Research Article

Strong Convergence of a Unified General Iteration for $k$-Strictly Pseudononsparing Mapping in Hilbert Spaces

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We introduce a unified general iterative method to approximate a fixed point of $k$-strictly pseudononsparing mapping. Under some suitable conditions, we prove that the iterative sequence generated by the proposed method converges strongly to a fixed point of a $k$-strictly pseudononsparing mapping with an idea of mean convergence, which also solves a class of variational inequalities as an optimality condition for a minimization problem. The results presented in this paper may be viewed as a refinement and as important generalizations of the previously known results announced by many other authors.

1. Introduction

Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ with inner product $\langle \cdot , \cdot \rangle$ and norm $\| \cdot \|$, respectively. Recall that a mapping $T : C \to C$ is said to be $k$-strict pseudocontractive if there exists a constant $k \in [0, 1)$ such that

$$
\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\| (I - T)x - (I - T)y \|^2,
$$

\forall x, y \in C. \quad (1)

If $k = 0$, $T$ is said to be nonexpansive mapping; that is,

$$
\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \quad (2)
$$

The set of fixed points of $T$ is denoted by $F(T)$; that is, $F(T) = \{ x \in C : Tx = x \}$. Recall also that a mapping $T : C \to C$ is said to be nonsparing if

$$
2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2, \quad \forall x, y \in C. \quad (3)
$$

It is shown in the study by Iemoto and Takahashi [1] that (3) is equivalent to

$$
\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2 \langle x - Tx, y - Ty \rangle, \quad \forall x, y \in C. \quad (4)
$$

Observe that every nonsparing mapping is quasinonexpansive; that is, $\|Tx - p\| \leq \|x - p\|$ for all $x \in C$ and all $p \in F(T)$. Following the terminology of Browder and Petryshyn [2], a mapping $T : C \to C$ is called $k$-strictly pseudononsparing if there exists a constant $k \in [0, 1)$ such that

$$
\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\| (I - T)x - (I - T)y \|^2
$$

$$
+ 2 \langle x - Tx, y - Ty \rangle, \quad \forall x, y \in C. \quad (5)
$$

Clearly, every nonsparing mapping is $k$-strictly pseudononsparing, but the converse is not true. This shows that the class of $k$-strictly pseudononsparing mappings is more general than the class of nonsparing mappings. Moreover, we remark also that the class of $k$-strictly pseudononsparing mappings is independent of the class of $k$-strict pseudocontractions.
Fixed point problem of nonlinear mappings recently becomes an attractive subject because of its application in solving variational inequalities and equilibrium problems arising in various fields of pure and applied sciences. Moreover, various iterative schemes and methods have been developed for finding fixed points of nonlinear mappings. It is worth mentioning that iterative methods for nonexpansive and nonspreading mappings have been extensively investigated. However, iterative methods for strict pseudo-contractions are far less developed than those for nonexpansive mappings though Browder and Petryshyn [2] initiated contractions are far less developed than those for nonexpansive mappings through Browder and Petryshyn [2] initiated contractions.

Moreover, various iterative schemes and methods have been developed for finding fixed points of nonlinear mappings. It is worth mentioning that iterative methods for nonexpansive and nonspreading mappings have been extensively investigated. However, iterative methods for strict pseudo-contractions are far less developed than those for nonexpansive mappings through Browder and Petryshyn [2] initiated contractions are far less developed than those for nonexpansive mappings through Browder and Petryshyn [2] initiated contractions.

where auxiliary mapping \( T_\beta = \beta I + (1 - \beta)T \). They proved that the sequences \( \{x_n\} \) and \( \{z_n\} \) converge strongly to \( P_{\Gamma(T)}u \), which is the metric projection of \( H \) onto \( F(T) \). Moreover, they considered the following Halpern type iterative scheme:

\[
x_{n+1} = \alpha_n u + (1 - \alpha_n) T_\beta x_n, \quad n \geq 1.
\]

They also proved that \( \{x_n\} \) generated by (9) converges strongly to \( q \in F(T) \) under some suitable conditions and hence resolved in the affirmative the open problem raised by Kurokawa and Takahashi [14] in their final remark for the case where the mapping \( T \) is averaged.

