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*Research Article*

# On Positive Solutions of a Fourth Order Nonlinear Neutral Delay Difference Equation

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The existence results of uncountably many bounded positive solutions for a fourth order nonlinear neutral delay difference equation are proved by means of the Krasnoselskii's fixed point theorem and Schauder's fixed point theorem. A few examples are included.

## 1. Introduction

In the past few decades, the researchers [1–31] and others studied oscillation, asymptotic behavior, and solvability for a lot of second and third order nonlinear difference equations, some of which are as follows:

$$\begin{gathered}
\Delta\left(a_n\Delta\left(x_n+px_{n-\tau}\right)\right)+F\left(n+1,x_{n+1-\sigma}\right)=0,\quad n\geq 1,\\
\Delta\left(a_n\Delta\left(x_n+bx_{n-\tau}\right)\right)\\
+f\left(n,x_{n-d_{1n}},\ldots,x_{n-d_{kn}}\right)=c_n,\quad n\geq n_0,\\
\Delta^3x_n+f\left(n,x_n,x_{n-r}\right)=0,\quad n\geq n_0,\\
\Delta\left(a_n\Delta^2\left(x_n+P_nx_{n-\tau}\right)\right)\\
+f\left(n,x_{n-d_{1n}},\ldots,x_{n-d_{kn}}\right)=g_n,\quad n\geq n_0,\\
\Delta^2\left(a_n\Delta\left(x_n+\gamma_nx_{n-\tau}\right)\right)+\Delta^2f\left(n,x_{n-d_{1n}},\ldots,x_{n-d_{kn}}\right)\\
+\Delta g\left(n,x_{n-d_{1n}},\ldots,x_{n-d_{kn}}\right)\\
=h\left(n,x_{n-d_{1n}},\ldots,x_{n-d_{kn}}\right),\quad n\geq n_0.
\end{gathered}\tag{1}$$

By employing a few famous tools in nonlinear analysis including the nonlinear alternative of Leray-Schauder type, Banach's fixed point theorem, Schauder's fixed point theorem, Krasnoselskii's fixed point theorem, coincidence degree theory and critical point theory, the authors [3, 4, 9, 13–16, 18, 25, 27, 28] and others proved the existence of nonoscillatory solutions, uncountably many bounded nonoscillatory solutions and periodic solutions for the difference equations above, where Lipschitz conditions were used in [14, 16]. Recently, the authors [32] used the Krasnoselskii's fixed point theorem to obtain $h$-asymptotic stability results about the zero solution for a very general first order nonlinear neutral differential equation with functional delay.

However, to our knowledge, no one studied the following fourth order nonlinear neutral delay difference equation:

$$\begin{gathered}
\Delta^3\left(a_n\Delta\left(x_n+\gamma_nx_{n-\tau}\right)\right)+\Delta^3f\left(n,x_{b_{1n}},\ldots,x_{b_{kn}}\right)\\
+\Delta^2g\left(n,x_{c_{1n}},\ldots,x_{c_{kn}}\right)+\Delta h\left(n,x_{d_{1n}},\ldots,x_{d_{kn}}\right)\\
+p\left(n,x_{o_{1n}},\ldots,x_{o_{kn}}\right)=r_n,\quad n\geq n_0,
\end{gathered}\tag{2}$$

where $\tau,k\in\mathbb{N}$, $n_0\in\mathbb{N}_0$, $f,g,h,p\in C(\mathbb{N}_{n_0}\times\mathbb{R}^k,\mathbb{R})$, $\{a_n\}_{n\in\mathbb{N}_{n_0}}$, and $\{\gamma_n\}_{n\in\mathbb{N}_{n_0}}$ are real sequences with $a_n\neq 0$ for $n\in\mathbb{N}_{n_0}$ and $\{b_{ln},c_{ln},d_{ln},o_{ln}:\ n\in\mathbb{N}_{n_0},l\in\{1,2,\ldots,k\}\}\subset\mathbb{Z}$ with

$$\begin{gathered}
\lim_{n\to\infty}b_{ln}=\lim_{n\to\infty}c_{ln}=\lim_{n\to\infty}d_{ln}=\lim_{n\to\infty}o_{ln}=+\infty,\\
l\in\{1,2,\ldots,k\}.
\end{gathered}\tag{3}$$

The purpose of this paper is to fill this gap in the literature and to study solvability of (2). Under certain conditions, we prove the existence of uncountably many bounded positive solutions of (2) by means of the Krasnoselskii's fixed point theorem and Schauder's fixed point theorem, respectively. Nine examples are included.

This paper is organized as follows. In Section 2 we present some notations, definitions, and lemmas. In Section 3 we establish nine sufficient conditions which guarantee the existence of uncountably many bounded positive solutions of (2) by using fixed point theorems and new techniques. In Section 4 we give nine examples to illustrate the effectiveness and applications of the results presented in Section 3.

## 2. Preliminaries

Throughout this paper, we assume that $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{Z}, \mathbb{N}$ and $\mathbb{N}_0$ stand for the sets of all integers, positive integers, and nonnegative integers, respectively:

$$\begin{gathered}\mathbb{N}_{n_0} = \{n : n \in \mathbb{N} \text{ with } n \geq n_0\}, \quad n_0 \in \mathbb{N}_0, \\ \alpha = \inf\left\{b_{ln}, d_{ln}, c_{ln}, o_{ln} : 1 \leq l \leq k,\ n \in \mathbb{N}_{n_0}\right\}, \\ \beta = \min\{n_0 - \tau, \alpha\}, \\ \mathbb{Z}_\beta = \{n : n \in \mathbb{Z} \text{ with } n \geq \beta\};\end{gathered} \tag{4}$$

$\Delta$ denotes the forward different operator defined by $\Delta x_n = x_{n+1} - x_n$ and $\Delta^i x_n = \Delta(\Delta^{i-1} x_n)$ for $i \in \{2, 3, 4\}$. Let $l_\beta^\infty$ denote the Banach space of all bounded sequences on $\mathbb{Z}_\beta$ with norm:

$$\begin{gathered}\|x\| = \sup_{n \in \mathbb{Z}_\beta} |x_n| \quad \text{for } x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in l_\beta^\infty, \\ \Omega_1(N, M) \\ = \left\{x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in l_\beta^\infty : \ N \leq x_n \leq M,\ n \in \mathbb{Z}_\beta\right\}; \\ \Omega_{2T}(N, M) = \left\{x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in l_\beta^\infty : \ \frac{N}{\gamma_n} \leq x_n \leq \frac{M}{\gamma_n},\right. \\ \left. n \geq T;\ \frac{N}{\gamma_T} \leq x_n \leq \frac{M}{\gamma_T},\ \beta \leq n < T\right\}; \\ \Omega_{3T}(N, M) = \left\{x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in l_\beta^\infty : \ \frac{-N}{\gamma_n} \leq x_n \leq \frac{-M}{\gamma_n},\right. \\ \left. n \geq T;\ \frac{-N}{\gamma_T} \leq x_n \leq \frac{-M}{\gamma_T},\ \beta \leq n < T\right\}.\end{gathered} \tag{5}$$

Obviously, $\Omega_1(N, M)$, $\Omega_{2T}(N, M)$, and $\Omega_{3T}(N, M)$, are closed bounded and convex subsets of $l_\beta^\infty$ for any $M > N > 0$.

By a solution of (2), we mean a real sequence $\{x_n\}_{n \in \mathbb{Z}_\beta}$ with a positive integer $T \geq n_0 + \tau + |\beta|$ such that (2) is satisfied for all $n \geq T$.

*Definition 1* (see [5]). A subset $D$ of $l_\beta^\infty$ is said to be uniformly Cauchy (or equi-Cauchy) if for every $\varepsilon > 0$ there exists a positive integer $K > \beta$ such that

$$|x_i - x_j| < \varepsilon, \tag{6}$$

whenever $i, j > K$ for any $x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in l_\beta^\infty$.

**Lemma 2** (see [5]). *Each bounded and uniformly Cauchy subset of $l_\beta^\infty$ is relatively compact.*

**Lemma 3** (Krasnoselskii's fixed point theorem). *Let $X$ be a Banach space, let $D$ be a bounded closed convex subset of $X$, and let $S, G$ be mappings from $D$ into $X$ such that $Sx + Gy \in D$ for every pair $x, y \in D$. If $S$ is a contraction and $G$ is completely continuous, then the equation*

$$Sx + Gx = x \tag{7}$$

*has a solution in $D$.*

**Lemma 4** (Schauder's fixed point theorem). *Let $D$ be a nonempty closed convex subset of a Banach space $X$, $T : D \to D$ continuous, and $f(D)$ relatively compact. Then $f$ has at least one fixed point in $D$.*

**Lemma 5.** *Let $\tau \in \mathbb{N}$, $n_0 \in \mathbb{N}_0$, $\{a_n\}_{n \in \mathbb{N}_{n_0}}$, $\{b_n\}_{n \in \mathbb{N}_{n_0}}$, and $\{c_n\}_{n \in \mathbb{N}_{n_0}}$ be nonnegative sequences. If*

$$\sum_{j=n_0}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} j c_j a_s b_t < +\infty, \tag{8}$$

*then*

$$\begin{aligned}&\sum_{i=1}^{\infty} \sum_{j=n_0+i\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} c_j a_s b_t \\ &\quad = \sum_{j=n_0+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \left[\frac{j - n_0}{\tau}\right] c_j a_s b_t \\ &\quad \leq \frac{1}{\tau} \sum_{j=n_0+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} j c_j a_s b_t,\end{aligned} \tag{9}$$

*where $[(j - n_0)/\tau]$ denotes the integer part of number $(j - n_0)/\tau$.*

*Proof.* Notice that

$$\begin{aligned}&\sum_{i=1}^{\infty} \sum_{j=n_0+i\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} c_j a_s b_t \\ &\quad = \sum_{j=n_0+\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} c_j a_s b_t + \sum_{j=n_0+2\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} c_j a_s b_t \\ &\quad + \sum_{j=n_0+3\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} c_j a_s b_t + \cdots\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=n_0+\tau}^{n_0+2\tau-1}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\left[\frac{j-n_0}{\tau}\right]c_j a_s b_t \\
&\quad + \sum_{j=n_0+2\tau}^{n_0+3\tau-1}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\left[\frac{j-n_0}{\tau}\right]c_j a_s b_t \\
&\quad + \sum_{j=n_0+3\tau}^{n_0+4\tau-1}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\left[\frac{j-n_0}{\tau}\right]c_j a_s b_t + \cdots \\
&= \sum_{j=n_0+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\left[\frac{j-n_0}{\tau}\right]c_j a_s b_t \\
&\le \frac{1}{\tau}\sum_{j=n_0+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty} j c_j a_s b_t.
\end{aligned} \tag{10}
$$

That is, (9) holds. This completes the proof. □

## 3. Existence of Uncountably Many Bounded Positive Solutions

Now we study the existence of uncountably many bounded positive solutions for (2) by using the Krasnoselskii's fixed point theorem and Schauder's fixed point theorem, respectively.

**Theorem 6.** *Assume that there exist constants $n_1 \in \mathbb{N}_{n_0}$, $M$, $N$, and $c$ with $M > N > 0$ and $c \in [0, (M-N)/2M)$ and nonnegative sequences $\{W_n\}_{n\in\mathbb{N}_{n_0}}$, $\{P_n\}_{n\in\mathbb{N}_{n_0}}$, $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$, and $\{R_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying*

$$
|\gamma_n| \le c, \quad \forall n \ge n_1, \tag{11}
$$

$$
\begin{gathered}
|f(n,u_1,u_2,\ldots,u_k)| \le W_n, \qquad |g(n,u_1,u_2,\ldots,u_k)| \le P_n, \\
|h(n,u_1,u_2,\ldots,u_k)| \le Q_n, \qquad |p(n,u_1,u_2,\ldots,u_k)| \le R_n, \\
\forall (n,u_1,u_2,\ldots,u_k) \in \mathbb{N}_{n_0} \times [N,M]^k,
\end{gathered} \tag{12}
$$

$$
\begin{aligned}
\max\Bigg\{&\sum_{t=n_0}^{\infty}\frac{W_t}{|a_t|}, \sum_{s=n_0}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t + P_t\right], \\
&\sum_{j=n_0}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t + |r_t|\right)\Bigg\} < +\infty.
\end{aligned} \tag{13}
$$

*Then (2) possesses uncountably many bounded positive solutions in $\Omega_1(N,M)$.*

*Proof.* Set $L \in (N+cM, M(1-c))$. It follows from (13) that there exists $T \ge n_0 + n_1 + \tau + |\beta|$ sufficiently large such that

$$
\begin{aligned}
&\sum_{t=T}^{\infty}\frac{W_t}{|a_t|} + \sum_{s=T}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t + P_t\right] \\
&\quad + \sum_{j=T}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t + |r_t|\right) \\
&\quad < \min\{M(1-c) - L, L - cM - N\}.
\end{aligned} \tag{14}
$$

Define two mappings $U_L$ and $S_L\colon \Omega_1(N,M) \to l_\beta^\infty$ by

$$
(U_L x)_n = \begin{cases} \dfrac{1}{2}L - \gamma_n x_{n-\tau}, & n \ge T, \\ (U_L x)_T, & \beta \le n < T, \end{cases} \tag{15}
$$

$$
\begin{aligned}
&(S_L x)_n \\
&= \begin{cases}
\dfrac{1}{2}L + \displaystyle\sum_{t=n}^{\infty}\frac{1}{a_t} f\left(t, x_{b_{1t}}, \ldots, x_{b_{kt}}\right) \\
\quad + \displaystyle\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1)h\left(t, x_{d_{1t}}, \ldots, x_{d_{kt}}\right)\right. \\
\qquad \left. - g\left(t, x_{c_{1t}}, \ldots, x_{c_{kt}}\right)\right] \\
\quad - \displaystyle\sum_{j=n}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left[p\left(t, x_{o_{1t}}, \ldots, x_{o_{kt}}\right) - r_t\right], \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n \ge T, \\
(S_L x)_T, \qquad\qquad\qquad\qquad\qquad\qquad \beta \le n < T
\end{cases}
\end{aligned} \tag{16}
$$

for each $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in \Omega_1(N,M)$.

Now we prove that

$$
\begin{gathered}
U_L x + S_L y \in \Omega_1(N,M), \quad \forall x, y \in \Omega_1(N,M), \\
\|U_L x - U_L y\| \le c\|x - y\|, \quad \forall x, y \in \Omega_1(N,M), \\
\|S_L x\| < M, \quad \forall x, y \in \Omega_1(N,M).
\end{gathered} \tag{17}
$$

In view of (11), (12), and (14)–(16), we conclude that for any $x = \{x_n\}_{n\in\mathbb{Z}_\beta}$, $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \Omega_1(N,M)$, and $n \ge T$,

$$
\begin{aligned}
&|(U_L x)_n + (S_L y)_n - L| \\
&= \Bigg| -\gamma_n x_{n-\tau} + \sum_{t=n}^{\infty}\frac{1}{a_t} f\left(t, y_{b_{1t}}, \ldots, y_{b_{kt}}\right) \\
&\quad + \sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1)h\left(t, y_{d_{1t}}, \ldots, y_{d_{kt}}\right)\right. \\
&\qquad \left. - g\left(t, y_{c_{1t}}, \ldots, y_{c_{kt}}\right)\right] \\
&\quad - \sum_{j=n}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j} \\
&\qquad \times \left[p\left(t, y_{o_{1t}}, \ldots, y_{o_{kt}}\right) - r_t\right]\Bigg|
\end{aligned}
$$

$$\begin{aligned}
&\le cM + \sum_{t=T}^{\infty}\frac{W_t}{|a_t|} + \sum_{s=T}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|} \times \left[(t-s+1)Q_t + P_t\right] \\
&\quad + \sum_{j=T}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t + |r_t|\right) \\
&< cM + \min\{M(1-c) - L, L - cM - N\},
\end{aligned}$$

$$\begin{aligned}
\|U_L x - U_L y\| &= \sup_{n\in\mathbb{Z}_\beta}\left|(U_L x)_n - (U_L y)_n\right| \\
&= \max\left\{\sup_{T>n\ge\beta}\left|(U_L x)_n - (U_L y)_n\right|, \sup_{n\ge T}\left|(U_L x)_n - (U_L y)_n\right|\right\} \\
&= \sup_{n\ge T}\left(|\gamma_n|\,|x_{n-\tau} - y_{n-\tau}|\right) \\
&\le c\|x-y\|,
\end{aligned}$$

$$\begin{aligned}
\|S_L x\| &= \sup_{n\in\mathbb{Z}_\beta}\left|(S_L x)_n\right| \\
&= \max\left\{\sup_{T>n\ge\beta}\left|(S_L x)_n\right|, \sup_{n\ge T}\left|(S_L x)_n\right|\right\} \\
&= \sup_{n\ge T}\left|\frac{1}{2}L + \sum_{t=n}^{\infty}\frac{1}{a_t}f\left(t, x_{b_{1t}}, \ldots, x_{b_{kt}}\right)\right. \\
&\quad + \sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1) \times h\left(t, x_{d_{1t}}, \ldots, x_{d_{kt}}\right) - g\left(t, x_{c_{1t}}, \ldots, x_{c_{kt}}\right)\right] \\
&\quad \left. - \sum_{j=n}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left[p\left(t, x_{o_{1t}}, \ldots, x_{o_{kt}}\right) - r_t\right]\right| \\
&\le \sup_{n\ge T}\left(\frac{1}{2}L + \sum_{t=n}^{\infty}\frac{W_t}{|a_t|} + \sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t + P_t\right] + \sum_{j=n}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t + |r_t|\right)\right) \\
&\le \frac{1}{2}L + \sum_{t=T}^{\infty}\frac{W_t}{|a_t|} + \sum_{s=T}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t + P_t\right] \\
&\quad + \sum_{j=T}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t + |r_t|\right) \\
&< \frac{1}{2}L + \min\{M(1-c) - L, L - cM - N\} \\
&< M,
\end{aligned} \tag{18}$$

which yield that (17) hold.

