{T}_1 + T^*)} (\theta_1 - \theta_2)^2 \, dx$$

$$+ \int_{\mathbb{R}} \left( \frac{3\pi(\phi_{1x} + \bar{n}_1 + \bar{n})}{2(\theta_1 + \bar{T}_1 + T^*)} (\theta_1 - \theta_2)^2 \right.$$

$$\left. + \frac{\bar{n}}{\theta_1 + \bar{T}_1 + T^*} (\theta_1 - \theta_2)^2 \right) \, dx$$

$$- \int_{\mathbb{R}} \bar{n} (\theta_1 - \theta_2) \mathcal{H}_{tx} \, dx$$

$$= \int_{\mathbb{R}} \left( \frac{3\pi(\phi_{1x} + \bar{n}_1 + \bar{n})}{4(\theta_1 + \bar{T}_1 + T^*)} \right) (\theta_1 - \theta_2)^2 \, dx$$

$$- \int_{\mathbb{R}} \left( \frac{\bar{n}}{\theta_1 + \bar{T}_1 + T^*} \right) (\theta_1 - \theta_2) (\theta_1 - \theta_2)_x \, dx$$

$$+ \int_{\mathbb{R}} G_3 \frac{3\pi(\phi_{1x} + \bar{n}_1 + \bar{n})}{2(\theta_1 + \bar{T}_1 + T^*)} (\theta_1 - \theta_2) \, dx$$

$$\leq C (\Phi_0 + \delta)$$

$$\times \int_{\mathbb{R}} \left( (\theta_1 - \theta_2)^2 + (\theta_1 - \theta_2)_x^2 + \mathcal{H}_x^2 + \mathcal{H}_t^2 \right) \, dx$$

$$+ C (\Phi_0 + \delta) e^{-\gamma_2 t},$$

where in the last inequality, we have used

$$\int_{\mathbb{R}} \left( \frac{3\pi(\phi_{1x} + \bar{n}_1 + \bar{n})}{4(\theta_1 + \bar{T}_1 + T^*)} \right) (\theta_1 - \theta_2)^2 \, dx$$

$$- \int_{\mathbb{R}} \left( \frac{\bar{n}}{\theta_1 + \bar{T}_1 + T^*} \right) (\theta_1 - \theta_2) (\theta_1 - \theta_2)_x \, dx \leq C (\Phi_0 + \delta) \int_{\mathbb{R}} \left( (\theta_1 - \theta_2)^2 + (\theta_1 - \theta_2)_x^2 \right) \, dx,$$

$$\int_{\mathbb{R}} G_3 \frac{3\pi(\phi_{1x} + \bar{n}_1 + \bar{n})}{2(\theta_1 + \bar{T}_1 + T^*)} (\theta_1 - \theta_2) \, dx$$

$$\leq C (\Phi_0 + \delta) \int_{\mathbb{R}} \left( (\theta_1 - \theta_2)^2 + (\theta_1 - \theta_2)_x^2 + \mathcal{H}_x^2 + \mathcal{H}_t^2 \right)$$

$$+ C (\Phi_0 + \delta) e^{-\gamma_2 t},$$

(73)

with the aid of

$$\sum_{i=1}^{2} (-1)^{i-1} \int_{\mathbb{R}} \left( -\frac{\phi_{ix} + \bar{j}_1 + \bar{T}}{\phi_{ix} + \bar{n}_1 + \bar{n}} \right)^{2} S_i(x,t) \right) \right)$$

$$\times \frac{3\pi(\phi_{1x} + \bar{n}_1 + \bar{n})}{2(\theta_1 + \bar{T}_1 + T^*)} (\theta_1 - \theta_2) \, dx$$

$$\leq C (\Phi_0 + \delta) \int_{\mathbb{R}} \left( (\theta_1 - \theta_2)^2 + \mathcal{H}_x^2 + \mathcal{H}_t^2 \right) \, dx$$

$$+ C (\Phi_0 + \delta) e^{-\gamma_2 t}.$$ 

(74)

Combine (69), (70), and (71), and choose proper positive constant $\lambda_1$ and $\Lambda_1$ such that

$$\lambda_1 \times (70) + \Lambda_1 \times ((71) + (72)) \sim \mathcal{H}_t^2 + \mathcal{H}_t^2 + \mathcal{H}_x^2 + (\theta_1 - \theta_2)^2.$$ 

(75)

Then, we have

$$\frac{d}{dt} \left\| (\mathcal{H}, \mathcal{H}_t, \mathcal{H}_x, (\theta_1 - \theta_2)) (\cdot, t) \right\|^2$$

$$+ C \left\| (\mathcal{H}, \mathcal{H}_t, \mathcal{H}_x, (\theta_1 - \theta_2), (\theta_1 - \theta_2)_x) (\cdot, t) \right\|^2 \leq C (\Phi_0 + \delta) e^{-\gamma_2 t},$$

(76)

which, together with Gronwall’s inequality, yields

$$\left\| (\mathcal{H}, \mathcal{H}_t, \mathcal{H}_x, (\theta_1 - \theta_2)) (\cdot, t) \right\|^2 \leq C (\Phi_0 + \delta) e^{-\gamma_2 t},$$