In 2013, Kangtunyakarn [16] further studied variational inequalities and fixed point problem of \( k \)-strictly pseudononspreading mapping \( T \) by modifying the auxiliary mapping with projection technique. To be more precise, he introduced the following iterative scheme:

\[
x_{n+1} = \alpha_n u + \beta_n (I - \lambda_n (I - T)) x_n + \gamma_n S x_n, \quad n \geq 1,
\]

where \( \alpha_n, \beta_n, \gamma_n \in (0, 1) \) such that \( \alpha_n + \beta_n + \gamma_n = 1 \) and \( \beta_n \in [c, d] \) \( c, d \in (0, 1) \) and \( S \) is a nonexpansive mapping generated by a finite family of defining operators, whose fixed point problems are equivalent to variational inequalities. Moreover, under some suitable conditions, he proved that the sequence \( \{x_n\} \) converges strongly to \( P_{\Omega} u \), where \( \Omega \) is the intersection of the set of fixed point problems and the set of solutions for variational inequalities.

Inspired and motivated by research going on in this area, we introduce a modified general iterative method for \( k \)-strictly pseudononspreading mapping, which is defined in the following way:

\[
x_{n+1} = \alpha_n \gamma f (x_n) + \beta_n x_n + [ (1 - \beta_n) I - \alpha_n B ] T_{\lambda_n} x_n,
\]

where \( \lambda_n \in (0, 1) \) and sequences \( \{\alpha_n\} \) and \( \{\beta_n\} \) in \([0, 1]\). Note that, if \( \beta_n = 0 \), scheme (11) reduces to general iterative method (6), which is mainly due to Marino and Xu [10]. If \( \beta_n = 0, \gamma = 1 \), and \( B = I \), scheme (11) reduces to viscosity approximate method introduced by Moudafi [17] and developed by Inchan [18], which also extends the Halpern type results of [19, 20] with an idea of mean convergence for \( k \)-strictly pseudononspreading mapping.

Our purpose is not only to modify the general iterative method (6) and projection method (10) to the case of a \( k \)-strictly pseudononspreading mapping, but also to establish a new strong convergence theorem with an idea of mean convergence for a \( k \)-strictly pseudononspreading mapping, which also solves a class of variational inequalities as an optimality condition for a minimization problem. Our main results presented in this paper improve and extend the corresponding results of [10, 14–17] and many others.

2. Preliminaries

Let \( C \) be a nonempty closed convex subset of real Hilbert \( H \) space with inner product \((\cdot, \cdot)\) and norm \( \|\cdot\| \), respectively. For
every point \( x \in H \), there exists a unique nearest point in \( C \), denoted by \( P_C \), such that
\[
\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \tag{12}
\]
Then \( P_C \) is called the metric projection of \( H \) onto \( C \). It is well known that \( P_C \) is a nonexpansive mapping and the following inequality holds:
\[
\langle x - u, u - y \rangle \geq 0, \quad \forall y \in C, \tag{13}
\]
if and only if \( u = P_C x \) for given \( x \in H \) and \( u \in C \).

Let \( A \) be a mapping from \( C \) into \( H \). The normal variational inequality problem is to find a point \( u \in C \) such that
\[
\langle Au, v - u \rangle \geq 0, \quad \forall v \in C. \tag{14}
\]

The set of all solutions of the variational inequality is denoted by \( VI(C, A) \). Note that \( u \in VI(C, A) \) if and only if \( u = P_C (I - \lambda A) u \) for some \( \lambda > 0 \).