In order to prove that $S_L$ is completely continuous in $\Omega_1(N, M)$, we have to show that $S_L$ is continuous in $\Omega_1(N, M)$ and $S_L(\Omega_1(N, M))$ is relatively compact. Suppose that $\{x^m\}_{m\in\mathbb{N}}$ is an arbitrary sequence in $\Omega_1(N, M)$ and $x \in \Omega_1(N, M)$ with $\lim_{m\to\infty}x^m = x$, where $x^m = \{x_n^m\}_{n\in\mathbb{Z}_\beta}$ for each $m \in \mathbb{N}$ and $x = \{x_n\}_{n\in\mathbb{Z}_\beta}$. With the help of (12), (13), $\lim_{m\to\infty}x^m = x$, and the continuity of $f, g, h$, and $p$, we know that for given $\varepsilon > 0$, there exist $T_1, T_2, T_3$, and $T_4 \in \mathbb{N}$ with $T_4 > T_3 > T_2 > T_1 > T$ satisfying

$$\begin{aligned}
&\sum_{t=T_1}^{\infty}\frac{W_t}{|a_t|} + \sum_{s=T_1}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t + P_t\right] + \sum_{j=T_1}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}R_t < \frac{\varepsilon}{18}; \\
&\max\left\{\sum_{t=T_3}^{\infty}\frac{t-s+1}{|a_j|}R_t : T \le j \le T_1, j \le s \le T_2\right\} < \frac{\varepsilon}{18T_1T_2}; \\
&\max\left\{\sum_{s=T_2}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}R_t : T \le j \le T_1\right\} < \frac{\varepsilon}{18T_1}; \\
&\max\left\{\sum_{t=T_2}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t + P_t\right] : T \le s \le T_1\right\} < \frac{\varepsilon}{18T_1}; \\
&\max\left\{\left|q\left(t, x_{b_{1t}}^m, \ldots, x_{b_{kt}}^m\right) - q\left(t, x_{b_{1t}}, \ldots, x_{b_{kt}}\right)\right| : q \in \{f, g, h, p\}\right\} \\
&\quad < \frac{\varepsilon}{18T_1T_2T_3(A+B+E)}, \quad \forall m \ge T_4,\ T \le t \le T_3,
\end{aligned} \tag{19}$$

where

$$A = \max\left\{\frac{1}{|a_s|} : T \le s \le T_1\right\},$$

$$B = \max\left\{\frac{t-s+1}{|a_s|} : T \le s \le T_1, s \le t \le T_2\right\},$$

$$E = \max\left\{\frac{t-s+1}{|a_j|} : T \le j \le T_1, j \le s \le T_2, s \le t \le T_3\right\}. \tag{20}$$

By virtue of (16), (19), and (20), we get that

$$\begin{aligned}
&\|S_L x^m - S_L x\| \\
&= \sup_{n\in Z_\beta} \left|(S_L x^m)_n - (S_L x)_n\right| \\
&= \max\left\{\sup_{T>n\ge\beta} \left|(S_L x^m)_n - (S_L x)_n\right|, \sup_{n\ge T} \left|(S_L x^m)_n - (S_L x)_n\right|\right\} \\
&= \sup_{n\ge T}\Bigg|\sum_{t=n}^{\infty}\frac{1}{a_t} f\left(t, x^m_{b_{1t}}, \ldots, x^m_{b_{kt}}\right) \\
&\quad + \sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1)\, h\left(t, x^m_{d_{1t}}, \ldots, x^m_{d_{kt}}\right) - g\left(t, x^m_{c_{1t}}, \ldots, x^m_{c_{kt}}\right)\right] \\
&\quad - \sum_{j=n}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left[p\left(t, x^m_{o_{1t}}, \ldots, x^m_{o_{kt}}\right) - r_t\right] \\
&\quad - \Bigg(\sum_{t=n}^{\infty}\frac{1}{a_t} f\left(t, x_{b_{1t}}, \ldots, x_{b_{kt}}\right) \\
&\quad + \sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1)\, h\left(t, x_{d_{1t}}, \ldots, x_{d_{kt}}\right) - g\left(t, x_{c_{1t}}, \ldots, x_{c_{kt}}\right)\right] \\
&\quad - \sum_{j=n}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left[p\left(t, x_{o_{1t}}, \ldots, x_{o_{kt}}\right) - r_t\right]\Bigg)\Bigg| \\
&\le \sum_{t=T}^{\infty}\frac{1}{|a_t|}\left|f\left(t, x^m_{b_{1t}}, \ldots, x^m_{b_{kt}}\right) - f\left(t, x_{b_{1t}}, \ldots, x_{b_{kt}}\right)\right| \\
&\quad + \sum_{s=T}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\Big[(t-s+1)\left|h\left(t, x^m_{d_{1t}}, \ldots, x^m_{d_{kt}}\right) - h\left(t, x_{d_{1t}}, \ldots, x_{d_{kt}}\right)\right| \\
&\quad + \left|g\left(t, x^m_{c_{1t}}, \ldots, x^m_{c_{kt}}\right) - g\left(t, x_{c_{1t}}, \ldots, x_{c_{kt}}\right)\right|\Big] \\
&\quad + \sum_{j=T}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left|p\left(t, x^m_{o_{1t}}, \ldots, x^m_{o_{kt}}\right) - p\left(t, x_{o_{1t}}, \ldots, x_{o_{kt}}\right)\right| \\
&= \sum_{t=T}^{T_1}\frac{1}{|a_t|}\left|f\left(t, x^m_{b_{1t}}, \ldots, x^m_{b_{kt}}\right) - f\left(t, x_{b_{1t}}, \ldots, x_{b_{kt}}\right)\right| \\
&\quad + \sum_{t=T_1+1}^{\infty}\frac{1}{|a_t|}\left|f\left(t, x^m_{b_{1t}}, \ldots, x^m_{b_{kt}}\right) - f\left(t, x_{b_{1t}}, \ldots, x_{b_{kt}}\right)\right| \\
&\quad + \sum_{s=T}^{T_1}\sum_{t=s}^{T_2}\frac{1}{|a_s|}\Big[(t-s+1)\left|h\left(t, x^m_{d_{1t}}, \ldots, x^m_{d_{kt}}\right) - h\left(t, x_{d_{1t}}, \ldots, x_{d_{kt}}\right)\right| \\
&\quad + \left|g\left(t, x^m_{c_{1t}}, \ldots, x^m_{c_{kt}}\right) - g\left(t, x_{c_{1t}}, \ldots, x_{c_{kt}}\right)\right|\Big] \\
&\quad + \sum_{s=T}^{T_1}\sum_{t=T_2+1}^{\infty}\frac{1}{|a_s|}\Big[(t-s+1)\left|h\left(t, x^m_{d_{1t}}, \ldots, x^m_{d_{kt}}\right) - h\left(t, x_{d_{1t}}, \ldots, x_{d_{kt}}\right)\right| \\
&\quad + \left|g\left(t, x^m_{c_{1t}}, \ldots, x^m_{c_{kt}}\right) - g\left(t, x_{c_{1t}}, \ldots, x_{c_{kt}}\right)\right|\Big] \\
&\quad + \sum_{s=T_1+1}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\Big[(t-s+1)\left|h\left(t, x^m_{d_{1t}}, \ldots, x^m_{d_{kt}}\right) - h\left(t, x_{d_{1t}}, \ldots, x_{d_{kt}}\right)\right| \\
&\quad + \left|g\left(t, x^m_{c_{1t}}, \ldots, x^m_{c_{kt}}\right) - g\left(t, x_{c_{1t}}, \ldots, x_{c_{kt}}\right)\right|\Big] \\
&\quad + \sum_{j=T}^{T_1}\sum_{s=j}^{T_2}\sum_{t=s}^{T_3}\frac{t-s+1}{|a_j|}\left|p\left(t, x^m_{o_{1t}}, \ldots, x^m_{o_{kt}}\right) - p\left(t, x_{o_{1t}}, \ldots, x_{o_{kt}}\right)\right| \\
&\quad + \sum_{j=T}^{T_1}\sum_{s=j}^{T_2}\sum_{t=T_3+1}^{\infty}\frac{t-s+1}{|a_j|}\left|p\left(t, x^m_{o_{1t}}, \ldots, x^m_{o_{kt}}\right) - p\left(t, x_{o_{1t}}, \ldots, x_{o_{kt}}\right)\right| \\
&\quad + \sum_{j=T}^{T_1}\sum_{s=T_2+1}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left|p\left(t, x^m_{o_{1t}}, \ldots, x^m_{o_{kt}}\right) - p\left(t, x_{o_{1t}}, \ldots, x_{o_{kt}}\right)\right|
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=T_1+1}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left|p\left(t,x_{o_{1t}}^m,\ldots,x_{o_{kt}}^m\right)\right.\\
&\qquad\left.-p\left(t,x_{o_{1t}},\ldots,x_{o_{kt}}\right)\right|\\
&\le\sum_{t=T}^{T_1}A\left|f\left(t,x_{b_{1t}}^m,\ldots,x_{b_{kt}}^m\right)-f\left(t,x_{b_{1t}},\ldots,x_{b_{kt}}\right)\right|\\
&+\sum_{s=T}^{T_1}\sum_{t=s}^{T_2}\left[B\left|h\left(t,x_{d_{1t}}^m,\ldots,x_{d_{kt}}^m\right)-h\left(t,x_{d_{1t}},\ldots,x_{d_{kt}}\right)\right|\right.\\
&\qquad\left.+A\left|g\left(t,x_{c_{1t}}^m,\ldots,x_{c_{kt}}^m\right)-g\left(t,x_{c_{1t}},\ldots,x_{c_{kt}}\right)\right|\right]\\
&+\sum_{j=T}^{T_1}\sum_{s=j}^{T_2}\sum_{t=s}^{T_3}E\left|p\left(t,x_{o_{1t}}^m,\ldots,x_{o_{kt}}^m\right)-p\left(t,x_{o_{1t}},\ldots,x_{o_{kt}}\right)\right|\\
&+2\sum_{t=T_1+1}^{\infty}\frac{W_t}{|a_t|}+2\sum_{s=T}^{T_1}\sum_{t=T_2+1}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right]\\
&+2\sum_{s=T_1+1}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right]\\
&+2\sum_{j=T}^{T_1}\sum_{s=j}^{T_2}\sum_{t=T_3+1}^{\infty}\frac{t-s+1}{|a_j|}R_t\\
&+2\sum_{j=T}^{T_1}\sum_{s=T_2+1}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}R_t\\
&+2\sum_{j=T_1+1}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}R_t\\
&<\frac{\varepsilon A\left(T_1-T+1\right)}{18T_1T_2T_3\left(A+B+E\right)}\\
&+\frac{\varepsilon\left(B+A\right)\left(T_1-T+1\right)\left(T_2-T+1\right)}{18T_1T_2T_3\left(A+B+E\right)}\\
&+\frac{\varepsilon E\left(T_1-T+1\right)\left(T_2-T+1\right)\left(T_3-T+1\right)}{18T_1T_2T_3\left(A+B+E\right)}\\
&+\frac{2\varepsilon\left(T_1-T+1\right)}{18T_1}\\
&+\frac{2\varepsilon\left(T_1-T+1\right)\left(T_2-T+1\right)}{18T_1T_2}\\
&+\frac{2\varepsilon\left(T_1-T+1\right)}{18T_1}+\frac{\varepsilon}{18}<\varepsilon,\\
&\qquad\forall m\ge T_4,
\end{aligned}
\tag{21}
$$

which means that $S_L$ is continuous in $\Omega_1(N,M)$.

Next we prove that $S_L(\Omega_1(N,M))$ is uniformly Cauchy. It follows from (13) that for given $\varepsilon>0$, there exists $V>T$ satisfying

$$
\begin{aligned}
&\sum_{t=V}^{\infty}\frac{W_t}{|a_t|}+\sum_{s=V}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right]\\
&\quad+\sum_{j=V}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t+|r_t|\right)<\frac{\varepsilon}{4}.
\end{aligned}
\tag{22}
$$

Using (16) and (22), we know that for any $x=\{x_n\}_{n\in\mathbb{Z}_\beta}\in\Omega_1(N,M)$ and $m,n\ge V$,

$$
\begin{aligned}
&\left|\left(S_Lx\right)_m-\left(S_Lx\right)_n\right|\\
&=\left|\sum_{t=m}^{\infty}\frac{1}{a_t}f\left(t,x_{b_{1t}},\ldots,x_{b_{kt}}\right)-\sum_{t=n}^{\infty}\frac{1}{a_t}f\left(t,x_{b_{1t}},\ldots,x_{b_{kt}}\right)\right.\\
&\quad+\sum_{s=m}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1)h\left(t,x_{d_{1t}},\ldots,x_{d_{kt}}\right)\right.\\
&\qquad\left.-g\left(t,x_{c_{1t}},\ldots,x_{c_{kt}}\right)\right]\\
&\quad-\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1)h\left(t,x_{d_{1t}},\ldots,x_{d_{kt}}\right)\right.\\
&\qquad\left.-g\left(t,x_{c_{1t}},\ldots,x_{c_{kt}}\right)\right]\\
&\quad+\sum_{j=n}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left[p\left(t,x_{o_{1t}},\ldots,x_{o_{kt}}\right)-r_t\right]\\
&\quad\left.-\sum_{j=m}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left[p\left(t,x_{o_{1t}},\ldots,x_{o_{kt}}\right)-r_t\right]\right|\\
&\le\sum_{t=m}^{\infty}\frac{W_t}{|a_t|}+\sum_{t=n}^{\infty}\frac{W_t}{|a_t|}\\
&\quad+\sum_{s=m}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right]\\
&\quad+\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right]\\
&\quad+\sum_{j=n}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t+|r_t|\right)\\
&\quad+\sum_{j=m}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t+|r_t|\right)
\end{aligned}
\tag{23}
$$

$$\begin{aligned}
&\le 2\sum_{t=V}^{\infty}\frac{W_t}{|a_t|}+2\sum_{s=V}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right]\\
&\quad+2\sum_{j=V}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t+|r_t|\right)\\
&<\varepsilon,
\end{aligned}\tag{24}$$

which yields that $S_L(\Omega_1(N,M))$ is uniformly Cauchy. It follows from Lemma 2 that $S_L(\Omega_1(N,M))$ is relatively compact. Consequently Lemma 3 guarantees that there exists $x=\{x_n\}_{n\in\mathbb{Z}_\beta}\in\Omega_1(N,M)$ satisfying $U_Lx+S_Lx=x$, which together with (15) and (16) gives that

$$\begin{aligned}
x_n&=L-\gamma_nx_{n-\tau}+\sum_{t=n}^{\infty}\frac{1}{a_t}f\left(t,x_{b_{1t}},\dots,x_{b_{kt}}\right)\\
&\quad+\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1)h\left(t,x_{d_{1t}},\dots,x_{d_{kt}}\right)\right.\\
&\qquad\left.-g\left(t,x_{c_{1t}},\dots,x_{c_{kt}}\right)\right]\\
&\quad-\sum_{j=n}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left[p\left(t,x_{o_{1t}},\dots,x_{o_{kt}}\right)-r_t\right],\\
&\qquad\forall n\ge T,
\end{aligned}\tag{25}$$

which yields that

$$\begin{aligned}
&\Delta\left(x_n+\gamma_nx_{n-\tau}\right)+\frac{1}{a_n}f\left(n,x_{b_{1n}},\dots,x_{b_{kn}}\right)\\
&\quad=-\sum_{t=n}^{\infty}\frac{1}{a_n}\left[(t-n+1)h\left(t,x_{d_{1t}},\dots,x_{d_{kt}}\right)\right.\\
&\qquad\left.-g\left(t,x_{c_{1t}},\dots,x_{c_{kt}}\right)\right]\\
&\qquad+\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_n}\left[p\left(t,x_{o_{1t}},\dots,x_{o_{kt}}\right)-r_t\right],\\
&\qquad\forall n\ge T,\\
&a_n\Delta\left(x_n+\gamma_nx_{n-\tau}\right)+f\left(n,x_{b_{1n}},\dots,x_{b_{kn}}\right)\\
&\quad=-\sum_{t=n}^{\infty}(t-n+1)h\left(t,x_{d_{1t}},\dots,x_{d_{kt}}\right)\\
&\qquad+\sum_{t=n}^{\infty}g\left(t,x_{c_{1t}},\dots,x_{c_{kt}}\right)\\
&\qquad+\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}(t-s+1)\left[p\left(t,x_{o_{1t}},\dots,x_{o_{kt}}\right)-r_t\right],\\
&\qquad\forall n\ge T,
\end{aligned}\tag{26}$$

which means that

$$\begin{aligned}
&\Delta^3\left(a_n\Delta\left(x_n+\gamma_nx_{n-\tau}\right)\right)+\Delta^3f\left(n,x_{b_{1n}},\dots,x_{b_{kn}}\right)\\
&\quad+\Delta^2g\left(n,x_{c_{1n}},\dots,x_{c_{kn}}\right)+\Delta h\left(n,x_{d_{1n}},\dots,x_{d_{kn}}\right)\\
&\quad+p\left(n,x_{o_{1n}},\dots,x_{o_{kn}}\right)=r_t,\quad\forall n\ge T;
\end{aligned}\tag{27}$$

that is, $x=\{x_n\}_{n\in\mathbb{Z}_\beta}$ is a bounded positive solution of (2) in $\Omega_1(N,M)$.

Finally we prove that (2) has uncountably many bounded positive solutions in $\Omega_1(N,M)$. Let $L_1,L_2\in(cM+N,M(1-c))$ and $L_1\ne L_2$. For each $j\in\{1,2\}$, we conclude similarly that there exist a positive integer $T_j\ge n_0+n_1+\tau+|\beta|$ and two mappings $U_{L_j}$ and $S_{L_j}$ satisfying (14)–(16), where $L$ and $T$ are replaced by $L_j$ and $T_j$, respectively, and $U_{L_j}+S_{L_j}$ has a fixed point $z_j=\{z_{jn}\}_{n\in\mathbb{Z}_\beta}$, which is a bounded positive solution (2) in $\Omega_1(N,M)$; that is,

$$\begin{aligned}
z_{jn}&=L_j-\gamma_nz_{jn-\tau}+\sum_{t=n}^{\infty}\frac{1}{a_t}f\left(t,z_{jb_{1t}},\dots,z_{jb_{kt}}\right)\\
&\quad+\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1)h\left(t,z_{jd_{1t}},\dots,z_{jd_{kt}}\right)\right.\\
&\qquad\left.-g\left(t,z_{jc_{1t}},\dots,z_{jc_{kt}}\right)\right]\\
&\quad-\sum_{j=n}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left[p\left(t,z_{jo_{1t}},\dots,z_{jo_{kt}}\right)-r_t\right],\\
&\qquad\forall n\ge T_j,\ \ j\in\{1,2\}.
\end{aligned}\tag{28}$$

Equation (13) ensures that there exists some $T_3>\max\{T_1,T_2\}$ satisfying

$$\begin{aligned}
&\sum_{t=T_3}^{\infty}\frac{W_t}{|a_t|}+\sum_{s=T_3}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right]\\
&\quad+\sum_{j=T_3}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}R_t<\frac{|L_1-L_2|}{4}.
\end{aligned}\tag{29}$$