(77)

for some positive constants $\gamma_1$ and $C$. In the completely same way, treating $\int_{\mathbb{R}} \lambda_2 (30)x \mathcal{H}_x^2 + \mathcal{H}_t^2 + \lambda_2 (31)x (3\pi(\phi_{1x} + \bar{n}_1 + \bar{n})/2(\theta_1 + \bar{T}_1 + T^*))((\theta_1 - \theta_2)_x) \, dx$ for proper positive constants $\lambda_2$ and $\Lambda_2$, we can show

$$\left\| (\mathcal{H}_x, \mathcal{H}_x, \mathcal{H}_x, (\theta_1 - \theta_2)_x) (\cdot, t) \right\|^2 \leq C (\Phi_0 + \delta) e^{-\gamma_2 t},$$

(78)

for some constant $\gamma_2$.

Moreover, from (30), (77), and (78), we obtain

$$\left\| \mathcal{H}_t \right\|^2 \leq C (\Phi_0 + \delta) e^{-\gamma_2 t},$$

(79)

for $\gamma_3 = \min\{\gamma_1, \gamma_2\}$. Finally, by $\int_{\mathbb{R}} (31)x (\theta_1 - \theta_2)_t \, dx$ and using (77)–(79), there is a positive constant $\gamma_4$ such that

$$\left\| (\theta_1 - \theta_2) \right\|^2 \leq C (\Phi_0 + \delta) e^{-\gamma_4 t},$$

(80)

while from (31) and (77)–(78), we have

$$\left\| (\theta_1 - \theta_2)_x \right\|^2 \leq C (\Phi_0 + \delta) e^{-\gamma_2 t},$$

(81)

with $\gamma_5 = \min\{\gamma_3, \gamma_4\}$. Combination of (77)–(80) and (81) yields (64). This completes the proof. \hfill \Box

In the following, using the idea of [4,15], we turn to derive the time-decay rate of $(\phi_1, \theta_1, \phi_2, \theta_2)$ by which we are able to obtain the algebraical decay rate of $(\phi_1, \theta_1, \phi_2, \theta_2)$ in large time.
Lemma 10. Let \((\varphi_1, \theta_1, \varphi_2, \theta_2)\) be the global classical solution of the IVP (25) and (26) with initial data satisfying \(\Phi_0 + \delta \ll 1\). If it holds for \((\varphi_1, \varphi_2, \theta_1, \theta_2)\) \((t > 0)\) that

\[
\sum_{k=0}^{3} (1 + t)^k \| \partial_x^k (\varphi_1, \varphi_2) \|^2 + \sum_{k=0}^{2} (1 + t)^{k+2} \| \partial_x^k (\theta_1, \theta_2) \|^2
\]

\[
+ \frac{1}{(1 + t)^{k+2}} \| \partial_x^k (\theta_1, \theta_2) \|^2
\]

\[
+ \sum_{k=0}^{2} (1 + t)^{k+1} \| \partial_x^k (\theta_1, \theta_2) \|^2
\]

\[
+ (1 + t)^3 \| \partial_x^3 (\theta_1, \theta_2) \|^2 \ll 1,
\]

then one has

\[
\sum_{k=0}^{3} (1 + t)^k \| \partial_x^k (\varphi_1, \varphi_2) \|^2 + \sum_{k=0}^{3} \int_{0}^{t} (1 + \tau)^k \| \partial_x^k (\varphi_1, \varphi_2) \|^2 \, d\tau
\]

\[
+ \sum_{k=0}^{2} (1 + t)^{k+1} \| \partial_x^k (\theta_1, \theta_2) \|^2
\]

\[
+ \sum_{k=0}^{2} \int_{0}^{t} (1 + \tau)^{k+1} \| \partial_x^k (\theta_1, \theta_2) \|^2 \, d\tau
\]

\[
+ \int_{0}^{t} \left( \sum_{k=0}^{1} (1 + \tau)^{k+2} \| \partial_x^k (\theta_1, \theta_2) \|^2
\]

\[
+ (1 + \tau)^3 \| \partial_x^3 (\theta_1, \theta_2) \|^2 \right) \, d\tau
\]

\[
\leq C(\Phi_0 + \delta),
\]

\[
\sum_{k=0}^{2} (1 + t)^{k+2} \| \partial_x^k (\varphi_1, \varphi_2) \|^2
\]

\[
+ \sum_{k=0}^{2} \int_{0}^{t} (1 + \tau)^{k+1} \| \partial_x^k (\varphi_1, \varphi_2) \|^2 \, d\tau
\]

\[
+ \sum_{k=0}^{1} (1 + t)^{k+2} \| \partial_x^k (\theta_1, \theta_2) \|^2
\]

\[
+ \sum_{k=0}^{1} \int_{0}^{t} (1 + \tau)^{k+2} \| \partial_x^k (\theta_1, \theta_2) \|^2 \, d\tau
\]

\[
+ \left( \int_{0}^{t} \sum_{k=0}^{1} (1 + \tau)^{k+2} \| \partial_x^k (\theta_1, \theta_2) \|^2 \right) \, d\tau
\]

\[
\leq C(\Phi_0 + \delta).
\]  

(82)

Since the proof is similar as that in [15], we can omit the details.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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