Recall that an operator \( B \) is strongly positive if there exists a constant \( \bar{\gamma} > 0 \) with the property
\[
\langle Bx, x \rangle \geq \bar{\gamma} \|x\|^2, \quad \forall x \in H. \tag{15}
\]
Recall also that an operator \( f : C \to C \) is a contraction, if there exists a constant \( \rho \in (0,1) \) such that
\[
\|f(x) - f(y)\| \leq \rho \|x - y\|, \quad \forall x, y \in C. \tag{16}
\]

In order to prove our main results, we need the following lemmas and propositions.

**Lemma 1.** Let \( H \) be a real Hilbert space. There hold the following well-known results:
\begin{enumerate}[(i)]  
  \item \( \|x + y\|^2 \leq \|x\|^2 + 2\langle y, (x + y) \rangle, \forall x, y \in H; \)
  \item \( \|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2, t \in [0,1], \forall x, y \in H. \)
\end{enumerate}

**Lemma 2** (see [6]). Let \( \{x_n\} \) and \( \{z_n\} \) be bounded sequences in Banach space \( E \) and let \( \beta_n \) be a sequence in \( [0,1] \) such that
\[
0 < \liminf_{n \to \infty} \beta_n P_{\lambda_n} \leq \limsup_{n \to \infty} \beta_n < 1. \tag{17}
\]
Suppose \( x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n \) and
\[
\limsup_{n \to \infty} \left( \|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \right) < 0, \quad \forall \eta \geq 0. \tag{18}
\]
Then \( \lim_{n \to \infty} \|z_n - x_n\| = 0. \)

**Lemma 3** (see [10]). Let \( B \) be a strongly positive linear bounded operator on a Hilbert space \( H \) with a coefficient \( \bar{\gamma} > 0 \) and \( 0 < \rho < \|B\|^{-1} \). Then
\[
\|I - \rho B\| \leq 1 - \rho \bar{\gamma}. \tag{19}
\]

**Lemma 4** (see [10]). Let \( C \) be a nonempty closed convex subset of a Hilbert space \( H \). Assume that \( f : C \to C \) is a contraction with a coefficient \( \rho \in (0,1) \) and \( B \) is a strongly positive linear bounded operator with a coefficient \( \bar{\gamma} > 0 \). Then, for \( 0 < \gamma < \bar{\gamma}/\rho \),
\[
\langle x - y, (B - \gamma f)x - (B - \gamma f)y \rangle \geq (\bar{\gamma} - \gamma \rho) \|x - y\|^2, \tag{20}
\]
\[
\forall x, y \in H.
\]
That is, \( B - \gamma f \) is strongly monotone with coefficient \( \bar{\gamma} - \gamma \rho \).

**Lemma 5** (see [15]). Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \), and let \( T : C \to C \) be a \( k \)-strictly pseudononsparing mapping. Then \( I - T \) is demiclosed at zero.

**Lemma 6** (see [15]). Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \), and let \( T : C \to C \) be a \( k \)-strictly pseudononsparing mapping. If \( F(T) \neq \emptyset \), then it is closed and convex.

**Lemma 7** (see [16]). Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \), and let \( T : C \to C \) be a \( k \)-strictly pseudononsparing mapping with \( F(T) \neq \emptyset \). Then \( F(T) = VI(C, (I - T)). \)

**Lemma 8** (see [21]). Assume \( \{a_n\} \) is a sequence of nonnegative real numbers such that
\[
a_{n+1} = (1 - \gamma_n) a_n + \gamma_n \delta_n, \quad n \geq 0, \tag{21}
\]
where \( \{\gamma_n\} \) is a sequence in \((0,1)\) and \( \{\delta_n\} \) is a sequence such that
\begin{enumerate}[(i)]  
  \item \( \sum_{n=1}^{\infty} \gamma_n = \infty; \)
  \item \( 0 < \lim \sup_{n \to \infty} \gamma_n \delta_n \leq \lim \sup_{n \to \infty} \delta_n < \infty. \)
\end{enumerate}
Then \( \lim_{n \to \infty} \alpha_n = 0. \)