Combining (11), (12), (28), and (29), we deduce that for any $n\ge T_3$,

$$\begin{aligned}
&|z_{1n}-z_{2n}|\\
&\quad=\left|L_1-L_2-\gamma_nz_{1n-\tau}+\gamma_nz_{2n-\tau}\right.\\
&\qquad+\sum_{t=n}^{\infty}\frac{1}{a_t}f\left(t,z_{1b_{1t}},\dots,z_{1b_{kt}}\right)\\
&\qquad-\sum_{t=n}^{\infty}\frac{1}{a_t}f\left(t,z_{2b_{1t}},\dots,z_{2b_{kt}}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1)\,h\left(t,z_{1d_{1t}},\ldots,z_{1d_{kt}}\right)\right.\\
&\qquad\left.-g\left(t,z_{1c_{1t}},\ldots,z_{1c_{kt}}\right)\right]\\
&-\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1)\,h\left(t,z_{2d_{1t}},\ldots,z_{2d_{kt}}\right)\right.\\
&\qquad\left.-g\left(t,z_{2c_{1t}},\ldots,z_{2c_{kt}}\right)\right]\\
&-\sum_{j=n}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left[p\left(t,z_{1o_{1t}},\ldots,z_{1o_{kt}}\right)-r_t\right]\\
&\left.+\sum_{j=n}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left[p\left(t,z_{2o_{1t}},\ldots,z_{2o_{kt}}\right)-r_t\right]\right|\\
\geq{}&\left|L_1-L_2\right|-\left|\gamma_n\right|\left|z_{1n-\tau}-z_{2n-\tau}\right|\\
&-\sum_{t=n}^{\infty}\frac{1}{a_t}\left|f\left(t,z_{1b_{1t}},\ldots,z_{1b_{kt}}\right)-f\left(t,z_{2b_{1t}},\ldots,z_{2b_{kt}}\right)\right|\\
&-\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1)\left|h\left(t,z_{1d_{1t}},\ldots,z_{1d_{kt}}\right)\right.\right.\\
&\qquad\left.-h\left(t,z_{2d_{1t}},\ldots,z_{2d_{kt}}\right)\right|\\
&\qquad+\left|g\left(t,z_{1c_{1t}},\ldots,z_{1c_{kt}}\right)\right.\\
&\qquad\left.\left.-g\left(t,z_{2c_{1t}},\ldots,z_{2c_{kt}}\right)\right|\right]\\
&-\sum_{j=n}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left|p\left(t,z_{1o_{1t}},\ldots,z_{1o_{kt}}\right)\right.\\
&\qquad\left.-p\left(t,z_{2o_{1t}},\ldots,z_{2o_{kt}}\right)\right|\\
\geq{}&\left|L_1-L_2\right|-c\left\|z_1-z_2\right\|\\
&-2\left(\sum_{t=n}^{\infty}\frac{W_t}{\left|a_t\right|}+\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\frac{1}{\left|a_s\right|}\left[(t-s+1)\,Q_t+P_t\right]\right.\\
&\qquad\left.+\sum_{j=n}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{\left|a_j\right|}R_t\right)\\
\geq{}&\left|L_1-L_2\right|-c\left\|z_1-z_2\right\|\\
&-2\left(\sum_{t=T_3}^{\infty}\frac{W_t}{\left|a_t\right|}+\sum_{s=T_3}^{\infty}\sum_{t=s}^{\infty}\frac{1}{\left|a_s\right|}\left[(t-s+1)\,Q_t+P_t\right]\right.\\
&\qquad\left.+\sum_{j=T_3}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{\left|a_j\right|}R_t\right)\\
>{}&\frac{\left|L_1-L_2\right|}{2}-c\left\|z_1-z_2\right\|,
\end{aligned}
\tag{30}
$$

which implies that

$$
\left\|z_1-z_2\right\|>\frac{\left|L_1-L_2\right|}{2\,(1+c)}>0. \tag{31}
$$

That is, $z_1\neq z_2$. Thus (2) has uncountably many bounded positive solutions in $\Omega_1(N,M)$. This completes the proof. □

**Theorem 7.** *Assume that there exist constants $n_1\in\mathbb{N}_{n_0}$, $M$, $N$, and $c$ with $M>N>0$ and $c\in[0,(M-N)/M)$ and nonnegative sequences $\{W_n\}_{n\in\mathbb{N}_{n_0}}$, $\{P_n\}_{n\in\mathbb{N}_{n_0}}$, $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$, and $\{R_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (12), (13), *and*

$$
0\leq\gamma_n\leq c,\quad\forall n\geq n_1. \tag{32}
$$

*Then* (2) *possesses uncountably many bounded positive solutions in $\Omega_1(N,M)$.*

*Proof.* Let $L\in(N+cM,M)$. It follows from (13) that there exists $T\geq n_0+n_1+\tau+|\beta|$ sufficiently large such that

$$
\begin{aligned}
&\sum_{t=T}^{\infty}\frac{W_t}{\left|a_t\right|}+\sum_{s=T}^{\infty}\sum_{t=s}^{\infty}\frac{1}{\left|a_s\right|}\left[(t-s+1)\,Q_t+P_t\right]\\
&\quad+\sum_{j=T}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{\left|a_j\right|}\left(R_t+\left|r_t\right|\right)\\
&\quad<\min\left\{M-L,\ L-cM-N\right\}.
\end{aligned}
\tag{33}
$$

Let the mappings $U_L$ and $S_L:\Omega_1(N,M)\to l_\beta^\infty$ be defined by (15) and (16), respectively. By means of (12), (15), (16), (32), and (33), we infer that for any $x=\{x_n\}_{n\in\mathbb{Z}_\beta}$, $y=\{y_n\}_{n\in\mathbb{Z}_\beta}\in\Omega_1(N,M)$, and $n\geq T$,

$$
\begin{aligned}
&\left(U_Lx\right)_n+\left(S_Ly\right)_n\\
&\quad=L-\gamma_nx_{n-\tau}+\sum_{t=n}^{\infty}\frac{1}{a_t}f\left(t,y_{b_{1t}},\ldots,y_{b_{kt}}\right)\\
&\qquad+\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1)\,h\left(t,y_{d_{1t}},\ldots,y_{d_{kt}}\right)\right.\\
&\qquad\qquad\left.-g\left(t,y_{c_{1t}},\ldots,y_{c_{kt}}\right)\right]\\
&\qquad-\sum_{j=n}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left[p\left(t,y_{o_{1t}},\ldots,y_{o_{kt}}\right)-r_t\right]
\end{aligned}
$$

$$
\begin{aligned}
&\le L + \sum_{t=T}^{\infty} \frac{W_t}{|a_t|} + \sum_{s=T}^{\infty}\sum_{t=s}^{\infty} \frac{1}{|a_s|}\left[(t-s+1)Q_t + P_t\right] \\
&\quad + \sum_{j=T}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|}\left(R_t + |r_t|\right) \\
&< L + \min\{M-L, L-cM-N\} \\
&\le M, \\
&(U_L x)_n + (S_L y)_n \\
&= L - \gamma_n x_{n-\tau} + \sum_{t=n}^{\infty} \frac{1}{a_t} f\left(t, y_{b_{1t}}, \ldots, y_{b_{kt}}\right) \\
&\quad + \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} \frac{1}{a_s}\left[(t-s+1)h\left(t, y_{d_{1t}}, \ldots, y_{d_{kt}}\right)\right. \\
&\qquad \left. - g\left(t, y_{c_{1t}}, \ldots, y_{c_{kt}}\right)\right] \\
&\quad - \sum_{j=n}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty} \frac{t-s+1}{a_j}\left[p\left(t, y_{o_{1t}}, \ldots, y_{o_{kt}}\right) - r_t\right] \\
&\ge L - cM - \sum_{t=T}^{\infty} \frac{W_t}{|a_t|} \\
&\quad - \sum_{s=T}^{\infty}\sum_{t=s}^{\infty} \frac{1}{|a_s|}\left[(t-s+1)Q_t + P_t\right] \\
&\quad - \sum_{j=T}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|}\left(R_t + |r_t|\right) \\
&> L - cM - \min\{M-L, L-cM-N\} \\
&\ge N;
\end{aligned}
\tag{34}
$$

that is, $U_L x + S_L y \in \Omega_1(N, M)$ for any $x, y \in \Omega_1(N, M)$. The rest of the proof is similar to that of Theorem 6 and is omitted. This completes the proof. □

**Theorem 8.** *Assume that there exist constants $n_1 \in \mathbb{N}_{n_0}$, $M$, $N$, and $c$ with $M > N > 0$ and $c \in [0, (M-N)/M)$ and four nonnegative sequences $\{W_n\}_{n\in\mathbb{N}_{n_0}}$, $\{P_n\}_{n\in\mathbb{N}_{n_0}}$, $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$, and $\{R_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (12), (13), *and*

$$
-c \le \gamma_n \le 0, \quad \forall n \ge n_1 \tag{35}
$$

*Then* (2) *possesses uncountably many bounded positive solutions in $\Omega_1(N, M)$.*

*Proof.* Let $L \in (N, (1-c)M)$. It follows from (13) that there exists $T \ge n_0 + n_1 + \tau + |\beta|$ sufficiently large such that

$$
\begin{aligned}
&\sum_{t=T}^{\infty} \frac{W_t}{|a_t|} + \sum_{s=T}^{\infty}\sum_{t=s}^{\infty} \frac{1}{|a_s|}\left[(t-s+1)Q_t + P_t\right] \\
&\quad + \sum_{j=T}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|}\left(R_t + |r_t|\right) \\
&< \min\{(1-c)M - L, L - N\}.
\end{aligned}
\tag{36}
$$

Let the mappings $U_L$ and $S_L : \Omega_1(N, M) \to l_\beta^\infty$ be defined by (15) and (16), respectively. Making use of (12), (15), (16), (35), and (36), we derive that for any $x = \{x_n\}_{n\in\mathbb{Z}_\beta}$, $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \Omega_1(N, M)$ and $n \ge T$,

$$
\begin{aligned}
&(U_L x)_n + (S_L y)_n \\
&= L - \gamma_n x_{n-\tau} + \sum_{t=n}^{\infty} \frac{1}{a_t} f\left(t, y_{b_{1t}}, \ldots, y_{b_{kt}}\right) \\
&\quad + \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} \frac{1}{a_s}\left[(t-s+1)h\left(t, y_{d_{1t}}, \ldots, y_{d_{kt}}\right)\right. \\
&\qquad \left. - g\left(t, y_{c_{1t}}, \ldots, y_{c_{kt}}\right)\right] \\
&\quad - \sum_{j=n}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty} \frac{t-s+1}{a_j}\left[p\left(t, y_{o_{1t}}, \ldots, y_{o_{kt}}\right) - r_t\right] \\
&\le L + cM + \sum_{t=T}^{\infty} \frac{W_t}{|a_t|} \\
&\quad + \sum_{s=T}^{\infty}\sum_{t=s}^{\infty} \frac{1}{|a_s|}\left[(t-s+1)Q_t + P_t\right] \\
&\quad + \sum_{j=T}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|}\left(R_t + |r_t|\right) \\
&< L + cM + \min\{(1-c)M - L, L - N\} \\
&\le M, \\
&(U_L x)_n + (S_L y)_n \\
&= L - \gamma_n x_{n-\tau} + \sum_{t=n}^{\infty} \frac{1}{a_t} f\left(t, y_{b_{1t}}, \ldots, y_{b_{kt}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1)\,h\left(t,y_{d_{1t}},\ldots,y_{d_{kt}}\right)\right.\\
&\qquad\left.-g\left(t,y_{c_{1t}},\ldots,y_{c_{kt}}\right)\right]\\
&-\sum_{j=n}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left[p\left(t,y_{o_{1t}},\ldots,y_{o_{kt}}\right)-r_t\right]\\
&\geq L-\sum_{t=T}^{\infty}\frac{W_t}{|a_t|}-\sum_{s=T}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)\,Q_t+P_t\right]\\
&-\sum_{j=T}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t+|r_t|\right)\\
&>L-\min\{(1-c)\,M-L,L-N\}\\
&\geq N;
\end{aligned}
\tag{37}
$$

that is, $U_Lx+S_Ly\in\Omega_1(N,M)$ for any $x,y\in\Omega_1(N,M)$. The rest of the proof is similar to that of Theorem 6 and is omitted. This completes the proof. □

**Theorem 9.** *Assume that there exist constants $n_1\in\mathbb{N}_{n_0}$, $M$, $N$, and $c$ with $M>N>0$ and $c>M/(M-N)$ and nonnegative sequences $\{W_n\}_{n\in\mathbb{N}_{n_0}}$, $\{P_n\}_{n\in\mathbb{N}_{n_0}}$, $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$, and $\{R_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (13):

$$\gamma_n\geq c,\quad\forall n\geq n_1, \tag{38}$$

$$
\begin{aligned}
&\left|f\left(n,u_1,u_2,\ldots,u_k\right)\right|\leq W_n,\qquad\left|g\left(n,u_1,u_2,\ldots,u_k\right)\right|\leq P_n,\\
&\left|h\left(n,u_1,u_2,\ldots,u_k\right)\right|\leq Q_n,\qquad\left|p\left(n,u_1,u_2,\ldots,u_k\right)\right|\leq R_n,\\
&\qquad\forall\left(n,u_1,u_2,\ldots,u_k\right)\in\mathbb{N}_{n_0}\times\left[0,\frac{M}{c}\right]^k.
\end{aligned}
\tag{39}
$$

*Then* (2) *possesses uncountably many bounded positive solutions in $l_\beta^\infty$.*

*Proof.* Let $L\in(N+M/c,M)$. It follows from (13) that there exists an integer $T\geq n_0+n_1+\tau+|\beta|$ satisfying

$$
\begin{aligned}
&\sum_{t=T}^{\infty}\frac{W_t}{|a_t|}+\sum_{s=T}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)\,Q_t+P_t\right]\\
&\quad+\sum_{j=T}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t+|r_t|\right)\\
&\quad<\min\left\{M-L,L-N-\frac{M}{c}\right\}.
\end{aligned}
\tag{40}
$$

Define two mappings $U_L$ and $S_L:\Omega_{2T}(N,M)\to l_\beta^\infty$ by

$$
\left(U_Lx\right)_n=\begin{cases}\dfrac{1}{\gamma_{n+\tau}}\left(L-x_{n+\tau}\right), & n\geq T,\\ \left(U_Lx\right)_T, & \beta\leq n<T\end{cases}
\tag{41}
$$

$$
\left(S_Lx\right)_n=\begin{cases}\dfrac{1}{\gamma_{n+\tau}}\left\{\displaystyle\sum_{t=n+\tau}^{\infty}\frac{1}{a_t}f\left(t,x_{b_{1t}},\ldots,x_{b_{kt}}\right)\right.\\
\qquad+\displaystyle\sum_{s=n+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1)\right.\\
\qquad\qquad\times h\left(t,x_{d_{1t}},\ldots,x_{d_{kt}}\right)\\
\qquad\qquad\left.-g\left(t,x_{c_{1t}},\ldots,x_{c_{kt}}\right)\right]\\
\qquad-\displaystyle\sum_{j=n+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\\
\qquad\qquad\times\left[p\left(t,x_{o_{1t}},\ldots,x_{o_{kt}}\right)\right.\\
\qquad\qquad\left.\left.-r_t\right]\right\},\\
\qquad\qquad\qquad n\geq T,\\
\left(S_Lx\right)_T,\qquad\qquad\beta\leq n<T,\end{cases}
\tag{42}
$$

for each $x=\{x_n\}_{n\in\mathbb{Z}_\beta}\in\Omega_{2T}(N,M)$. By means of (13) and (39)–(42), we get that for any $x=\{x_n\}_{n\in\mathbb{Z}_\beta}$, $y=\{y_n\}_{n\in\mathbb{Z}_\beta}\in\Omega_{2T}(N,M)$, and $n\geq T$,

$$
\begin{aligned}
&\left(U_Lx\right)_n+\left(S_Ly\right)_n\\
&=\frac{1}{\gamma_{n+\tau}}\left(L-x_{n+\tau}+\sum_{t=n+\tau}^{\infty}\frac{1}{a_t}f\left(t,y_{b_{1t}},\ldots,y_{b_{kt}}\right)\right.\\
&\quad+\sum_{s=n+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1)\,h\left(t,y_{d_{1t}},\ldots,y_{d_{kt}}\right)\right.\\
&\qquad\left.-g\left(t,y_{c_{1t}},\ldots,y_{c_{kt}}\right)\right]\\
&\quad-\sum_{j=n+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left[p\left(t,y_{o_{1t}},\ldots,y_{o_{kt}}\right)\right.\\
&\qquad\left.\left.-r_t\right]\right)\\
&\leq\frac{1}{\gamma_{n+\tau}}\left(L-\frac{N}{\gamma_{n+\tau}}+\sum_{t=T+\tau}^{\infty}\frac{W_t}{|a_t|}\right.\\
&\quad+\sum_{s=T+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)\,Q_t+P_t\right]\\
&\quad\left.+\sum_{j=T+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left(R_t+|r_t|\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&< \frac{1}{\gamma_{n+\tau}}\left(L+\min\left\{M-L, L-N-\frac{M}{c}\right\}\right)\\
&\le \frac{M}{\gamma_{n+\tau}},\\
&(U_L x)_n + (S_L y)_n\\
&= \frac{1}{\gamma_{n+\tau}}\Bigg(L - x_{n+\tau} + \sum_{t=n+\tau}^{\infty}\frac{1}{a_t} f\left(t, y_{b_{1t}}, \ldots, y_{b_{kt}}\right)\\
&\qquad + \sum_{s=n+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1)\, h\left(t, y_{d_{1t}}, \ldots, y_{d_{kt}}\right)\right.\\
&\qquad\qquad \left. - g\left(t, y_{c_{1t}}, \ldots, y_{c_{kt}}\right)\right]\\
&\qquad - \sum_{j=n+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left[p\left(t, y_{o_{1t}}, \ldots, y_{o_{kt}}\right) - r_t\right]\Bigg)\\
&\ge \frac{1}{\gamma_{n+\tau}}\Bigg(L - \frac{M}{\gamma_{n+\tau}} - \sum_{t=T+\tau}^{\infty}\frac{W_t}{|a_t|}\\
&\qquad - \sum_{s=T+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)\, Q_t + P_t\right]\\
&\qquad - \sum_{j=T+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left(R_t + |r_t|\right)\Bigg)\\
&> \frac{1}{\gamma_{n+\tau}}\left(L - \frac{M}{c} - \min\left\{M-L, L-N-\frac{M}{c}\right\}\right)\\
&\ge \frac{N}{\gamma_{n+\tau}},
\end{aligned}
\tag{43}
$$

which yield that $U_L x + S_L y \in \Omega_{2T}(N, M)$ for any $x, y \in \Omega_{2T}(N, M)$. By virtue of (38) and (41), we infer that

$$
\begin{aligned}
\|U_L x - U_L y\| &= \sup_{n\in\mathbb{Z}_\beta}\left|(U_L x)_n - (U_L y)_n\right|\\
&= \max\left\{\sup_{T>n\ge\beta}\left|(U_L x)_n - (U_L y)_n\right|,\right.\\
&\qquad \left.\sup_{n\ge T}\left|(U_L x)_n - (U_L y)_n\right|\right\}\\
&= \sup_{n\ge T}\left(\frac{1}{|\gamma_{n+\tau}|}\left|x_{n+\tau} - y_{n+\tau}\right|\right)\\
&\le \frac{1}{c}\|x-y\|;
\end{aligned}
\tag{44}
$$

that is, $U_L$ is a contraction in $\Omega_{2T}(N, M)$ because $c > M/(M-N) > 1$.

In order to prove that $S_L$ is completely continuous in $\Omega_{2T}(N, M)$, we have to show that $S_L$ is continuous in $\Omega_{2T}(N, M)$ and $S_L(\Omega_{2T}(N, M)$ is relatively compact. Suppose that $\{x^m\}_{m\in\mathbb{N}}$ is an arbitrary sequence in $\Omega_{2T}(N, M)$ and $x \in \Omega_{2T}(N, M)$ with $\lim_{m\to\infty} x^m = x$, where $x^m = \{x_n^m\}_{n\in\mathbb{Z}_\beta}$ for each $m \in \mathbb{N}$ and $x = \{x_n\}_{n\in\mathbb{Z}_\beta}$. On account of (13), (39), $\lim_{m\to\infty} x^m = x$, and the continuity of $f, g, h$, and $p$, we obtain that for given $\varepsilon > 0$, there exist $T_1$, $T_2$, $T_3$, and $T_4 \in \mathbb{N}$ with $T_4 > T_3 > T_2 > T_1 > T$ satisfying

$$
\begin{aligned}
&\sum_{t=T_1+\tau}^{\infty}\frac{W_t}{|a_t|} + \sum_{s=T_1+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)\, Q_t + P_t\right]\\
&\qquad + \sum_{j=T_1+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}R_t < \frac{\varepsilon c}{18};\\
&\max\left\{\sum_{t=T_3+\tau}^{\infty}\frac{t-s+1}{|a_j|}R_t : T \le j \le T_1+\tau,\quad j \le s \le T_2+\tau\right\}\\
&\qquad < \frac{\varepsilon c}{18 T_1 T_2};\\
&\max\left\{\sum_{s=T_2+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}R_t : T \le j \le T_1+\tau\right\}\\
&\qquad < \frac{\varepsilon c}{18 T_1};\\
&\max\left\{\sum_{t=T_2+\tau}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)\, Q_t + P_t\right]\right.\\
&\qquad\qquad \left. : T \le s \le T_1+\tau\right\} < \frac{\varepsilon c}{18 T_1};\\
&\max\left\{\left|q\left(t, x_{b_{1t}}^m, \ldots, x_{b_{kt}}^m\right)\right.\right.\\
&\qquad \left.\left. - q\left(t, x_{b_{1t}}, \ldots, x_{b_{kt}}\right)\right| : q \in \{f, g, h, p\}\right\}\\
&\qquad < \frac{\varepsilon c}{18 T_1 T_2 T_3 (A+B+E)},\quad \forall m \ge T_4,\ T \le t \le T_3,
\end{aligned}
\tag{45}
$$

where

$$
\begin{aligned}
A &= \max\left\{\frac{1}{|a_s|} : T+\tau \le s \le T_1+\tau\right\},\\
B &= \max\left\{\frac{t-s+1}{|a_s|} : T+\tau \le s \le T_1+\tau,\ s \le t \le T_2+\tau\right\},\\
E &= \max\left\{\frac{t-s+1}{|a_j|} : T+\tau \le j \le T_1+\tau,\right.\\
&\qquad \left. j \le s \le T_2+\tau,\ s \le t \le T_3+\tau\right\}.
\end{aligned}
\tag{46}
$$

It follows from (42)–(46) that

$$
\begin{aligned}
&\left\|S_L x^m - S_L x\right\| \\
&= \sup_{n\in\mathbb{Z}_\beta} \left|\left(S_L x^m\right)_n - \left(S_L x\right)_n\right| \\
&= \max\left\{\sup_{T>n\geq\beta} \left|\left(S_L x^m\right)_n - \left(S_L x\right)_n\right|, \right. \\
&\qquad \left. \sup_{n\geq T} \left|\left(S_L x^m\right)_n - \left(S_L x\right)_n\right|\right\} \\
&= \sup_{n\geq T}\left|\frac{1}{\gamma_{n+\tau}}\left(\sum_{t=n+\tau}^{\infty}\frac{1}{a_t} f\left(t, x_{b_{1t}}^m, \ldots, x_{b_{kt}}^m\right)\right.\right. \\
&\qquad + \sum_{s=n+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1)\, h\left(t, x_{d_{1t}}^m, \ldots, x_{d_{kt}}^m\right)\right. \\
&\qquad \left. - g\left(t, x_{c_{1t}}^m, \ldots, x_{c_{kt}}^m\right)\right] \\
&\qquad \left. - \sum_{j=n+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left[p\left(t, x_{o_{1t}}^m, \ldots, x_{o_{kt}}^m\right) - r_t\right]\right) \\
&\qquad - \frac{1}{\gamma_{n+\tau}}\left(\sum_{t=n+\tau}^{\infty}\frac{1}{a_t} f\left(t, x_{b_{1t}}, \ldots, x_{b_{kt}}\right)\right. \\
&\qquad + \sum_{s=n+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1)\, h\left(t, x_{d_{1t}}, \ldots, x_{d_{kt}}\right)\right. \\
&\qquad \left. - g\left(t, x_{c_{1t}}, \ldots, x_{c_{kt}}\right)\right] \\
&\qquad - \sum_{j=n+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j} \\
&\qquad \left.\left. \times\left[p\left(t, x_{o_{1t}}, \ldots, x_{o_{kt}}\right) - r_t\right]\right)\right| \\
&\leq \frac{1}{c}\left(\sum_{t=T+\tau}^{\infty}\frac{1}{|a_t|}\left|f\left(t, x_{b_{1t}}^m, \ldots, x_{b_{kt}}^m\right)\right.\right. \\
&\qquad \left. - f\left(t, x_{b_{1t}}, \ldots, x_{b_{kt}}\right)\right| \\
&\qquad + \sum_{s=T+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)\left|h\left(t, x_{d_{1t}}^m, \ldots, x_{d_{kt}}^m\right)\right.\right. \\
&\qquad \left. - h\left(t, x_{d_{1t}}, \ldots, x_{d_{kt}}\right)\right| \\
&\qquad + \left|g\left(t, x_{c_{1t}}^m, \ldots, x_{c_{kt}}^m\right)\right. \\
&\qquad \left.\left. - g\left(t, x_{c_{1t}}, \ldots, x_{c_{kt}}\right)\right|\right] \\
&\qquad + \sum_{j=T+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left|p\left(t, x_{o_{1t}}^m, \ldots, x_{o_{kt}}^m\right)\right. \\
&\qquad \left.\left. - p\left(t, x_{o_{1t}}, \ldots, x_{o_{kt}}\right)\right|\right) \\
&\leq \frac{1}{c}\left(\sum_{t=T+\tau}^{T_1+\tau-1}\frac{1}{|a_t|}\left|f\left(t, x_{b_{1t}}^m, \ldots, x_{b_{kt}}^m\right) - f\left(t, x_{b_{1t}}, \ldots, x_{b_{kt}}\right)\right|\right. \\
&\qquad + \sum_{t=T_1+\tau}^{\infty}\frac{1}{|a_t|}\left|f\left(t, x_{b_{1t}}^m, \ldots, x_{b_{kt}}^m\right)\right. \\
&\qquad \left. - f\left(t, x_{b_{1t}}, \ldots, x_{b_{kt}}\right)\right| \\
&\qquad + \sum_{s=T+\tau}^{T_1+\tau-1}\sum_{t=s}^{T_2+\tau-1}\frac{1}{|a_s|}\left[(t-s+1)\left|h\left(t, x_{d_{1t}}^m, \ldots, x_{d_{kt}}^m\right)\right.\right. \\
&\qquad \left. - h\left(t, x_{d_{1t}}, \ldots, x_{d_{kt}}\right)\right| \\
&\qquad + \left|g\left(t, x_{c_{1t}}^m, \ldots, x_{c_{kt}}^m\right)\right. \\
&\qquad \left.\left. - g\left(t, x_{c_{1t}}, \ldots, x_{c_{kt}}\right)\right|\right] \\
&\qquad + \sum_{s=T+\tau}^{T_1+\tau-1}\sum_{t=T_2+\tau}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)\left|h\left(t, x_{d_{1t}}^m, \ldots, x_{d_{kt}}^m\right)\right.\right. \\
&\qquad \left. - h\left(t, x_{d_{1t}}, \ldots, x_{d_{kt}}\right)\right| \\
&\qquad + \left|g\left(t, x_{c_{1t}}^m, \ldots, x_{c_{kt}}^m\right)\right. \\
&\qquad \left.\left. - g\left(t, x_{c_{1t}}, \ldots, x_{c_{kt}}\right)\right|\right] \\
&\qquad + \sum_{s=T_1+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)\left|h\left(t, x_{d_{1t}}^m, \ldots, x_{d_{kt}}^m\right)\right.\right. \\
&\qquad \left. - h\left(t, x_{d_{1t}}, \ldots, x_{d_{kt}}\right)\right| \\
&\qquad + \left|g\left(t, x_{c_{1t}}^m, \ldots, x_{c_{kt}}^m\right)\right. \\
&\qquad \left.\left. - g\left(t, x_{c_{1t}}, \ldots, x_{c_{kt}}\right)\right|\right] \\
&\qquad + \sum_{j=T+\tau}^{T_1+\tau-1}\sum_{s=j}^{T_2+\tau-1}\sum_{t=s}^{T_3+\tau-1}\frac{t-s+1}{|a_j|}\left|p\left(t, x_{o_{1t}}^m, \ldots, x_{o_{kt}}^m\right)\right. \\
&\qquad \left. - p\left(t, x_{o_{1t}}, \ldots, x_{o_{kt}}\right)\right| \\
&\qquad + \sum_{j=T+\tau}^{T_1+\tau-1}\sum_{s=j}^{T_2+\tau-1}\sum_{t=T_3+\tau}^{\infty}\frac{t-s+1}{|a_j|}\left|p\left(t, x_{o_{1t}}^m, \ldots, x_{o_{kt}}^m\right)\right. \\
&\qquad \left. - p\left(t, x_{o_{1t}}, \ldots, x_{o_{kt}}\right)\right| \\
&\qquad + \sum_{j=T+\tau}^{T_1+\tau-1}\sum_{s=T_2+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left|p\left(t, x_{o_{1t}}^m, \ldots, x_{o_{kt}}^m\right)\right. \\
&\qquad \left. - p\left(t, x_{o_{1t}}, \ldots, x_{o_{kt}}\right)\right|
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=T_1+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left|p\left(t,x_{o_{1t}}^m,\ldots,x_{o_{kt}}^m\right)\right.\\
&\qquad\left.-p\left(t,x_{o_{1t}},\ldots,x_{o_{kt}}\right)\right|\Bigg)\\
&\le\frac{1}{c}\Bigg(\sum_{t=T+\tau}^{T_1+\tau-1}A\left|f\left(t,x_{b_{1t}}^m,\ldots,x_{b_{kt}}^m\right)\right.\\
&\qquad\left.-f\left(t,x_{b_{1t}},\ldots,x_{b_{kt}}\right)\right|\\
&\quad+\sum_{s=T+\tau}^{T_1+\tau-1}\sum_{t=s}^{T_2+\tau-1}\left[B\left|h\left(t,x_{d_{1t}}^m,\ldots,x_{d_{kt}}^m\right)\right.\right.\\
&\qquad\left.-h\left(t,x_{d_{1t}},\ldots,x_{d_{kt}}\right)\right|\\
&\qquad+A\left|g\left(t,x_{c_{1t}}^m,\ldots,x_{c_{kt}}^m\right)\right.\\
&\qquad\left.\left.-g\left(t,x_{c_{1t}},\ldots,x_{c_{kt}}\right)\right|\right]\\
&\quad+\sum_{j=T+\tau}^{T_1+\tau-1}\sum_{s=j}^{T_2+\tau-1}\sum_{t=s}^{T_3+\tau-1}E\left|p\left(t,x_{o_{1t}}^m,\ldots,x_{o_{kt}}^m\right)\right.\\
&\qquad\left.-p\left(t,x_{o_{1t}},\ldots,x_{o_{kt}}\right)\right|\\
&\quad+2\sum_{t=T_1+\tau}^{\infty}\frac{W_t}{|a_t|}\\
&\quad+2\sum_{s=T+\tau}^{T_1+\tau-1}\sum_{t=T_2+\tau}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right]\\
&\quad+2\sum_{s=T_1+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right]\\
&\quad+2\sum_{j=T+\tau}^{T_1+\tau-1}\sum_{s=j}^{T_2+\tau-1}\sum_{t=T_3+\tau}^{\infty}\frac{t-s+1}{|a_j|}R_t\\
&\quad+2\sum_{j=T+\tau}^{T_1+\tau-1}\sum_{s=T_2+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}R_t\\
&\quad+2\sum_{j=T_1+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}R_t\Bigg)\\
&<\frac{\varepsilon A(T_1-T)}{18T_1T_2T_3(A+B+E)}\\
&\quad+\frac{\varepsilon(B+A)(T_1-T)(T_2-T)}{18T_1T_2T_3(A+B+E)}\\
&\quad+\frac{\varepsilon E(T_1-T)(T_2-T)(T_3-T)}{18T_1T_2T_3(A+B+E)}\\
&\quad+\frac{2\varepsilon(T_1-T)}{18T_1}+\frac{2\varepsilon(T_1-T)(T_2-T)}{18T_1T_2}\\
&\quad+\frac{2\varepsilon(T_1-T)}{18T_1}+\frac{\varepsilon}{18}<\varepsilon,\quad\forall m\ge T_4,
\end{aligned}
\tag{47}
$$

which means that $S_L$ is continuous in $\Omega_{2T}(N,M)$. It follows from (13) that for given $\varepsilon>0$, there exists $V>T$ satisfying

$$
\begin{aligned}
&\sum_{t=V+\tau}^{\infty}\frac{W_t}{|a_t|}\\
&\quad+\sum_{s=V+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right]\\
&\quad+\sum_{j=V+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t+|r_t|\right)<\frac{\varepsilon}{4}.
\end{aligned}
\tag{48}
$$

Next we prove that $S_L(\Omega_{2T}(N,M))$ is uniformly Cauchy. In view of (42) and (48), we infer that for any $x=\{x_n\}_{n\in\mathbb{Z}_\beta}\in\Omega_{2T}(N,M)$ and $m,n\ge V$,

$$
\begin{aligned}
&\left|(S_Lx)_m-(S_Lx)_n\right|\\
&=\Bigg|\frac{1}{\gamma_{m+\tau}}\sum_{t=m+\tau}^{\infty}\frac{1}{a_t}f\left(t,x_{b_{1t}},\ldots,x_{b_{kt}}\right)\\
&\quad-\frac{1}{\gamma_{n+\tau}}\sum_{t=n+\tau}^{\infty}\frac{1}{a_t}f\left(t,x_{b_{1t}},\ldots,x_{b_{kt}}\right)\\
&\quad+\frac{1}{\gamma_{m+\tau}}\sum_{s=m+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1)h\left(t,x_{d_{1t}},\ldots,x_{d_{kt}}\right)\right.\\
&\qquad\left.-g\left(t,x_{c_{1t}},\ldots,x_{c_{kt}}\right)\right]\\
&\quad-\frac{1}{\gamma_{n+\tau}}\sum_{s=n+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1)h\left(t,x_{d_{1t}},\ldots,x_{d_{kt}}\right)\right.\\
&\qquad\left.-g\left(t,x_{c_{1t}},\ldots,x_{c_{kt}}\right)\right]\\
&\quad+\frac{1}{\gamma_{n+\tau}}\sum_{j=n+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left[p\left(t,x_{o_{1t}},\ldots,x_{o_{kt}}\right)-r_t\right]\\
&\quad-\frac{1}{\gamma_{m+\tau}}\sum_{j=m+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left[p\left(t,x_{o_{1t}},\ldots,x_{o_{kt}}\right)-r_t\right]\Bigg|
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{c}\Bigg(\sum_{t=m+\tau}^{\infty}\frac{W_t}{|a_t|}+\sum_{t=n+\tau}^{\infty}\frac{W_t}{|a_t|}\\
&\quad+\sum_{s=m+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right]\\
&\quad+\sum_{s=n+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right]\\
&\quad+\sum_{j=n+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t+|r_t|\right)\\
&\quad+\sum_{j=m+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t+|r_t|\right)\Bigg)\\
&\le \frac{2}{c}\Bigg(\sum_{t=V+\tau}^{\infty}\frac{W_t}{|a_t|}\\
&\quad+\sum_{s=V+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right]\\
&\quad+\sum_{j=V+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t+|r_t|\right)\Bigg)\\
&< \varepsilon.
\end{aligned}
\tag{49}
$$

Note that (40) and (42) yield that

$$
\begin{aligned}
&\|S_Lx\|\\
&=\sup_{n\in\mathbb{Z}_\beta}\left|(S_Lx)_n\right|\\
&=\max\left\{\sup_{T>n\ge\beta}\left|(S_Lx)_n\right|,\ \sup_{n\ge T}\left|(S_Lx)_n\right|\right\}\\
&=\sup_{n\ge T}\Bigg|\frac{1}{\gamma_{n+\tau}}\Bigg(\sum_{t=n+\tau}^{\infty}\frac{1}{a_t}f\left(t,x_{b_{1t}},\dots,x_{b_{kt}}\right)\\
&\qquad+\sum_{s=n+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\Big[(t-s+1)h\left(t,x_{d_{1t}},\dots,x_{d_{kt}}\right)\\
&\qquad\qquad-g\left(t,x_{c_{1t}},\dots,x_{c_{kt}}\right)\Big]\\
&\qquad-\sum_{j=n+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left[p\left(t,x_{o_{1t}},\dots,x_{o_{kt}}\right)-r_t\right]\Bigg)\Bigg|\\
&\le\frac{1}{c}\Bigg(\sum_{t=T+\tau}^{\infty}\frac{W_t}{|a_t|}\\
&\qquad+\sum_{s=T+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right]\\
&\qquad+\sum_{j=T+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t+|r_t|\right)\Bigg)\\
&<\frac{1}{c}\min\left\{M-L,\ L-N-\frac{M}{c}\right\}\\
&<\frac{M}{c},\quad\forall x=\{x_n\}_{n\in\mathbb{Z}_\beta}\in\Omega_{2T}(N,M),
\end{aligned}
\tag{50}
$$

which gives that $S_L(\Omega_{2T}(N,M))$ is bounded. Thus Lemma 3 means that there exists $x=\{x_n\}_{n\in\mathbb{Z}_\beta}\in\Omega_{2T}(N,M)$ such that $U_Lx+S_Lx=x$, which is a bounded positive solution of (2) in $\Omega_{2T}(N,M)\subset l_\beta^\infty$.