### 3. Main Results

**Theorem 9.** Let \( C \) be a nonempty closed convex subset of a Hilbert space \( H \) and let \( T : C \to C \) be a \( k \)-strictly pseudononsparing mapping such that \( F(T) \neq \emptyset \). Let \( f : C \to C \) be a contraction with a coefficient \( \rho \in (0,1) \) and let \( B \) be a strongly positive bounded linear operator with \( \bar{\gamma} > 0 \). For a given point \( x_0 \in C \) and \( 0 < \gamma < \bar{\gamma}/\rho \), assume that \( \alpha_n, \beta_n, \lambda_n \in [0,1] \) satisfying the following conditions:
\begin{enumerate}[(i)]  
  \item \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty; \)
  \item \( 0 < \lim \inf_{n \to \infty} \beta_n \leq \lim \sup_{n \to \infty} \beta_n < 1; \)
  \item \( \lambda_n \in (0,1 - k) \) and \( \lim_{n \to \infty} \lambda_n = 0. \)
\end{enumerate}
Then the sequence \( \{x_n\} \) generated by (11) converges strongly to \( q \in F(T) \), which is the unique solution of the following variational inequality:
\[
\langle (B - \gamma f)q, q - w \rangle \leq 0, \quad \forall w \in F(T). \tag{22}
\]

**Proof.** First, we show that sequences \( \{x_n\} \) and \( \{T x_n\} \) are bounded. Indeed, from the property of \( k \)-strictly pseudononsparing mapping defined on \( T \) and \( p \in F(T) \), we have
\[
\|T x_n - T p\|^2 = \|x_n - p - [(I - T)x_n - (I - T)p]\|^2 \\
= \|x_n - p\|^2 - 2 \langle x_n - p, (I - T)x_n \rangle + \| (I - T)x_n \|^2 \\
\leq \|x_n - p\|^2 + k \| (I - T)x_n - (I - T)p \|^2 \\
+ 2 \langle (I - T)x_n, (I - T)p \rangle \\
= \|x_n - p\|^2 + k \| (I - T)x_n \|^2, \tag{23}
\]
which implies that
\[(1-k) \| (I-T)x_n \|^2 \leq 2 \langle x_n - p, (I-T)x_n \rangle. \tag{22} \]
From \( T_{\lambda_n} = P_C[I - \lambda_n(I-T)] \) and (22), we obtain
\[
\| T_{\lambda_n} x_n - p \|^2 \\
\leq \| (x_n - p) - \lambda_n [(I-T)x_n - (I-T)p] \|^2 \\
= \| x_n - p \|^2 - 2 \lambda_n \langle x_n - p, (I-T)x_n \rangle + \lambda_n^2 \| (I-T)x_n \|^2 \\
\leq \| x_n - p \|^2 - \lambda_n [1 - k] \| (I-T)x_n \|^2 + \lambda_n^2 \| (I-T)x_n \|^2 \\
= \| x_n - p \|^2 - \lambda_n [1 - k] \| (I-T)x_n \|^2 \leq \| x_n - p \|^2. \tag{23} \]

By (i) and Lemma 3, we have that \((1 - \beta_n) I - \alpha_n B\) is positive and \(\| (1 - \beta_n) I - \alpha_n B \| \leq 1 - \beta_n - \alpha_n \bar{\gamma}\) for all \(n \geq 1\) (see, i.e., [8]). It follows from (11) and (23) that
\[
\| x_{n+1} - p \| \\
= \| \alpha_n (\gamma f(x_n) - Bp) + \beta_n (x_n - p) \\
+ [(1 - \beta_n) I - \alpha_n B] (T_{\lambda_n} x_n - p) \| \\
\leq \alpha_n \| \gamma f(x_n) - Bp \| \\
+ \beta_n \| x_n - p \| + [(1 - \beta_n - \alpha_n \bar{\gamma}) \| T_{\lambda_n} x_n - p \| \\
\leq \alpha_n \gamma f(x_n) - f(p) \| + \alpha_n \| \gamma f(p) - Bp \| \\
+ \beta_n \| x_n - p \| + [(1 - \beta_n - \alpha_n \bar{\gamma}) \| x_n - p \| \\
\leq \max \left\{ \| x_0 - p \|, \frac{1}{\gamma - \gamma \bar{\gamma}} \| \gamma f(p) - Bp \| \right\} \tag{24} \]

Therefore, \( \{ x_n \} \) is bounded and so is \( \{ T_{\lambda_n} x_n \} \). On the other hand, we estimate
\[
\| T x_n - p \|^2 \\
\leq \| x_n - p \|^2 + k \| (I-T)x_n - (I-T)p \|^2 \\
+ 2 \langle x_n - T x_n, p - Tp \rangle \\
= \| x_n - p \|^2 + k \| (x_n - p) - (T x_n - p) \|^2 \\
= \| x_n - p \|^2 + k \left( \| x_n - p \|^2 - 2 \langle x_n - p, T x_n - p \rangle \right) \\
+ \| T x_n - p \|^2, \tag{26} \]
which implies that
\[
(1-k) \| T x_n - p \|^2 \leq (1 + k) \| x_n - p \|^2 \\
+ 2k \| x_n - p \| \| T x_n - p \|. \tag{27} \]

From (27), we can obtain
\[
0 \geq (1-k) \| T x_n - p \|^2 \\
- (1 + k) \| x_n - p \|^2 - 2k \| x_n - p \| \| T x_n - p \| \\
= (1-k) \left( \| T x_n - p \|^2 + \| x_n - p \| \| T x_n - p \| \right) \\
- (1 + k) \| x_n - p \|^2 + \| x_n - p \| \| T x_n - p \| \\
= (1-k) \| T x_n - p \| \left( \| T x_n - p \| + \| x_n - p \| \right) \\
- (1 + k) \| x_n - p \| \left( \| x_n - p \| + \| T x_n - p \| \right). \tag{28} \]

It follows that
\[
\| T x_n - p \| \leq \frac{1 + k}{1 - k} \| x_n - p \|. \tag{29} \]

Combining (25) and (29), we conclude that \( \{ T x_n \} \) is bounded. Next, we will show that \( \lim_{n \to \infty} \| x_{n+1} - x_n \| = 0 \). To do this, define a sequence \( \{ x_n \} \) by
\[
x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n, \quad n \geq 1. \tag{30} \]

Observe that
\[
z_{n+1} - z_n \\
= \frac{x_{n+2} - \beta_{n+1} x_{n+1}}{1 - \beta_{n+1}} \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \\
= \frac{\alpha_{n+1} \gamma f(x_n) + [(1 - \beta_{n+1}) I - \alpha_{n+1} B] w_{n+1}}{1 - \beta_{n+1}} \\
- \frac{\alpha_n \gamma f(x_n) + [(1 - \beta_n) I - \alpha_n B] w_n}{1 - \beta_n} \tag{31} \]

where \( w_{n+1} = T_{\lambda_n} x_n \), and
\[
\| w_{n+1} - w_n \| \\
\leq \left( \| (I - \lambda_{n+1} (I-T)) x_{n+1} - (I - \lambda_n (I-T)) x_n \| \\
\leq \| x_{n+1} - x_n - \lambda_{n+1} (I-T) x_{n+1} + \lambda_n (I-T) x_n \| \tag{32} \]