Let $L_1,L_2\in(N+M/c,M)$ and $L_1\ne L_2$. For any $j\in\{1,2\}$, we deduce similarly that there exist a positive integer $T_j\ge n_0+n_1+\tau+|\beta|$, a closed bounded and convex subset $\Omega_{2T_j}(N,M)$ of $l_\beta^\infty$, and two mappings $U_{L_j}$ and $S_{L_j}$ satisfying (40)–(42), where $L$ and $T$ are replaced by $L_j$ and $T_j$, respectively, and $U_{L_j}+S_{L_j}$ possesses a fixed point $z_j=\{z_{jn}\}_{n\in\mathbb{Z}_\beta}\in\Omega_{2T_j}(N,M)$, which is a bounded positive solution of (2); that is,

$$
\begin{aligned}
&z_{jn}\\
&=\frac{1}{\gamma_{n+\tau}}\Bigg(L_j-z_{jn+\tau}+\sum_{t=n+\tau}^{\infty}\frac{1}{a_t}f\left(t,z_{jb_{1t}},\dots,z_{jb_{kt}}\right)\\
&\qquad+\sum_{s=n+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\Big[(t-s+1)h\left(t,z_{jd_{1t}},\dots,z_{jd_{kt}}\right)\\
&\qquad\qquad-g\left(t,z_{jc_{1t}},\dots,z_{jc_{kt}}\right)\Big]\\
&\qquad-\sum_{j=n+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left[p\left(t,z_{jo_{1t}},\dots,z_{jo_{kt}}\right)-r_t\right]\Bigg),\\
&\qquad\qquad\forall n\ge T_j,\ j\in\{1,2\}.
\end{aligned}
\tag{51}
$$

Observe that (13) implies that there exists $T_3\in\mathbb{N}$ with $T_3>\max\{T_1,T_2\}$ satisfying

$$
\begin{aligned}
&\sum_{t=T_3+\tau}^{\infty}\frac{W_t}{|a_t|}\\
&\quad+\sum_{s=T_3+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right]\\
&\quad+\sum_{j=T_3+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}R_t<\frac{|L_1-L_2|}{4},
\end{aligned}
\tag{52}
$$

which together with (51) yields that for each $n \geq T_3$

$$\begin{aligned}
&\left|z_{1n} - z_{2n} + \frac{z_{1n+\tau} - z_{2n+\tau}}{\gamma_{n+\tau}}\right| \\
&= \left|\frac{1}{\gamma_{n+\tau}}\left(L_1 - L_2 + \sum_{t=n+\tau}^{\infty}\frac{1}{a_t} f\left(t, z_{1b_{1t}}, \ldots, z_{1b_{kt}}\right)\right.\right. \\
&\qquad - \sum_{t=n+\tau}^{\infty}\frac{1}{a_t} f\left(t, z_{2b_{1t}}, \ldots, z_{2b_{kt}}\right) \\
&\qquad + \sum_{s=n+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1)\, h\left(t, z_{1d_{1t}}, \ldots, z_{1d_{kt}}\right)\right. \\
&\qquad\qquad \left. - g\left(t, z_{1c_{1t}}, \ldots, z_{1c_{kt}}\right)\right] \\
&\qquad - \sum_{s=n+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1)\, h\left(t, z_{2d_{1t}}, \ldots, z_{2d_{kt}}\right)\right. \\
&\qquad\qquad \left. - g\left(t, z_{2c_{1t}}, \ldots, z_{2c_{kt}}\right)\right] \\
&\qquad - \sum_{j=n+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left[p\left(t, z_{1o_{1t}}, \ldots, z_{1o_{kt}}\right) - r_t\right] \\
&\qquad \left.\left. + \sum_{j=n+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left[p\left(t, z_{2o_{1t}}, \ldots, z_{2o_{kt}}\right) - r_t\right]\right)\right| \\
&\geq \frac{1}{\gamma_{n+\tau}}\left(\left|L_1 - L_2\right| - \sum_{t=n+\tau}^{\infty}\frac{1}{a_t}\left|f\left(t, z_{1b_{1t}}, \ldots, z_{1b_{kt}}\right)\right.\right. \\
&\qquad\qquad \left. - f\left(t, z_{2b_{1t}}, \ldots, z_{2b_{kt}}\right)\right| \\
&\qquad - \sum_{s=n+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1)\left|h\left(t, z_{1d_{1t}}, \ldots, z_{1d_{kt}}\right)\right.\right. \\
&\qquad\qquad \left. - h\left(t, z_{2d_{1t}}, \ldots, z_{2d_{kt}}\right)\right| \\
&\qquad\qquad + \left|g\left(t, z_{1c_{1t}}, \ldots, z_{1c_{kt}}\right)\right. \\
&\qquad\qquad\qquad \left.\left. - g\left(t, z_{2c_{1t}}, \ldots, z_{2c_{kt}}\right)\right|\right] \\
&\qquad - \sum_{j=n+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left|p\left(t, z_{1o_{1t}}, \ldots, z_{1o_{kt}}\right)\right. \\
&\qquad\qquad \left.\left. - p\left(t, z_{2o_{1t}}, \ldots, z_{2o_{kt}}\right)\right|\right) \\
&\geq \frac{1}{\gamma_{n+\tau}}\left(\left|L_1 - L_2\right| - 2\sum_{t=T_3+\tau}^{\infty}\frac{W_t}{|a_t|}\right. \\
&\qquad - 2\sum_{s=T_3+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)\, Q_t + P_t\right] \\
&\qquad \left. - 2\sum_{j=T_3+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}R_t\right) \\
&\geq \frac{1}{\gamma_{n+\tau}}\left(\left|L_1 - L_2\right| - \frac{\left|L_1 - L_2\right|}{2}\right) \\
&> \frac{\left|L_1 - L_2\right|}{2\gamma_{n+\tau}},
\end{aligned} \tag{53}$$

which implies that

$$\left|z_{1n} - z_{2n} + \frac{z_{1n+\tau} - z_{2n+\tau}}{\gamma_{n+\tau}}\right| > 0, \quad \forall n \geq T_3; \tag{54}$$

that is, $z_1 \neq z_2$. Consequently, (2) has uncountably many bounded positive solutions in $l_\beta^\infty$. This completes the proof. □

**Theorem 10.** *Assume that there exist constants $n_1 \in \mathbb{N}_{n_0}$, $M$, $N$, $c$, and $d$ with $d > c > 1$ and $M(1-1/c) > N(1-1/d)$ and nonnegative sequences $\{W_n\}_{n\in\mathbb{N}_{n_0}}$, $\{P_n\}_{n\in\mathbb{N}_{n_0}}$, $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$, and $\{R_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (13)*:*

$$c \leq \gamma_n \leq d, \quad \forall n \geq n_1, \tag{55}$$

$$\begin{gathered}
\left|f(n, u_1, u_2, \ldots, u_k)\right| \leq W_n, \qquad \left|g(n, u_1, u_2, \ldots, u_k)\right| \leq P_n, \\
\left|h(n, u_1, u_2, \ldots, u_k)\right| \leq Q_n, \qquad \left|p(n, u_1, u_2, \ldots, u_k)\right| \leq R_n, \\
\forall (n, u_1, u_2, \ldots, u_k) \in \mathbb{N}_{n_0} \times \left[\frac{N}{d}, \frac{M}{c}\right]^k.
\end{gathered} \tag{56}$$

*Then* (2) *possesses uncountably many bounded positive solutions in $l_\beta^\infty$.*

*Proof.* Let $L \in (N + M/c, M + N/d)$. It follows from (13) that there exists an integer $T \geq n_0 + n_1 + \tau + |\beta|$ satisfying

$$\begin{aligned}
&\sum_{t=T}^{\infty}\frac{W_t}{|a_t|} + \sum_{s=T}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)\, Q_t + P_t\right] \\
&\quad + \sum_{j=T}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t + |r_t|\right) \\
&\quad < \min\left\{M + \frac{N}{d} - L,\ L - N - \frac{M}{c}\right\}.
\end{aligned} \tag{57}$$

Let the mappings $U_L$ and $S_L : \Omega_{2T}(N, M) \to l_\beta^\infty$ be defined by (41) and (42), respectively. Using (13) and (41), (42) and

(57), we obtain that for any $x = \{x_n\}_{n\in\mathbb{Z}_\beta}$, $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \Omega_{2T}(N, M)$ and $n \geq T$,

$$\begin{aligned}
&(U_L x)_n + (S_L y)_n \\
&= \frac{1}{\gamma_{n+\tau}}\Bigg(L - x_{n+\tau} + \sum_{t=n+\tau}^{\infty}\frac{1}{a_t} f\left(t, y_{b_{1t}}, \ldots, y_{b_{kt}}\right) \\
&\qquad + \sum_{s=n+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1)\, h\left(t, y_{d_{1t}}, \ldots, y_{d_{kt}}\right)\right. \\
&\qquad\qquad \left. - g\left(t, y_{c_{1t}}, \ldots, y_{c_{kt}}\right)\right] \\
&\qquad - \sum_{j=n+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left[p\left(t, y_{o_{1t}}, \ldots, y_{o_{kt}}\right) - r_t\right]\Bigg) \\
&\leq \frac{1}{\gamma_{n+\tau}}\Bigg(L - \frac{N}{\gamma_{n+\tau}} + \sum_{t=T+\tau}^{\infty}\frac{W_t}{|a_t|} \\
&\qquad + \sum_{s=T+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)\, Q_t + P_t\right] \\
&\qquad + \sum_{j=T+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left(R_t + |r_t|\right)\Bigg) \\
&< \frac{1}{\gamma_{n+\tau}}\left(L - \frac{N}{d} + \min\left\{M + \frac{N}{d} - L,\ L - N - \frac{M}{c}\right\}\right) \\
&\leq \frac{M}{\gamma_{n+\tau}},
\end{aligned}$$

$$\begin{aligned}
&(U_L x)_n + (S_L y)_n \\
&= \frac{1}{\gamma_{n+\tau}}\Bigg(L - x_{n+\tau} + \sum_{t=n+\tau}^{\infty}\frac{1}{a_t} f\left(t, y_{b_{1t}}, \ldots, y_{b_{kt}}\right) \\
&\qquad + \sum_{s=n+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1)\, h\left(t, y_{d_{1t}}, \ldots, y_{d_{kt}}\right)\right. \\
&\qquad\qquad \left. - g\left(t, y_{c_{1t}}, \ldots, y_{c_{kt}}\right)\right] \\
&\qquad - \sum_{j=n+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left[p\left(t, y_{o_{1t}}, \ldots, y_{o_{kt}}\right) - r_t\right]\Bigg) \\
&\geq \frac{1}{\gamma_{n+\tau}}\Bigg(L - \frac{M}{\gamma_{n+\tau}} - \sum_{t=T+\tau}^{\infty}\frac{W_t}{|a_t|} \\
&\qquad - \sum_{s=T+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)\, Q_t + P_t\right] \\
&\qquad - \sum_{j=T+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left(R_t + |r_t|\right)\Bigg) \\
&> \frac{1}{\gamma_{n+\tau}}\left(L - \frac{M}{c} - \min\left\{M + \frac{N}{d} - L,\ L - N - \frac{M}{c}\right\}\right) \\
&\geq \frac{N}{\gamma_{n+\tau}},
\end{aligned} \tag{58}$$

which imply that $U_L x + S_L y \in \Omega_{2T}(N, M)$ for any $x, y \in \Omega_{2T}(N, M)$. The rest of the proof is similar to that of Theorem 9 and is omitted. This completes the proof. □

**Theorem 11.** *Assume that there exist constants $n_1 \in \mathbb{N}_{n_0}$, $M$, $N$, and $c$ with $M > N > 0$ and $c > M/(M - N)$ and nonnegative sequences $\{W_n\}_{n\in\mathbb{N}_{n_0}}$, $\{P_n\}_{n\in\mathbb{N}_{n_0}}$, $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$, and $\{R_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (13), (39), *and*

$$\gamma_n \leq -c, \quad \forall n \geq n_1. \tag{59}$$

*Then* (2) *possesses uncountably many bounded positive solutions in $l_\beta^\infty$.*

*Proof.* Let $L \in (N,\ M(1-1/c))$. It follows from (13) that there exists an integer $T \geq n_0 + n_1 + \tau + |\beta|$ satisfying

$$\begin{aligned}
&\sum_{t=T}^{\infty}\frac{W_t}{|a_t|} + \sum_{s=T}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)\, Q_t + P_t\right] \\
&\qquad + \sum_{j=T}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t + |r_t|\right) \\
&\qquad < \min\left\{L - N,\ M - \frac{M}{c} - L\right\}.
\end{aligned} \tag{60}$$

Define two mappings $U_L$ and $S_L : \Omega_{3T}(N, M) \to l_\beta^\infty$ by (42) and

$$(U_L x)_n = \begin{cases} \dfrac{1}{\gamma_{n+\tau}}\left(-L - x_{n+\tau}\right), & n \geq T, \\ (U_L x)_T, & \beta \leq n < T, \end{cases} \tag{61}$$

for each $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in \Omega_{3T}(N, M)$. It follows from (42) and (59)–(61) that for any $x = \{x_n\}_{n\in\mathbb{Z}_\beta}$, $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \Omega_{3T}(N, M)$, and $n \geq T$,

$$\begin{aligned}
&(U_L x)_n + (S_L y)_n \\
&= \frac{1}{\gamma_{n+\tau}}\Bigg(-L - x_{n+\tau} + \sum_{t=n+\tau}^{\infty}\frac{1}{a_t} f\left(t, y_{b_{1t}}, \ldots, y_{b_{kt}}\right) \\
&\qquad + \sum_{s=n+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1)\, h\left(t, y_{d_{1t}}, \ldots, y_{d_{kt}}\right)\right. \\
&\qquad\qquad \left. - g\left(t, y_{c_{1t}}, \ldots, y_{c_{kt}}\right)\right] \\
&\qquad - \sum_{j=n+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left[p\left(t, y_{o_{1t}}, \ldots, y_{o_{kt}}\right)\right. \\
&\qquad\qquad \left. - r_t\right]\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{\gamma_{n+\tau}}\left(-L+\frac{M}{\gamma_{n+\tau}}-\sum_{t=T+\tau}^{\infty}\frac{W_t}{|a_t|}\right.\\
&\qquad -\sum_{s=T+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right]\\
&\qquad \left.-\sum_{j=T+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left(R_t+|r_t|\right)\right)\\
&< \frac{1}{\gamma_{n+\tau}}\left(-L-\frac{M}{c}-\min\left\{L-N, M-\frac{M}{c}-L\right\}\right)\\
&\le \frac{-M}{\gamma_{n+\tau}},\\
&(U_Lx)_n+(S_Ly)_n\\
&= \frac{1}{\gamma_{n+\tau}}\left(-L-x_{n+\tau}+\sum_{t=n+\tau}^{\infty}\frac{1}{a_t}f\left(t, y_{b_{1t}},\dots,y_{b_{kt}}\right)\right.\\
&\qquad +\sum_{s=n+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1)h\left(t, y_{d_{1t}},\dots,y_{d_{kt}}\right)\right.\\
&\qquad \left.-g\left(t, y_{c_{1t}},\dots,y_{c_{kt}}\right)\right]\\
&\qquad \left.-\sum_{j=n+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left[p\left(t, y_{o_{1t}},\dots,y_{o_{kt}}\right)-r_t\right]\right)\\
&\ge \frac{1}{\gamma_{n+\tau}}\left(-L-x_{n+\tau}+\sum_{t=T+\tau}^{\infty}\frac{W_t}{|a_t|}\right.\\
&\qquad +\sum_{s=T+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right]\\
&\qquad \left.+\sum_{j=T+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left(R_t+|r_t|\right)\right)\\
&> \frac{1}{\gamma_{n+\tau}}\left(-L+\min\left\{L-N, M-\frac{M}{c}-L\right\}\right)\\
&\ge \frac{-N}{\gamma_{n+\tau}},
\end{aligned}
\tag{62}
$$

which yield that $U_Lx+S_Ly\in\Omega_{3T}(N,M)$. The rest of the proof is similar to that of Theorem 9 and hence is omitted. This completes the proof. □

**Theorem 12.** *Assume that there exist constants $n_1\in\mathbb{N}_{n_0}$, $M$, $N$, $c$, and $d$ with $M>N>0$ and $M(1-1/c)>N(1+1/d)$, $d>c>1$, and nonnegative sequences $\{W_n\}_{n\in\mathbb{N}_{n_0}}$, $\{P_n\}_{n\in\mathbb{N}_{n_0}}$, $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$, and $\{R_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (13), (56), *and*

$$
-d\le\gamma_n\le -c,\quad \forall n\ge n_1. \tag{63}
$$

*Then* (2) *possesses uncountably many bounded positive solutions in $l_\beta^\infty$.*

*Proof.* Let $L\in(N(1+1/d), M(1-1/c))$. It follows from (13) that there exists an integer $T\ge n_0+n_1+\tau+|\beta|$ satisfying

$$
\begin{aligned}
&\sum_{t=T}^{\infty}\frac{W_t}{|a_t|}\\
&\quad +\sum_{s=T}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right]\\
&\quad +\sum_{j=T}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t+|r_t|\right)\\
&\quad <\min\left\{L-\frac{N}{d}-N, M-\frac{M}{c}-L\right\}.
\end{aligned}
\tag{64}
$$

Let the mappings $U_L$ and $S_L:\Omega_{3T}(N,M)\to l_\beta^\infty$ be defined by (42) and (61), respectively. By means of (42), (56), (61), and (64), we deduce that for any $x=\{x_n\}_{n\in\mathbb{Z}_\beta}$, $y=\{y_n\}_{n\in\mathbb{Z}_\beta}\in\Omega_{3T}(N,M)$, and $n\ge T$,

$$
\begin{aligned}
&(U_Lx)_n+(S_Ly)_n\\
&= \frac{1}{\gamma_{n+\tau}}\left(-L-x_{n+\tau}\right.\\
&\qquad +\sum_{t=n+\tau}^{\infty}\frac{1}{a_t}f\left(t, y_{b_{1t}},\dots,y_{b_{kt}}\right)\\
&\qquad +\sum_{s=n+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1)h\left(t, y_{d_{1t}},\dots,y_{d_{kt}}\right)\right.\\
&\qquad \left.-g\left(t, y_{c_{1t}},\dots,y_{c_{kt}}\right)\right]\\
&\qquad \left.-\sum_{j=n+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left[p\left(t, y_{o_{1t}},\dots,y_{o_{kt}}\right)-r_t\right]\right)\\
&\le \frac{1}{\gamma_{n+\tau}}\left(-L+\frac{M}{\gamma_{n+\tau}}-\sum_{t=T+\tau}^{\infty}\frac{W_t}{|a_t|}\right.\\
&\qquad -\sum_{s=T+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right]\\
&\qquad \left.-\sum_{j=T+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left(R_t+|r_t|\right)\right)\\
&< \frac{1}{\gamma_{n+\tau}}\left(-L-\frac{M}{c}-\min\left\{L-\frac{N}{d}-N,\ M-\frac{M}{c}-L\right\}\right)\\
&\le \frac{-M}{\gamma_{n+\tau}},
\end{aligned}
$$

$$
\begin{aligned}
&(U_L x)_n + (S_L y)_n \\
&= \frac{1}{\gamma_{n+\tau}}\Bigg(-L - x_{n+\tau} + \sum_{t=n+\tau}^{\infty}\frac{1}{a_t} f\left(t, y_{b_{1t}}, \ldots, y_{b_{kt}}\right) \\
&\qquad + \sum_{s=n+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\Big[(t-s+1)\, h\left(t, y_{d_{1t}}, \ldots, y_{d_{kt}}\right) \\
&\qquad\qquad - g\left(t, y_{c_{1t}}, \ldots, y_{c_{kt}}\right)\Big] \\
&\qquad - \sum_{j=n+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left[p\left(t, y_{o_{1t}}, \ldots, y_{o_{kt}}\right) - r_t\right]\Bigg) \\
&\geq \frac{1}{\gamma_{n+\tau}}\Bigg(-L + \frac{N}{d} - x_{n+\tau} + \sum_{t=T+\tau}^{\infty}\frac{W_t}{|a_t|} \\
&\qquad + \sum_{s=T+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)\, Q_t + P_t\right] \\
&\qquad + \sum_{j=T+\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left(R_t + |r_t|\right)\Bigg) \\
&> \frac{1}{\gamma_{n+\tau}}\left(-L + \frac{N}{d} + \min\left\{L - \frac{N}{d} - N,\ M - \frac{M}{c} - L\right\}\right) \\
&\geq \frac{-N}{\gamma_{n+\tau}},
\end{aligned}
\tag{65}
$$

which mean that $U_L x + S_L y \in \Omega_{3T}(N, M)$. The rest of the proof is similar to that of Theorem 9 and hence is omitted. This completes the proof. □