\[
\leq \| x_{n+1} - x_n \| + \lambda_{n+1} \| (I-T) x_{n+1} - (I-T) x_n \| \\
+ |\lambda_{n+1} - \lambda_n| \| (I-T) x_n \|. \]
From (31) and (32), we obtain
\[
\| z_{n+1} - z_n \| \leq \alpha_{n+1} \left( \frac{1}{1 - \beta_n} \right) \| \gamma f (x_{n+1}) - B w_{n+1} \|
+ \| w_{n+1} - w_n \| + \alpha_n \left( \frac{1}{1 - \beta_n} \right) \| \gamma f (x_n) - B w_n \|
\]
(33)
\[
+ \| x_{n+1} - x_n \| + \alpha_n \left( \frac{1}{1 - \beta_n} \right) \| \gamma f (x_n) - B w_n \|
+ \lambda_{n+1} \| (I - T) x_{n+1} - (I - T) x_n \|
+ | \lambda_{n+1} - \lambda_n | \| (I - T) x_n \|.
\]

It follows from conditions (i)–(iii) and Lemma 2 that
\[
\lim_{n \to \infty} \| z_n - x_n \| = 0.
\] (34)

From (30) and (34) and condition (ii), we have
\[
\lim_{n \to \infty} \| x_{n+1} - x_n \| = \lim_{n \to \infty} (1 - \lambda_n) \| z_n - x_n \| = 0.
\] (35)

Moreover, note that \( w_n = T_{\lambda_n} x_n \) and
\[
\| x_n - w_n \|
\leq \| x_n - x_{n+1} \| + \| x_{n+1} - w_n \|
= \| x_n - x_{n+1} \|
+ \alpha_n \| \gamma f (x_n) + \beta_n x_n + [(1 - \beta_n) I - \alpha_n B] w_n - w_n \|
\leq \| x_n - x_{n+1} \| + \alpha_n \| \gamma f (x_n) - B w_n \| + \beta_n \| x_n - w_n \|.
\] (36)

which implies that
\[
\| x_n - w_n \| \leq \frac{1}{1 - \beta_n} \| x_n - x_{n+1} \| + \frac{\alpha_n}{1 - \beta_n} \| \gamma f (x_n) - B w_n \|.
\] (37)

Combining conditions (i)–(ii) and (35), we obtain
\[
\lim_{n \to \infty} \| x_n - w_n \| = \lim_{n \to \infty} \| x_n - T_{\lambda_n} x_n \| = 0.
\] (38)

That is,
\[
\lim_{n \to \infty} \| x_n - P_C [I - \lambda_n (I - T)] x_n \| = 0.
\] (39)

Next, we will prove that \( \limsup_{n \to \infty} \langle \gamma f (q) - B q, x_n - q \rangle \leq 0 \), where \( q = P_{F(T)} [I - (I + \gamma f) \lambda] q \). To show this inequality, take a subsequence \( \{ x_{n_j} \} \) of \( \{ x_n \} \) such that
\[
\limsup_{n \to \infty} \langle \gamma f (q) - B q, x_n - q \rangle
= \lim_{j \to \infty} \langle \gamma f (q) - B q, x_{n_j} - q \rangle.
\] (40)

Without loss of generality, we may assume that \( \{ x_{n_j} \} \) converges weakly to \( w \); that is, \( x_{n_j} \rightharpoonup w \) as \( j \to \infty \), where \( w \in C \). We will show that \( w \in F(T) \). From Lemmas 5 and 7, we have \( F(T) = C(\lambda, I - T) \). Assume that \( w \neq P_C [I - \lambda_n (I - T)] w \). By condition (iii), (38), and Opial’s property, we obtain
\[
\liminf_{j \to \infty} \| x_{n_j} - w \|
< \liminf_{j \to \infty} \| x_{n_j} - P_C [I - \lambda_n (I - T)] w \|
\leq \liminf_{j \to \infty} \left( \| x_{n_j} - T_{\lambda_n} x_{n_j} \| + \| P_C [I - \lambda_n (I - T)] x_{n_j} \| \right)
+ \| P_C [I - \lambda_n (I - T)] w \|
\leq \liminf_{j \to \infty} \left( \| x_{n_j} - T_{\lambda_n} x_{n_j} \| + \| x_{n_j} - w \| + \| T_{\lambda_n} x_{n_j} - (I - T) w \| \right)
\leq \liminf_{j \to \infty} \| x_{n_j} - w \|.
\] (41)

This is a contradiction. Then \( w \in F(T) \). Since \( x_{n_j} \rightharpoonup w \) as \( j \to \infty \), we have
\[
\lim_{j \to \infty} \langle \gamma f (q) - B q, x_{n_j} - q \rangle
= \lim_{j \to \infty} \langle \gamma f (q) - B q, x_{n_j} - q \rangle
= \langle \gamma f (q) - B q, w - q \rangle \leq 0.
\] (42)

On the other hand, we will show the uniqueness of a solution of the variational inequality
\[
\langle (B - \gamma f) x, x - w \rangle \leq 0, \ w \in F(T).
\] (43)

Suppose \( q \in F(T) \) and \( \tilde{q} \in F(T) \) both are solutions to (43); then
\[
\langle (B - \gamma f) q, q - \tilde{q} \rangle \leq 0,
\langle (B - \gamma f) \tilde{q}, \tilde{q} - q \rangle \leq 0.
\] (44)

Adding up (44), we get
\[
\langle (B - \gamma f) q - (B - \gamma f) \tilde{q}, q - \tilde{q} \rangle \leq 0.
\] (45)

From Lemma 4, the strong monotonicity of \( B - \gamma f \), we obtain \( q = \tilde{q} \) and the uniqueness is proved.
Finally, we show that \( \{x_n\} \) converges strongly to \( q \) as \( n \to \infty \). From (11), (23), and Lemma 1, we have (note that \( w_n = T_\lambda x_{n'} \))

\[
\|x_{n+1} - q\|^2 = \langle \alpha_n f(x_n) + \beta_n x_n \rangle + \alpha_n \langle f(x_n) - Bq, x_{n+1} - q \rangle + \alpha_n \langle f(x_n) - Bq, x_{n+1} - q \rangle
\]

\[
= \alpha_n \langle \gamma f(x_n) - Bq, x_{n+1} - q \rangle + \beta_n \langle x_n - q, x_{n+1} - q \rangle
\]

\[
\leq \alpha_n \gamma \|f(x_n) - f(q)\| \|x_{n+1} - q\| + \alpha_n \gamma \|f(q) - Bq\| \|w_n - q\| \|x_{n+1} - q\|
\]

\[
\leq \alpha_n \gamma \rho \|x_n - q\| \|x_{n+1} - q\| + \alpha_n \gamma \|f(q) - Bq\| \|x_n - q\| \|x_{n+1} - q\|
\]

\[
= \left[1 - (\overline{\gamma} - \gamma \rho) \alpha_n \right] \|x_n - q\| \|x_{n+1} - q\| + \alpha_n \gamma \|f(q) - Bq\| \|x_n - q\| \|x_{n+1} - q\|
\]

\[
\leq \frac{1 - (\overline{\gamma} - \gamma \rho) \alpha_n}{2} \left( \|x_n - q\|^2 + \|x_{n+1} - q\|^2 \right) + \alpha_n \gamma \|f(q) - Bq\| \|x_n - q\| \|x_{n+1} - q\|
\]

\[
\leq \frac{1 - (\overline{\gamma} - \gamma \rho) \alpha_n}{2} \|x_n - q\|^2 + \alpha_n \gamma \|f(q) - Bq\| \|x_n - q\| \|x_{n+1} - q\|
\]

It follows that

\[
\|x_{n+1} - q\|^2 \leq \left[1 - (\overline{\gamma} - \gamma \rho) \alpha_n \right] \|x_n - q\|^2 + 2\alpha_n \gamma \|f(q) - Bq\| \|x_n - q\| \|x_{n+1} - q\|
\]

Together with \( 0 < \gamma < \overline{\gamma}/\rho \), condition (i), and (42), we can arrive at the desired conclusion \( \lim_{n \to \infty} \|x_n - q\| = 0 \) by Lemma 8. This completes the proof.