**Theorem 13.** *Assume that there exist constants $n_1 \in \mathbb{N}_{n_0}$, $M$, and $N$ with $M > N > 0$ and nonnegative sequences $\{W_n\}_{n\in\mathbb{N}_{n_0}}$, $\{P_n\}_{n\in\mathbb{N}_{n_0}}$, $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$, and $\{R_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (12), (13), *and*

$$
\gamma_n = 1, \quad \forall n \geq n_1. \tag{66}
$$

*Then* (2) *possesses uncountably many bounded positive solutions in $\Omega_1(N, M)$*

*Proof.* Let $L \in (N, M)$. Equation (13) ensures that there exists $T \geq n_0 + n_1 + \tau + |\beta|$ sufficiently large such that

$$
\begin{aligned}
&\sum_{t=T}^{\infty}\frac{W_t}{|a_t|} + \sum_{s=T}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)\, Q_t + P_t\right] \\
&\quad + \sum_{j=T}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t + |r_t|\right) \\
&\quad < \min\{M - L,\ L - N\}.
\end{aligned}
\tag{67}
$$

Define a mapping $S_L : \Omega_1(N, M) \to l_\beta^\infty$ by

$$
(S_L x)_n = \begin{cases}
L + \displaystyle\sum_{i=1}^{\infty}\sum_{t=n+(2i-1)\tau}^{n+2i\tau-1}\frac{1}{a_t} f\left(t, x_{b_{1t}}, \ldots, x_{b_{kt}}\right) \\
\quad + \displaystyle\sum_{i=1}^{\infty}\sum_{s=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{t=s}^{\infty}\frac{1}{a_s}\Big[(t-s+1) \\
\qquad \times h\left(t, x_{d_{1t}}, \ldots, x_{d_{kt}}\right) \\
\qquad - g\left(t, x_{c_{1t}}, \ldots, x_{c_{kt}}\right)\Big] \\
\quad - \displaystyle\sum_{i=1}^{\infty}\sum_{j=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j} \\
\qquad \times \left[p\left(t, x_{o_{1t}}, \ldots, x_{o_{kt}}\right) - r_t\right], \\
\qquad\qquad n \geq T, \\
(S_L x)_T, \qquad \beta \leq n < T,
\end{cases}
\tag{68}
$$

for each $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in \Omega_1(N, M)$. In view of (12), (67), and (68), we deduce that for every $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in \Omega_1(N, M)$ and $n \geq T$,

$$
\begin{aligned}
&\left|(S_L x)_n - L\right| \\
&= \Bigg|\sum_{i=1}^{\infty}\sum_{t=n+(2i-1)\tau}^{n+2i\tau-1}\frac{1}{a_t} f\left(t, x_{b_{1t}}, \ldots, x_{b_{kt}}\right) \\
&\quad + \sum_{i=1}^{\infty}\sum_{s=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{t=s}^{\infty}\frac{1}{a_s}\Big[(t-s+1)\, h\left(t, x_{d_{1t}}, \ldots, x_{d_{kt}}\right) \\
&\qquad - g\left(t, x_{c_{1t}}, \ldots, x_{c_{kt}}\right)\Big] \\
&\quad - \sum_{i=1}^{\infty}\sum_{j=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j} \\
&\qquad \times \left[p\left(t, x_{o_{1t}}, \ldots, x_{o_{kt}}\right) - r_t\right]\Bigg| \\
&\leq \sum_{i=1}^{\infty}\sum_{t=n+(2i-1)\tau}^{n+2i\tau-1}\frac{W_t}{|a_t|} \\
&\quad + \sum_{i=1}^{\infty}\sum_{s=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)\, Q_t + P_t\right] \\
&\quad + \sum_{i=1}^{\infty}\sum_{j=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t + |r_t|\right) \\
&\leq \sum_{t=T}^{\infty}\frac{W_t}{|a_t|} + \sum_{s=T}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)\, Q_t + P_t\right] \\
&\quad + \sum_{j=T}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t + |r_t|\right) \\
&< \min\{M - L, L - N\},
\end{aligned}
\tag{69}
$$

which means that $S_L(\Omega_1(N,M)) \subseteq \Omega_1(N,M)$ and $\|S_L x\| \leq M$ for all $x \in \Omega_1(N,M)$. It follows from (13) that for each $\varepsilon > 0$, there exists $V > T$ satisfying (22). Using (22) and (68), we obtain that for any $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in \Omega_1(N,M)$ and $m,n > V$,

$$\begin{aligned}
&\left|(S_L x)_m - (S_L x)_n\right| \\
&= \Bigg|\sum_{i=1}^{\infty}\sum_{t=m+(2i-1)\tau}^{m+2i\tau-1}\frac{1}{a_t}f\left(t,x_{b_{1t}},\ldots,x_{b_{kt}}\right) \\
&\quad -\sum_{i=1}^{\infty}\sum_{t=n+(2i-1)\tau}^{n+2i\tau-1}\frac{1}{a_t}f\left(t,x_{b_{1t}},\ldots,x_{b_{kt}}\right) \\
&\quad +\sum_{i=1}^{\infty}\sum_{s=m+(2i-1)\tau}^{m+2i\tau-1}\sum_{t=s}^{\infty}\frac{1}{a_s}\Big[(t-s+1)h\left(t,x_{d_{1t}},\ldots,x_{d_{kt}}\right) \\
&\qquad -g\left(t,x_{c_{1t}},\ldots,x_{c_{kt}}\right)\Big] \\
&\quad -\sum_{i=1}^{\infty}\sum_{s=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{t=s}^{\infty}\frac{1}{a_s}\Big[(t-s+1)h\left(t,x_{d_{1t}},\ldots,x_{d_{kt}}\right) \\
&\qquad -g\left(t,x_{c_{1t}},\ldots,x_{c_{kt}}\right)\Big] \\
&\quad +\sum_{i=1}^{\infty}\sum_{j=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j} \\
&\qquad \times\left[p\left(t,x_{o_{1t}},\ldots,x_{o_{kt}}\right)-r_t\right] \\
&\quad -\sum_{i=1}^{\infty}\sum_{j=m+(2i-1)\tau}^{m+2i\tau-1}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j} \\
&\qquad \times\left[p\left(t,x_{o_{1t}},\ldots,x_{o_{kt}}\right)-r_t\right]\Bigg| \\
&\leq \sum_{i=1}^{\infty}\sum_{t=m+(2i-1)\tau}^{m+2i\tau-1}\frac{W_t}{|a_t|}+\sum_{i=1}^{\infty}\sum_{t=n+(2i-1)\tau}^{n+2i\tau-1}\frac{W_t}{|a_t|} \\
&\quad +\sum_{i=1}^{\infty}\sum_{s=m+(2i-1)\tau}^{m+2i\tau-1}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right] \\
&\quad +\sum_{i=1}^{\infty}\sum_{s=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right] \\
&\quad +\sum_{i=1}^{\infty}\sum_{j=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t+|r_t|\right) \\
&\quad +\sum_{i=1}^{\infty}\sum_{j=m+(2i-1)\tau}^{m+2i\tau-1}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t+|r_t|\right) \\
&\leq 2\sum_{t=V}^{\infty}\frac{W_t}{|a_t|}+2\sum_{s=V}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right] \\
&\quad +2\sum_{j=V}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t+|r_t|\right) \\
&< \varepsilon,
\end{aligned} \tag{70}$$

which yields that $S_L(\Omega_1(N,M))$ is uniformly Cauchy.

Now we prove that $S_L$ is continuous in $\Omega_1(N,M)$. Suppose that $\{x^m\}_{m\in N}$ is an arbitrary sequence in $\Omega_1(N,M)$ and $x \in \Omega_1(N,M)$ with $\lim_{m\to\infty}x^m = x$, where $x^m = \{x_n^m\}_{n\in Z_\beta}$ for each $m \in N$ and $x = \{x_n\}_{n\in\mathbb{Z}_\beta}$. Using (12), (13), $\lim_{m\to\infty}x^m = x$, and the continuity of $f, g, h$, and $p$, we conclude that for given $\varepsilon > 0$, there exist $T_1, T_2, T_3$, and $T_4 \in \mathbb{N}$ with $T_4 > T_3 > T_2 > T_1 > T$ satisfying (19) and (20). It follows from (19), (20), and (68) that

$$\begin{aligned}
&\|S_L x^m - S_L x\| \\
&= \sup_{n\in\mathbb{Z}_\beta}\left|(S_L x^m)_n - (S_L x)_n\right| \\
&= \max\Bigg\{\sup_{T>n\geq\beta}\left|(S_L x^m)_n - (S_L x)_n\right|, \\
&\qquad \sup_{n\geq T}\left|(S_L x^m)_n - (S_L x)_n\right|\Bigg\} \\
&= \sup_{n\geq T}\Bigg|L+\sum_{i=1}^{\infty}\sum_{t=n+(2i-1)\tau}^{n+2i\tau-1}\frac{1}{a_t}f\left(t,x_{b_{1t}}^m,\ldots,x_{b_{kt}}^m\right) \\
&\quad +\sum_{i=1}^{\infty}\sum_{s=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{t=s}^{\infty}\frac{1}{a_s}\Big[(t-s+1)h\left(t,x_{d_{1t}}^m,\ldots,x_{d_{kt}}^m\right) \\
&\qquad -g\left(t,x_{c_{1t}}^m,\ldots,x_{c_{kt}}^m\right)\Big] \\
&\quad -\sum_{i=1}^{\infty}\sum_{j=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left[p\left(t,x_{o_{1t}}^m,\ldots,x_{o_{kt}}^m\right)-r_t\right] \\
&\quad -\Bigg(L+\sum_{i=1}^{\infty}\sum_{t=n+(2i-1)\tau}^{n+2i\tau-1}\frac{1}{a_t}f\left(t,x_{b_{1t}},\ldots,x_{b_{kt}}\right) \\
&\qquad +\sum_{i=1}^{\infty}\sum_{s=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{t=s}^{\infty}\frac{1}{a_s}\Big[(t-s+1) \\
&\qquad \times h\left(t,x_{d_{1t}},\ldots,x_{d_{kt}}\right) \\
&\qquad -g\left(t,x_{c_{1t}},\ldots,x_{c_{kt}}\right)\Big] \\
&\qquad -\sum_{i=1}^{\infty}\sum_{j=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}
\end{aligned}$$

$$\times\left[p\left(t,x_{o_{1t}},\ldots,x_{o_{kt}}\right)-r_t\right]\Big)\Big|$$

$$\leq\sum_{i=1}^{\infty}\sum_{t=n+(2i-1)\tau}^{n+2i\tau-1}\frac{1}{|a_t|}\Big|f\left(t,x_{b_{1t}}^m,\ldots,x_{b_{kt}}^m\right)-f\left(t,x_{b_{1t}},\ldots,x_{b_{kt}}\right)\Big|$$

$$+\sum_{i=1}^{\infty}\sum_{s=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\Big[(t-s+1)\Big|h\left(t,x_{d_{1t}}^m,\ldots,x_{d_{kt}}^m\right)-h\left(t,x_{d_{1t}},\ldots,x_{d_{kt}}\right)\Big|+\Big|g\left(t,x_{c_{1t}}^m,\ldots,x_{c_{kt}}^x\right)-g\left(t,x_{c_{1t}},\ldots,x_{c_{kt}}\right)\Big|\Big]$$

$$+\sum_{i=1}^{\infty}\sum_{j=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\times\Big|p\left(t,x_{o_{1t}}^m,\ldots,x_{o_{kt}}^m\right)-p\left(t,x_{o_{1t}},\ldots,x_{o_{kt}}\right)\Big|$$

$$\leq\sum_{t=T}^{\infty}\frac{1}{|a_t|}\Big|f\left(t,x_{b_{1t}}^m,\ldots,x_{b_{kt}}^m\right)-f\left(t,x_{b_{1t}},\ldots,x_{b_{kt}}\right)\Big|$$

$$+\sum_{s=T}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\Big[(t-s+1)\Big|h\left(t,x_{d_{1t}}^m,\ldots,x_{d_{kt}}^m\right)-h\left(t,x_{d_{1t}},\ldots,x_{d_{kt}}\right)\Big|+\Big|g\left(t,x_{c_{1t}}^m,\ldots,x_{c_{kt}}^x\right)-g\left(t,x_{c_{1t}},\ldots,x_{c_{kt}}\right)\Big|\Big]$$

$$+\sum_{j=T}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\Big|p\left(t,x_{o_{1t}}^m,\ldots,x_{o_{kt}}^m\right)-p\left(t,x_{o_{1t}},\ldots,x_{o_{kt}}\right)\Big|$$

$$<\varepsilon, \tag{71}$$

which implies that $S_L$ is continuous in $\Omega_1(N,M)$. Thus Lemma 4 means that $S_L$ possesses a fixed point $x=\{x_n\}_{n\in\mathbb{Z}_\beta}\in\Omega_1(N,M)$; that is,

$$x_n=L+\sum_{i=1}^{\infty}\sum_{t=n+(2i-1)\tau}^{n+2i\tau-1}\frac{1}{a_t}f\left(t,x_{b_{1t}},\ldots,x_{b_{kt}}\right)+\sum_{i=1}^{\infty}\sum_{s=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{t=s}^{\infty}\frac{1}{a_s}\Big[(t-s+1)\times h\left(t,x_{d_{1t}},\ldots,x_{d_{kt}}\right)-g\left(t,x_{c_{1t}},\ldots,x_{c_{kt}}\right)\Big]$$

$$-\sum_{i=1}^{\infty}\sum_{j=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\times\left[p\left(t,x_{o_{1t}},\ldots,x_{o_{kt}}\right)-r_t\right],\quad\forall n\geq T+\tau,$$

$$x_{n-\tau}=L+\sum_{i=1}^{\infty}\sum_{t=n+(2i-2)\tau}^{n+(2i-1)\tau-1}\frac{1}{a_t}f\left(t,x_{b_{1t}},\ldots,x_{b_{kt}}\right)+\sum_{i=1}^{\infty}\sum_{s=n+(2i-2)\tau}^{n+(2i-1)\tau-1}\sum_{t=s}^{\infty}\frac{1}{a_s}\Big[(t-s+1)\times h\left(t,x_{d_{1t}},\ldots,x_{d_{kt}}\right)-g\left(t,x_{c_{1t}},\ldots,x_{c_{kt}}\right)\Big]$$

$$-\sum_{i=1}^{\infty}\sum_{j=n+(2i-2)\tau}^{n+(2i-1)\tau-1}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\times\left[p\left(t,x_{o_{1t}},\ldots,x_{o_{kt}}\right)-r_t\right],\quad\forall n\geq T+\tau, \tag{72}$$

which imply that

$$x_n+x_{n-\tau}=2L+\sum_{t=n}^{\infty}\frac{1}{a_t}f\left(t,x_{b_{1t}},\ldots,x_{b_{kt}}\right)+\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\Big[(t-s+1)h\left(t,x_{d_{1t}},\ldots,x_{d_{kt}}\right)-g\left(t,x_{c_{1t}},\ldots,x_{c_{kt}}\right)\Big]$$

$$-\sum_{j=n}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left[p\left(t,x_{o_{1t}},\ldots,x_{o_{kt}}\right)-r_t\right],\quad\forall n\geq T+\tau, \tag{73}$$

which means that

$$\begin{aligned}
&\Delta\left(a_n\Delta\left(x_n+x_{n-\tau}\right)\right)+\Delta f\left(n,x_{b_{1n}},\ldots,x_{b_{kn}}\right)\\
&\quad+g\left(n,x_{c_{1n}},\ldots,x_{c_{kn}}\right)\\
&=\sum_{t=n}^{\infty}h\left(n,x_{d_{1n}},\ldots,x_{d_{kn}}\right)\\
&\quad-\sum_{t=n}^{\infty}(t-n+1)\left[p\left(t,x_{o_{1t}},\ldots,x_{o_{kt}}\right)-r_t\right],\\
&\qquad\qquad\forall n\geq T+\tau,\\
&\Delta^3\left(a_n\Delta\left(x_n+x_{n-\tau}\right)\right)+\Delta^3 f\left(n,x_{b_{1n}},\ldots,x_{b_{kn}}\right)\\
&\quad+\Delta^2 g\left(n,x_{c_{1n}},\ldots,x_{c_{kn}}\right)+\Delta h\left(n,x_{d_{1n}},\ldots,x_{d_{kn}}\right)\\
&\quad+p\left(n,x_{o_{1n}},\ldots,x_{o_{kn}}\right)=r_n,\quad\forall n\geq T+\tau,
\end{aligned}\tag{74}$$

which yields that $x=\{x_n\}_{n\in\mathbb{Z}_\beta}\in\Omega_1(N,M)$ is bounded positive solution of (2). The rest of the proof is similar to that of Theorem 6 and is omitted. This completes the proof. □

**Theorem 14.** *Assume that there exist constants $n_1\in\mathbb{N}_{n_0}$, $M$, and $N$ with $M>N>0$ and nonnegative sequences $\{W_n\}_{n\in\mathbb{N}_{n_0}}$, $\{P_n\}_{n\in\mathbb{N}_{n_0}}$, $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$, and $\{R_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (12) *and*

$$\gamma_n=-1,\quad\forall n\geq n_1;\tag{75}$$

$$\begin{aligned}
\max\Bigg\{&\sum_{i=1}^{\infty}\sum_{t=n_0+i\tau}^{\infty}\frac{W_t}{|a_t|},\\
&\sum_{i=1}^{\infty}\sum_{s=n_0+i\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[P_t+(t-s+1)Q_t\right],\\
&\sum_{i=1}^{\infty}\sum_{j=n_0+i\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\\
&\qquad\times\left(R_t+|r_t|\right)\Bigg\}<+\infty.
\end{aligned}\tag{76}$$

*Then* (2) *possesses uncountably many bounded positive solution in $\Omega_1(N,M)$.*

*Proof.* Let $L\in(N,M)$. It follows from (76) that there exists $T\geq n_0+n_1+\tau+|\beta|$ sufficiently large such that

$$\begin{aligned}
&\sum_{i=1}^{\infty}\sum_{t=T+i\tau}^{\infty}\frac{W_t}{|a_t|}\\
&\quad+\sum_{i=1}^{\infty}\sum_{s=T+i\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right]\\
&\quad+\sum_{i=1}^{\infty}\sum_{j=T+i\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t+|r_t|\right)\\
&<\min\{M-L,L-N\}.
\end{aligned}\tag{77}$$

Define a mapping $S_L:\Omega_1(N,M)\to l_\beta^\infty$ by

$$\begin{aligned}
&\left(S_Lx\right)_n\\
&=\begin{cases}
L-\sum_{i=1}^{\infty}\sum_{t=n+i\tau}^{\infty}\frac{1}{a_t}f\left(t,x_{b_{1t}},\ldots,x_{b_{kt}}\right)\\
\quad-\sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\\
\qquad\times\left[(t-s+1)h\left(t,x_{d_{1t}},\ldots,x_{d_{kt}}\right)\right.\\
\qquad\left.-g\left(t,x_{c_{1t}},\ldots,x_{c_{kt}}\right)\right]\\
\quad-\sum_{i=1}^{\infty}\sum_{j=n+i\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\\
\qquad\times\left[p\left(t,x_{o_{1t}},\ldots,x_{o_{kt}}\right)-r_t\right],\\
\qquad\qquad n\geq T,\\
(S_Lx)_T,\qquad\qquad\beta\leq n<T,
\end{cases}
\end{aligned}\tag{78}$$

for all $x=\{x_n\}_{n\in\mathbb{Z}_\beta}\in\Omega_1(N,M)$. By virtue of (12), (77), and (78), we know that for every $x=\{x_n\}_{n\in\mathbb{Z}_\beta}\in\Omega_1(N,M)$ and $n\geq T$,

$$\begin{aligned}
&\left|\left(S_Lx\right)_n-L\right|\\
&=\Bigg|-\sum_{i=1}^{\infty}\sum_{t=n+i\tau}^{\infty}\frac{1}{a_t}f\left(t,x_{b_{1t}},\ldots,x_{b_{kt}}\right)\\
&\quad-\sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1)h\left(t,x_{d_{1t}},\ldots,x_{d_{kt}}\right)\right.\\
&\qquad\left.-g\left(t,x_{c_{1t}},\ldots,x_{c_{kt}}\right)\right]\\
&\quad-\sum_{i=1}^{\infty}\sum_{j=n+i\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\\
&\qquad\times\left[p\left(t,x_{o_{1t}},\ldots,x_{o_{kt}}\right)-r_t\right]\Bigg|\\
&\leq\sum_{i=1}^{\infty}\sum_{t=n+i\tau}^{\infty}\frac{W_t}{|a_t|}\\
&\quad+\sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{\infty}\sum_{j=n+i\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t+|r_t|\right)\\
&\le\sum_{i=1}^{\infty}\sum_{t=T+i\tau}^{\infty}\frac{W_t}{|a_t|}\\
&+\sum_{i=1}^{\infty}\sum_{s=T+i\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right]\\
&+\sum_{i=1}^{\infty}\sum_{j=T+i\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t+|r_t|\right)\\
&<\min\{M-L,L-N\},
\end{aligned}
\tag{79}
$$

which means that $S_L(\Omega_1(N,M))\subseteq\Omega_1(N,M)$ and $\|S_Lx\|\le M$ for each $x\in\Omega_1(N,M)$.