**Theorem 10.** Let \( C \) be a nonempty closed convex subset of a Hilbert space \( H \) and let \( T : C \to C \) be a k-strictly pseudononspreading mapping such that \( F(T) \neq \emptyset \). Let \( f : C \to C \) be a contraction with a coefficient \( \rho \in (0,1) \). Let \( \{x_n\} \) be a sequence generated by \( x_0 \in C \) in the following manner:

\[
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) T_\lambda x_{n'}, \quad n \geq 1,
\]

where \( \{\alpha_n\}, \{\beta_n\}, \) and \( \{\lambda_n\} \) are sequences in \((0,1)\) satisfying the following conditions:

(i) \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \);

(ii) \( 0 < \lim \inf_{n \to \infty} \beta_n \leq \lim \sup_{n \to \infty} \beta_n < 1 \);

(iii) \( \lambda_n \in (0,1) \) and \( \lim_{n \to \infty} \lambda_n = 0 \).

Then the sequence \( \{x_n\} \) converges strongly to \( q \in F(T) \), which is the unique solution of the following variational inequality:

\[
\langle (I - f)q, q - w \rangle \leq 0, \quad \forall w \in F(T).
\]

**Proof.** Putting \( B = I \) and \( \gamma = 1 \), general iterative scheme (II) reduces to viscosity iteration (48). The desired conclusion follows immediately from Theorem 9. This completes the proof.

**Theorem 11.** Let \( C \) be a nonempty closed convex subset of a Hilbert space \( H \) and let \( T : C \to C \) be a nonspreading mapping (or quasinonexpansive) such that \( F(T) \neq \emptyset \). Let \( f : C \to C \) be a contraction with a coefficient \( \rho \in (0,1) \) and let \( B \) be a strongly positive bounded linear operator with \( \overline{\gamma} > 0 \) and \( 0 < \gamma < \overline{\gamma}/\rho \). Let \( \{x_n\} \) be a sequence generated by \( x_0 \in C \) in the following manner:

\[
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \beta_n) \alpha_n B T_\lambda x_{n'}, \quad n \geq 1,
\]

where \( \{\alpha_n\} \) and \( \{\beta_n\} \) are two sequences in \((0,1)\) satisfying the following conditions:

(i) \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \);

(ii) \( 0 < \lim \inf_{n \to \infty} \beta_n \leq \lim \sup_{n \to \infty} \beta_n < 1 \);

(iii) \( \lambda_n \in (0,1) \) and \( \lim_{n \to \infty} \lambda_n = 0 \).

Then the sequence \( \{x_n\} \) converges strongly to \( q \in F(T) \), which is the unique solution of the following variational inequality:

\[
\langle (B - \gamma f)q, q - w \rangle \leq 0, \quad \forall w \in F(T).
\]

**Proof.** Clearly, every nonspreading mapping \( T \) is 0-strictly pseudononspreading, which is also quasinonexpansive. Therefore, the desired conclusion follows immediately from Theorem 9. This completes the proof.

**Remark 12.** Theorems 9 and 10 extend the Halpern type methods of [14, 15] and viscosity methods of Moudafi [17] to more general unified general iterative methods for \( k \)-strictly pseudononspreading mapping, which also solves a class of variational inequalities related to an optimality problem.

**Remark 13.** Theorems 9 and 10 improve and extend the main results of Kangtunyakarn [16] for \( k \)-strictly pseudononspreading mapping in different directions.

**Remark 14.** The auxiliary mapping \( T_\beta \) of Osilike and Isiogugu [15] is generalized to the averaged mapping \( T_\lambda \) presented in scheme (II) with variable coefficient and projection operator based on the equivalence between variational inequality and fixed point problem.
Conflict of Interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

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