Next we prove that $S_L(\Omega_1(N,M))$ is uniformly Cauchy. It follows from (76) that for any given $\varepsilon>0$ there exists $T^*\ge T$ with

$$
\begin{aligned}
&\sum_{i=1}^{\infty}\sum_{t=T^*+i\tau}^{\infty}\frac{W_t}{|a_t|}\\
&\quad+\sum_{i=1}^{\infty}\sum_{s=T^*+i\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right]\\
&\quad+\sum_{i=1}^{\infty}\sum_{j=T^*+i\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t+|r_t|\right)\\
&\quad<\frac{\varepsilon}{2}.
\end{aligned}
\tag{80}
$$

It follows from (12), (78), and (80) that for any $x=\{x_n\}_{n\in\mathbb{Z}_\beta}\in\Omega_1(N,M)$ and $m,n\ge T^*$,

$$
\begin{aligned}
&\left|(S_Lx)_m-(S_Lx)_n\right|\\
&=\Bigg|\sum_{i=1}^{\infty}\sum_{t=n+i\tau}^{\infty}\frac{1}{a_t}f\left(t,x_{b_{1t}},\ldots,x_{b_{kt}}\right)\\
&\quad-\sum_{i=1}^{\infty}\sum_{t=m+i\tau}^{\infty}\frac{1}{a_t}f\left(t,x_{b_{1t}},\ldots,x_{b_{kt}}\right)\\
&\quad+\sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\Big[(t-s+1)h\left(t,x_{d_{1t}},\ldots,x_{d_{kt}}\right)\\
&\qquad\qquad-g\left(t,x_{c_{1t}},\ldots,x_{c_{kt}}\right)\Big]\\
&\quad-\sum_{i=1}^{\infty}\sum_{s=m+i\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\Big[(t-s+1)h\left(t,x_{d_{1t}},\ldots,x_{d_{kt}}\right)\\
&\qquad\qquad-g\left(t,y_{c_{1t}},\ldots,y_{c_{kt}}\right)\Big]\\
&\quad+\sum_{i=1}^{\infty}\sum_{j=n+i\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left[p\left(t,x_{o_{1t}},\ldots,x_{o_{kt}}\right)-r_t\right]\\
&\quad-\sum_{i=1}^{\infty}\sum_{j=m+i\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\\
&\qquad\qquad\times\left[p\left(t,x_{o_{1t}},\ldots,x_{o_{kt}}\right)-r_t\right]\Bigg|\\
&\le\sum_{i=1}^{\infty}\sum_{t=n+i\tau}^{\infty}\frac{W_t}{|a_t|}+\sum_{i=1}^{\infty}\sum_{t=m+i\tau}^{\infty}\frac{W_t}{|a_t|}\\
&\quad+\sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right]\\
&\quad+\sum_{i=1}^{\infty}\sum_{s=m+i\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right]\\
&\quad+\sum_{i=1}^{\infty}\sum_{j=n+i\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t+|r_t|\right)\\
&\quad+\sum_{i=1}^{\infty}\sum_{j=m+i\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t+|r_t|\right)\\
&\le 2\sum_{i=1}^{\infty}\sum_{t=T^*+i\tau}^{\infty}\frac{W_t}{|a_t|}\\
&\quad+2\sum_{i=1}^{\infty}\sum_{s=T^*+i\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right]\\
&\quad+2\sum_{i=1}^{\infty}\sum_{j=T^*+i\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\left(R_t+|r_t|\right)\\
&<\varepsilon,
\end{aligned}
\tag{81}
$$

which yields that $S_L(\Omega_1(N,M))$ is uniformly Cauchy.

Now we prove that $S_L$ is continuous in $\Omega_1(N,M)$. Suppose that $\{x^m\}_{m\in\mathbb{N}}$ is an arbitrary sequence in $\Omega_1(N,M)$ and $x\in\Omega_1(N,M)$ with $\lim_{m\to\infty}x^m=x$, where $x^m=\{x_n^m\}_{n\in\mathbb{Z}_\beta}$ for each $m\in\mathbb{N}$ and $x=\{x_n\}_{n\in\mathbb{Z}_\beta}$. By (12), (76), $\lim_{m\to\infty}x^m=x$, and the continuity of $f,g,h$, and $p$, we get that for given $\varepsilon>0$, there exist $T_1,T_2,T_3,T_4$, and $T_5\in\mathbb{N}$ with $T_5>T_4>T_3>T_2>T_1>T$ satisfying

$$
\begin{aligned}
&\sum_{i=T_1}^{\infty}\sum_{t=T+i\tau}^{\infty}\frac{W_t}{|a_t|}\\
&\quad+\sum_{i=T_1}^{\infty}\sum_{s=T+i\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right]
\end{aligned}
$$

$$+\sum_{i=T_1}^{\infty}\sum_{j=T+i\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}R_t<\frac{\varepsilon}{18};$$

$$\sum_{t=T+T_2\tau}^{\infty}\frac{W_t}{|a_t|}+\sum_{s=T+T_2\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right]$$
$$+\sum_{j=T+T_2\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}R_t<\frac{\varepsilon}{18T_1};$$

$$\max\left\{\sum_{t=T+T_3\tau}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)Q_t+P_t\right]+\sum_{s=T+T_3\tau}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}R_t : T+\tau\le s\le T+T_2\tau, T+\tau\le j\le T+T_2\tau\right\}<\frac{\varepsilon}{18T_1T_2\tau};$$

$$\max\left\{\sum_{t=T+T_4\tau}^{\infty}\frac{t-s+1}{|a_j|}R_t : T+\tau\le j\le T+T_2\tau, j\le s\le T+T_3\tau\right\}<\frac{\varepsilon}{18T_1T_2T_3\tau^2};$$

$$\sum_{i=1}^{T_1-1}\sum_{t=T+i\tau}^{T+T_2\tau-1}A\left|f\left(t,x_{b_{1t}}^m,\dots,x_{b_{kt}}^m\right)-f\left(t,x_{b_{1t}},\dots,x_{b_{kt}}\right)\right|$$
$$+\sum_{i=1}^{T_1-1}\sum_{s=T+i\tau}^{T+T_2\tau-1}\sum_{t=s}^{T+T_3\tau-1}\left[B\left|h\left(t,x_{d_{1t}}^m,\dots,x_{d_{kt}}^m\right)-h\left(t,x_{d_{1t}},\dots,x_{d_{kt}}\right)\right|+A\left|g\left(t,x_{c_{1t}}^m,\dots,x_{c_{kt}}^m\right)-g\left(t,x_{c_{1t}},\dots,x_{c_{kt}}\right)\right|\right]$$
$$+\sum_{i=1}^{T_1-1}\sum_{j=T+i\tau}^{T+T_2\tau-1}\sum_{s=j}^{T+T_3\tau-1}\sum_{t=s}^{T+T_4\tau-1}E\left|p\left(t,x_{o_{1t}}^m,\dots,x_{o_{kt}}^m\right)-p\left(t,x_{o_{1t}},\dots,x_{o_{kt}}\right)\right|<\frac{\varepsilon}{18}, \tag{82}$$

where

$$A=\max\left\{\frac{1}{|a_s|} : T\le s\le T+T_2\tau\right\},$$
$$B=\max\left\{\frac{t-s+1}{|a_s|} : T\le s\le T+T_2\tau, s\le t\le T+T_3\tau\right\},$$
$$E=\max\left\{\frac{t-s+1}{|a_j|} : T\le j\le T+T_2\tau, j\le s\le T+T_3\tau,\ s\le t\le T+T_4\tau\right\}. \tag{83}$$

In terms of (12), (78), and (83), we have

$$\begin{aligned}
&\|S_Lx^m-S_Lx\|\\
&=\sup_{n\in\mathbb{Z}_\beta}\left|(S_Lx^m)_n-(S_Lx)_n\right|\\
&=\max\left\{\sup_{\beta\ge n>T}\left|(S_Lx^m)_n-(S_Lx)_n\right|,\ \sup_{n\ge T}\left|(S_Lx^m)_n-(S_Lx)_n\right|\right\}\\
&=\sup_{n\ge T}\left|L-\sum_{i=1}^{\infty}\sum_{t=n+i\tau}^{\infty}\frac{1}{a_t}f\left(t,x_{b_{1t}}^m,\dots,x_{b_{kt}}^m\right)\right.\\
&\quad-\sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1)h\left(t,x_{d_{1t}}^m,\dots,x_{d_{kt}}^m\right)-g\left(t,x_{c_{1t}}^m,\dots,x_{c_{kt}}^m\right)\right]\\
&\quad-\sum_{i=1}^{\infty}\sum_{j=n+i\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\left[p\left(t,x_{o_{1t}}^m,\dots,x_{o_{kt}}^m\right)-r_t\right]\\
&\quad-\left(L-\sum_{i=1}^{\infty}\sum_{t=n+i\tau}^{\infty}\frac{1}{a_t}f\left(t,x_{b_{1t}},\dots,x_{b_{kt}}\right)\right.\\
&\qquad-\sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left[(t-s+1)h\left(t,x_{d_{1t}},\dots,x_{d_{kt}}\right)-g\left(t,x_{c_{1t}},\dots,x_{c_{kt}}\right)\right]\\
&\qquad\left.\left.-\sum_{i=1}^{\infty}\sum_{j=n+i\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{a_j}\times\left[p\left(t,x_{o_{1t}},\dots,x_{o_{kt}}\right)-r_t\right]\right)\right|\\
&\le\sum_{i=1}^{\infty}\sum_{t=T+i\tau}^{\infty}\frac{1}{|a_t|}\left|f\left(t,x_{b_{1t}}^m,\dots,x_{b_{kt}}^m\right)-f\left(t,x_{b_{1t}},\dots,x_{b_{kt}}\right)\right|\\
&\quad+\sum_{i=1}^{\infty}\sum_{s=T+i\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\left[(t-s+1)\left|h\left(t,x_{d_{1t}}^m,\dots,x_{d_{kt}}^m\right)\right.\right.
\end{aligned}$$

$$
\begin{aligned}
&\qquad -h\left(t, x_{d_{1t}}, \ldots, x_{d_{kt}}\right)\Big| \\
&\qquad + \Big|g\left(t, x_{c_{1t}}^{m}, \cdots, x_{c_{kt}}^{m}\right) \\
&\qquad -g\left(t, x_{c_{1t}}, \ldots, x_{c_{kt}}\right)\Big|\Big] \\
&+ \sum_{i=T_1}^{\infty}\sum_{j=T+i\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|}\Big|p\left(t, x_{o_{1t}}^{m}, \ldots, x_{o_{kt}}^{m}\right) \\
&\qquad -p\left(t, x_{o_{1t}}, \ldots, x_{o_{kt}}\right)\Big| \\
&\le \sum_{i=1}^{T_1-1}\sum_{t=T+i\tau}^{T+T_2\tau-1}\frac{1}{|a_t|}\Big|f\left(t, x_{b_{1t}}^{m}, \ldots, x_{b_{kt}}^{m}\right) \\
&\qquad -f\left(t, x_{b_{1t}}, \ldots, x_{b_{kt}}\right)\Big| \\
&+ \sum_{i=1}^{T_1-1}\sum_{t=T+T_2\tau}^{\infty}\frac{1}{|a_t|}\Big|f\left(t, x_{b_{1t}}^{m}, \ldots, x_{b_{kt}}^{m}\right) \\
&\qquad -f\left(t, x_{b_{1t}}, \ldots, x_{b_{kt}}\right)\Big| \\
&+ \sum_{i=T_1}^{\infty}\sum_{t=T+i\tau}^{\infty}\frac{1}{|a_t|}\Big|f\left(t, x_{b_{1t}}^{m}, \ldots, x_{b_{kt}}^{m}\right) \\
&\qquad -f\left(t, x_{b_{1t}}, \ldots, x_{b_{kt}}\right)\Big| \\
&+ \sum_{i=1}^{T_1-1}\sum_{s=T+i\tau}^{T+T_2\tau-1}\sum_{t=s}^{T+T_3\tau-1}\frac{1}{|a_s|} \\
&\qquad \times\Big[(t-s+1)\Big|h\left(t, x_{d_{1t}}^{m}, \ldots, x_{d_{kt}}^{m}\right) \\
&\qquad -h\left(t, x_{d_{1t}}, \ldots, x_{d_{kt}}\right)\Big| \\
&\qquad + \Big|g\left(t, x_{c_{1t}}^{m}, \ldots, x_{c_{kt}}^{m}\right) \\
&\qquad -g\left(t, x_{c_{1t}}, \ldots, x_{c_{kt}}\right)\Big|\Big] \\
&+ \sum_{i=1}^{T_1-1}\sum_{s=T+i\tau}^{T+T_2\tau-1}\sum_{t=T+T_3\tau}^{\infty}\frac{1}{|a_s|} \\
&\qquad \times\Big[(t-s+1)\Big|h\left(t, x_{d_{1t}}^{m}, \ldots, x_{d_{kt}}^{m}\right) \\
&\qquad -h\left(t, x_{d_{1t}}, \ldots, x_{d_{kt}}\right)\Big| \\
&\qquad + \Big|g\left(t, x_{c_{1t}}^{m}, \ldots, x_{c_{kt}}^{m}\right) \\
&\qquad -g\left(t, x_{c_{1t}}, \ldots, x_{c_{kt}}\right)\Big|\Big] \\
&+ \sum_{i=1}^{T_1-1}\sum_{s=T+T_2\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\Big[(t-s+1)\Big|h\left(t, x_{d_{1t}}^{m}, \ldots, x_{d_{kt}}^{m}\right) \\
&\qquad -h\left(t, x_{d_{1t}}, \ldots, x_{d_{kt}}\right)\Big| \\
&\qquad + \Big|g\left(t, x_{c_{1t}}^{m}, \ldots, x_{c_{kt}}^{m}\right) \\
&\qquad -g\left(t, x_{c_{1t}}, \ldots, x_{c_{kt}}\right)\Big|\Big] \\
&+ \sum_{i=T_1}^{\infty}\sum_{s=T+i\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{|a_s|}\Big[(t-s+1)\Big|h\left(t, x_{d_{1t}}^{m}, \ldots, x_{d_{kt}}^{m}\right) \\
&\qquad -h\left(t, x_{d_{1t}}, \ldots, x_{d_{kt}}\right)\Big| \\
&\qquad + \Big|g\left(t, x_{c_{1t}}^{m}, \ldots, x_{c_{kt}}^{m}\right) \\
&\qquad -g\left(t, x_{c_{1t}}, \ldots, x_{c_{kt}}\right)\Big|\Big] \\
&+ \sum_{i=1}^{T_1-1}\sum_{j=T+i\tau}^{T+T_2\tau-1}\sum_{s=j}^{T+T_3\tau-1}\sum_{t=s}^{T+T_4\tau-1}\frac{t-s+1}{|a_j|} \\
&\qquad \times\Big|p\left(t, x_{o_{1t}}^{m}, \ldots, x_{o_{kt}}^{m}\right) \\
&\qquad -p\left(t, x_{o_{1t}}, \ldots, x_{o_{kt}}\right)\Big| \\
&+ \sum_{i=1}^{T_1-1}\sum_{j=T+i\tau}^{T+T_2\tau-1}\sum_{s=j}^{T+T_3\tau-1}\sum_{t=T+T_4\tau}^{\infty}\frac{t-s+1}{|a_j|} \\
&\qquad \times\Big|p\left(t, x_{o_{1t}}^{m}, \ldots, x_{o_{kt}}^{m}\right) \\
&\qquad -p\left(t, x_{o_{1t}}, \ldots, x_{o_{kt}}\right)\Big| \\
&+ \sum_{i=1}^{T_1-1}\sum_{j=T+i\tau}^{T+T_2\tau-1}\sum_{s=T+T_3\tau}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|} \\
&\qquad \times\Big|p\left(t, x_{o_{1t}}^{m}, \ldots, x_{o_{kt}}^{m}\right) \\
&\qquad -p\left(t, x_{o_{1t}}, \ldots, x_{o_{kt}}\right)\Big| \\
&+ \sum_{i=1}^{T_1-1}\sum_{j=T+T_2\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|} \\
&\qquad \times\Big|p\left(t, x_{o_{1t}}^{m}, \ldots, x_{o_{kt}}^{m}\right) \\
&\qquad -p\left(t, x_{o_{1t}}, \ldots, x_{o_{kt}}\right)\Big| \\
&+ \sum_{i=T_1}^{\infty}\sum_{j=T+i\tau}^{\infty}\sum_{s=j}^{\infty}\sum_{t=s}^{\infty}\frac{t-s+1}{|a_j|} \\
&\qquad \times\Big|p\left(t, x_{o_{1t}}^{m}, \ldots, x_{o_{kt}}^{m}\right) \\
&\qquad -p\left(t, x_{o_{1t}}, \ldots, x_{o_{kt}}\right)\Big| \\
&\le \sum_{i=1}^{T_1-1}\sum_{t=T+i\tau}^{T+T_2\tau-1}A\Big|f\left(t, x_{b_{1t}}^{m}, \ldots, x_{b_{kt}}^{m}\right) - f\left(t, x_{b_{1t}}, \ldots, x_{b_{kt}}\right)\Big| \\
&+ \sum_{i=1}^{T_1-1}\sum_{s=T+i\tau}^{T+T_2\tau-1}\sum_{t=s}^{T+T_3\tau-1}\Big[B\Big|h\left(t, x_{d_{1t}}^{m}, \ldots, x_{d_{kt}}^{m}\right) \\
&\qquad -h\left(t, x_{d_{1t}}, \ldots, x_{d_{kt}}\right)\Big|
\end{aligned}
$$

$$
\begin{aligned}
&\quad + A\left|g\left(t, x_{c_{1t}}^{m}, \ldots, x_{c_{kt}}^{m}\right)\right. \\
&\qquad \left.\left. - g\left(t, x_{c_{1t}}, \ldots, x_{c_{kt}}\right)\right|\right] \\
&+ \sum_{i=1}^{T_1-1} \sum_{j=T+i\tau}^{T+T_2\tau-1} \sum_{s=j}^{T+T_3\tau-1} \sum_{t=s}^{T+T_4\tau-1} E\left|p\left(t, x_{o_{1t}}^{m}, \cdots, x_{o_{kt}}^{m}\right)\right. \\
&\qquad \left. - p\left(t, x_{o_{1t}}, \cdots, x_{o_{kt}}\right)\right| \\
&+ 2\sum_{i=1}^{T_1-1} \sum_{t=T+T_2\tau}^{\infty} \frac{W_t}{|a_t|} + 2\sum_{i=T_1}^{\infty} \sum_{t=T+i\tau}^{\infty} \frac{W_t}{|a_t|} \\
&+ 2\sum_{i=1}^{T_1-1} \sum_{s=T+i\tau}^{T+T_2\tau-1} \sum_{t=T+T_3\tau}^{\infty} \frac{1}{|a_s|}\left[(t-s+1)Q_t + P_t\right] \\
&+ 2\sum_{i=1}^{T_1-1} \sum_{s=T+T_2\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|}\left[(t-s+1)Q_t + P_t\right] \\
&+ 2\sum_{i=T_1}^{\infty} \sum_{s=T+i\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{|a_s|}\left[(t-s+1)Q_t + P_t\right] \\
&+ 2\sum_{i=1}^{T_1-1} \sum_{j=T+i\tau}^{T+T_2\tau-1} \sum_{s=j}^{T+T_3\tau-1} \sum_{t=T+T_4\tau}^{\infty} \frac{t-s+1}{|a_j|} R_t \\
&+ 2\sum_{i=1}^{T_1-1} \sum_{j=T+i\tau}^{T+T_2\tau-1} \sum_{s=T+T_3\tau}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} R_t \\
&+ 2\sum_{i=1}^{T_1-1} \sum_{j=T+T_2\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} R_t \\
&+ 2\sum_{i=T_1}^{\infty} \sum_{j=T+i\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{|a_j|} R_t \\
&< \frac{\varepsilon}{18} + \frac{2(T_1-1)(T_2-1)\tau\varepsilon}{18T_1T_2\tau} \\
&+ \frac{2(T_1-1)\varepsilon}{18T_1} \\
&+ \frac{2(T_1-1)(T_2-1)(T_3-1)\tau^2\varepsilon}{18T_1T_2T_3\tau^2} \\
&+ \frac{2\varepsilon}{18} < \varepsilon, \quad \forall m \geq T_5,
\end{aligned} \tag{84}
$$

which yields that $S_L$ is continuous in $\Omega_1(N, M)$. It follows from Lemmas 2 and 4 that $S_L$ has a fixed point $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in \Omega_1(N, M)$; that is,

$$
\begin{aligned}
x_n &= L - \sum_{i=1}^{\infty} \sum_{t=n+i\tau}^{\infty} \frac{1}{a_t} f\left(t, x_{b_{1t}}, \ldots, x_{b_{kt}}\right) \\
&\quad - \sum_{i=1}^{\infty} \sum_{s=n+i\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s}\left[(t-s+1)\right. \\
&\qquad \times h\left(t, x_{d_{1t}}, \ldots, x_{d_{kt}}\right) \\
&\qquad \left. - g\left(t, x_{c_{1t}}, \ldots, x_{c_{kt}}\right)\right] \\
&\quad + \sum_{i=1}^{\infty} \sum_{j=n+i\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} \\
&\qquad \times \left[p\left(t, x_{o_{1t}}, \ldots, x_{o_{kt}}\right) - r_t\right], \\
&\qquad \forall n \geq T + \tau, \\
x_{n-\tau} &= L - \sum_{i=1}^{\infty} \sum_{t=n+(i-1)\tau}^{\infty} \frac{1}{a_t} f\left(t, x_{b_{1t}}, \ldots, x_{b_{kt}}\right) \\
&\quad - \sum_{i=1}^{\infty} \sum_{s=n+(i-1)\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} \\
&\qquad \times \left[(t-s+1) h\left(t, x_{d_{1t}}, \ldots, x_{d_{kt}}\right)\right. \\
&\qquad \left. - g\left(t, x_{c_{1t}}, \ldots, x_{c_{kt}}\right)\right] \\
&\quad + \sum_{i=1}^{\infty} \sum_{j=n+(i-1)\tau}^{\infty} \sum_{s=j}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_j} \\
&\qquad \times \left[p\left(t, x_{o_{1t}}, \ldots, x_{o_{kt}}\right) - r_t\right], \\
&\qquad \forall n \geq T + \tau,
\end{aligned} \tag{85}
$$

which imply that

$$
\begin{aligned}
&\Delta\left(x_n - x_{n-\tau}\right) \\
&\quad = -\frac{1}{a_n} f\left(n, x_{b_{1n}}, \ldots, x_{b_{kn}}\right) \\
&\qquad - \sum_{t=n}^{\infty} \frac{1}{a_n}\left[(t-n+1) h\left(t, x_{d_{1t}}, \ldots, x_{d_{kt}}\right)\right. \\
&\qquad\qquad \left. - g\left(t, x_{c_{1t}}, \ldots, x_{c_{kt}}\right)\right] \\
&\qquad + \sum_{s=n}^{\infty} \sum_{t=s}^{\infty} \frac{t-s+1}{a_n}\left[p\left(t, x_{o_{1t}}, \ldots, x_{o_{kt}}\right) - r_t\right], \\
&\qquad\qquad \forall n \geq T + \tau,
\end{aligned} \tag{86}
$$

which yields that

$$
\begin{aligned}
&\Delta^3\left(a_n \Delta\left(x_n - x_{n-\tau}\right)\right) + \Delta^3 f\left(n, x_{b_{1n}}, \ldots, x_{b_{kn}}\right) \\
&\quad + \Delta^2 g\left(n, x_{c_{1n}}, \ldots, x_{c_{kn}}\right) + \Delta h\left(n, x_{d_{1n}}, \ldots, x_{d_{kn}}\right) \\
&\quad + p\left(n, x_{o_{1n}}, \ldots, x_{o_{kn}}\right) = r_n, \quad \forall n \geq T + \tau;
\end{aligned} \tag{87}
$$

that is, $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in \Omega_1(N, M)$ is a bounded positive solution of (2). The rest of the proof is similar to that of Theorem 6 and is omitted. This is completes proof. □

## 4. Applications

Now we display nine examples as applications of the results presented in Section 3.

*Example 1.* Consider the fourth order nonlinear neutral delay difference equation:

$$\begin{aligned}
&\Delta^3\left(\left(n^5+2n\right)\Delta\left(x_n+\frac{(-1)^n n^2}{4n^2+n+3}x_{n-\tau}\right)\right)\\
&\quad+\Delta^3\left(nx_{n^2-n}^{17}+\sqrt{n}x_{n^2-n}^{5}-\sqrt{n-3}\right)\\
&\quad+\Delta^2\left(\frac{(-1)^n x_{n-3}^{21}+5n^2+1}{\sqrt{n^7-17}\left(x_{n-3}^4+3n^2+2\right)}\right)\\
&\quad+\Delta\left(\frac{12\sqrt{n}-x_{2n-1}^{15}\cos^2\left(n^3x_{2n-1}^5\right)}{2+n^5\left|x_{2n-1}\right|}\right)\\
&\quad+\frac{(-1)^{n-1}n\sin^{11}\left(nx_{n^3-3n}^9\right)}{\sqrt{n^2+2n+5}\left(1+x_{n^3-3n}^{22}\right)+n^5}\\
&\quad=\frac{4n^3+2n-1}{(n+1)^{12}\sqrt{3n^2+7}},\\
&\qquad\forall n\geq 7.
\end{aligned}\tag{88}$$

It follows from Theorem 6 that (88) possesses uncountably many bounded positive solutions in $\Omega_1(N,M)$.

*Example 2.* Consider the fourth order nonlinear neutral delay difference equation:

$$\begin{aligned}
&\Delta^3\left((-1)^{n+1}n^{5/2}\Delta\left(x_n+\frac{6n-3}{10n+2}x_{n-\tau}\right)\right)\\
&\quad+\Delta^3\left(\frac{x_{n^3-5}^5+2n^3}{x_{n^3-5}^8+(n+3)^4}\right)\\
&\quad+\Delta^2\left(\frac{(-1)^{n-1}n^2}{x_{2n-1}^{14}+n^5+\left|\cos\left(x_{2n-1}^2n^3\right)\right|}\right)\\
&\quad+\Delta\left(\frac{x_{n-1}^4\sin\sqrt{n+x_{n-1}^2}-n^4}{n^7+x_{n-1}^6\cos^4\sqrt{\left|x_{n-1}^3\right|+n}}\right)\\
&\quad+\frac{x_{5n-9}^8n^6}{n^9+\left|nx_{5n-9}\sin^3\left(n^2+1\right)\right|}\\
&\quad=\frac{\cos\left(n^3-2n+3\right)}{n^4+\sqrt{3n-1}},\\
&\qquad\forall n\geq 1.
\end{aligned}\tag{89}$$

It follows from Theorem 7 that (89) possesses uncountably many bounded positive solutions in $\Omega_1(N,M)$.

*Example 3.* Consider the fourth order nonlinear neutral delay difference equation:

$$\begin{aligned}
&\Delta^3\left(\left(n^7-n^3+2\right)\right.\\
&\qquad\left.\times\Delta\left(x_n-\frac{4n^2\cos^4\left(3n^2+2\right)}{5n^2+2n}x_{n-\tau}\right)\right)\\
&\quad+\Delta^3\left(\sqrt{n-9}x_{n^3-5}^3x_{n+11}^5\right)\\
&\quad+\Delta^2\left(\frac{\sqrt{n+3}x_{n^3-4}^{21}-\sqrt{2n+5}x_{n^2-1}^{12}}{n^5+3nx_{n^3-4}^2x_{n^2-1}^2}\right)\\
&\quad+\Delta\left(\frac{\left(x_{n-7}+x_{n^4-1}+n\right)^4}{\left|\left(n+x_{n-7}\right)^5\left(n+x_{n^4-1}\right)^3\right|+1}\right)\\
&\quad+\frac{n^2x_{n^2-3}^3x_{n-4}^5+\ln n}{n^2+x_{n-4}^4+\left(n+x_{n^2-3}\right)^6}\\
&\quad=\frac{(-1)^{n+1}\left(4n^5-3\right)}{n^9+4n^2+5},\\
&\qquad\forall n\geq 9.
\end{aligned}\tag{90}$$

Theorem 8 implies that (90) possesses uncountably many bounded positive solutions in $\Omega_1(N,M)$.

*Example 4.* Consider the fourth order nonlinear neutral delay difference equation:

$$\begin{aligned}
&\Delta^3\left(n^3\Delta\left(x_n+\sqrt{\ln\left(3n^3-134n\right)}x_{n-\tau}\right)\right)\\
&\quad+\Delta^3\left(\frac{2n+x_{2n-3}^3\cos\sqrt{n}}{\left(n+x_{2n-3}\right)^4(n+1)^5+72}\right)\\
&\quad+\Delta^2\left(\frac{nx_{n^5}^7\sin^3\left(n^2x_{n^5}^3\right)}{\left(\sqrt{n+2}+3\right)^6}\right)\\
&\quad+\Delta\left(\frac{x_{n+6}^2-(-1)^{n+1}n}{\sqrt{n^2+1}+\left(x_{n+6}+\sqrt{n}\right)^{10}}\right)\\
&\quad+\frac{n^2+x_{2n-18}^{27}}{n^{13}+x_{2n-18}^2\left|\cos^3\left(nx_{2n-18}^4\right)\right|}\\
&\quad=\frac{n^7-n^{11}-2}{n^{15}+36},\\
&\qquad\forall n\geq 15.
\end{aligned}\tag{91}$$

Theorem 9 yields that (91) has uncountably many bounded positive solutions in $l_\beta^\infty$.

*Example 5.* Consider the fourth order nonlinear neutral delay difference equation:

$$\begin{aligned}
&\Delta^3\left((-1)^n n\ln^2 n\Delta\left(x_n+\left(1+\frac{1}{n}\right)^n x_{n-\tau}\right)\right)\\
&\quad+\Delta^3\left(\frac{\sqrt{n}x_{n-4}^3-x_{n-4}^2+1}{n^5+\ln\left(1+x_{n-4}^2\right)}\right)\\
&\quad+\Delta^2\left(\frac{n^2x_{n-9}^3-nx_{n-9}-\sqrt{n}}{n^8+\left(x_{n-9}^4-n\right)^4}\right)\\
&\quad+\Delta\left(\frac{x_{n-6}^5-(-1)^n n}{n^6+\ln^3\left(n+x_{n-6}^8\right)}\right)\\
&\quad+\frac{\cos\left(n-\ln\left(1+x_{n-7}^2\right)\right)}{n^5+\cos^3\left(nx_{n-7}^7\right)+1}\\
&\quad=\frac{n^2-n+(-1)^n}{n^9+3n+1},\\
&\qquad\forall n\geq 10.
\end{aligned}\tag{92}$$

Theorem 10 guarantees that (92) possesses uncountably many bounded positive solutions in $l_\beta^\infty$.

*Example 6.* Consider the fourth order nonlinear neutral delay difference equation:

$$\begin{aligned}
&\Delta^3\left(\frac{n^4+2n}{\ln(n+5)}\Delta\left(x_n-(n-16)^3x_{n-\tau}\right)\right)\\
&\quad+\Delta^3\left(\sqrt{n^2-2n}\cos^2\left(nx_{n-5}^2\right)\right)\\
&\quad+\Delta^2\left(\frac{x_{n^2-2n+3}^3\ln^2(n-13)}{n^{11}+\left|x_{n^2-2n+3}+1\right|}\right)\\
&\quad+\Delta\left(\frac{nx_{n-19}^4\sin\left(x_{n-19}^2+n\right)}{n^9+2n+x_{n-19}^6}\right)\\
&\quad+\frac{n^2\sin^{16}\left(n+x_{n-13}^4\right)}{\left(n+x_{n-13}^2\right)^8}\\
&=\frac{(-1)^{n+1}\left(n^3+11\right)}{(n+2)^5(n+5)^4},\quad\forall n\geq 18.
\end{aligned}\tag{93}$$

Theorem 11 ensures that (93) possesses uncountably many bounded positive solutions in $l_\beta^\infty$.

*Example 7.* Consider the fourth order nonlinear neutral delay difference equation:

$$\begin{aligned}
&\Delta^3\left(-n^8\Delta\left(x_n-\left(3-\sin^9 n\right)x_{n-\tau}\right)\right)\\
&\quad+\Delta^3\left(\frac{n^5\ln\left(n+x_{n-30}^2\right)-x_{n-30}^3}{n^{25}+x_{n-30}^8}\right)\\
&\quad+\Delta^2\left(\frac{n^3x_{n^2-1}^3-nx_{n^2-1}^2-\cos\left(n^5\right)}{n^{81}+\left(nx_{n^2-1}^3-1\right)^2}\right)\\
&\quad+\Delta\left(\frac{n^6x_{n-2}^3-n+3}{n^{16}+\left(n^3-x_{n-2}^7\right)^2}\right)\\
&\quad+\frac{n^8+3n^5x_{n-4}^2-x_{n-4}^3}{n^{58}+n^{22}-\sin\left(n^5x_{n-4}^3\right)}\\
&\quad=\frac{n^{25}+(-1)^n n^{11}-1}{n^{91}+33n+5},\quad\forall n\geq 1.
\end{aligned}\tag{94}$$

Theorem 12 guarantees that (94) possesses uncountably many bounded positive solutions in $l_\beta^\infty$.

*Example 8.* Consider the fourth order nonlinear neutral delay difference equation:

$$\begin{aligned}
&\Delta^3\left((-1)^{n-1}(n+2)^3\Delta\left(x_n+x_{n-\tau}\right)\right)\\
&\quad+\Delta^3\left(\frac{n^4x_{n^2-9}^5x_{n-4}^2}{n^3+\left|x_{n^2-9}^{25}-nx_{n-4}^{37}\right|}\right)\\
&\quad+\Delta^2\left(\frac{n^4+x_{n-1}^2x_{n^3-3}^3}{n^9+\ln^3\left(n^2+x_{n^3-3}^6\right)}\right)\\
&\quad+\Delta\left(\frac{\sin\left(n^9x_{n-2}^8x_{5n-6}^9\right)}{n^5+\left(n+2x_{n-2}^{82}x_{5n-6}^{19}\right)^2}\right)\\
&\quad+\frac{\ln\left(n^2+x_{n^2-17}^6x_{n-6}^{14}\right)}{n^{16}+\left(2x_{n^2-17}+x_{n-6}\right)^2}\\
&=\frac{(-1)^{n(n+1)/2}\cos^7\left(3n^5+1\right)}{n^5+n^3+3},\\
&\qquad\forall n\geq 8.
\end{aligned}\tag{95}$$

It follows from Theorem 13 that (95) possesses uncountably many bounded positive solutions in $\Omega(N,M)$.

*Example 9.* Consider the fourth order nonlinear neutral delay difference equation:

$$\begin{aligned}
&\Delta^3\left(n^{17}\Delta\left(x_n - x_{n-\tau}\right)\right)\\
&\quad+\Delta^3\left(\sqrt{n}x_{9n-5}^3x_{2n-1}^2-\sin^3\left(\arctan\left(n^2+1\right)\right)\right)\\
&\quad+\Delta^2\left(\frac{n^3+x_{n-7}^4x_{n^2-2}^5}{\left(n+x_{n-7}^6\right)^{13}+\left|x_{n-7}-nx_{n^2-2}^2\right|}\right)\\
&\quad+\Delta\left(\frac{x_{n-10}^3\ln^4\left(n^2+x_{n^3-3}^2\right)}{\left(n+x_{n-10}^2x_{n^3-3}^2\right)^9(n+2)^{31}}\right)\\
&\quad+\frac{\left(3-n^2\right)x_{3n-2}^2-nx_{n-9}^9}{n^{60}+\left|x_{3n-2}x_{n-9}\right|}\\
&\quad=\frac{(-1)^n\left(n^2-1\right)\cos^5\left(n-\sqrt{n}\right)}{n^{19}+15n^8+1},\\
&\qquad\qquad\forall n\geq 11.
\end{aligned}\tag{96}$$

Theorem 14 means that (96) possesses uncountably many bounded positive solutions in $\Omega(N, M)$.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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