Research Article

Global and Blow-Up Solutions for a Class of Nonlinear Parabolic Problems under Robin Boundary Condition

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We discuss the global and blow-up solutions of the following nonlinear parabolic problems with a gradient term under Robin boundary conditions: $(b(u))_t = \nabla \cdot (h(t)k(x)a(u)\nabla u) + f(x, u, |\nabla u|^2, t)$, in $D \times (0, T)$, $(\partial u/\partial n) + \gamma u = 0$, on $\partial D \times (0, T)$, $u(x, 0) = u_0(x) > 0$, in $\bar{D}$, where $D \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain with smooth boundary ∂D . Under some appropriate assumption on the functions f, h, k, b , and a and initial value u_0 , we obtain the sufficient conditions for the existence of a global solution, an upper estimate of the global solution, the sufficient conditions for the existence of a blow-up solution, an upper bound for “blow-up time,” and an upper estimate of “blow-up rate.” Our approach depends heavily on the maximum principles.

1. Introduction

The study of global and blow-up solutions for nonlinear parabolic equations has received a lot of attention in the past several decades (see [1–4]). In most works, so far, a variety of approaches have been developed in dealing with different nonlinear parabolic problems, such as the existence of global solution, blow-up solution, an upper bound for “blow-up time,” an upper estimate of “blow-up rate,” or global solution. So far, some applications in physics, chemistry, and biology are relevant to blow-up phenomena which can be found in [5–11]. In this paper, we consider the global and blow-up solutions of the following nonlinear parabolic equation with Robin boundary condition:

$$\begin{aligned} (b(u))_t &= \nabla \cdot (h(t)k(x)a(u)\nabla u) + f(x, u, q, t), \\ &\quad \text{in } D \times (0, T), \\ \frac{\partial u}{\partial n} + \gamma u &= 0, \quad \text{on } \partial D \times (0, T), \\ u(x, 0) &= u_0(x) > 0, \quad \text{in } \bar{D}, \end{aligned} \quad (1)$$

where $q = |\nabla u|^2$, $D \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain with smooth boundary ∂D , $\partial u/\partial n$ represents the outward normal derivative on ∂D , γ is positive constant, u_0 is the initial value,

T is the maximal existence time of u , and $\bar{D}$ is the closure of D . Set $\mathbb{R}^+ = (0, +\infty)$. We assume, throughout the paper, that $b(s)$ is a positive $C^3(\mathbb{R}^+)$ function, $b'(s) > 0$ for any $s \in \mathbb{R}^+$, $a(s)$ is a positive $C^2(\mathbb{R}^+)$ function, $k(x)$ is a positive $C^1(\bar{D})$ function, $h(t)$ is a positive $C^1(\mathbb{R}^+)$ function, $f(x, s, d, t)$ is a nonnegative $C^1(\bar{D} \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+)$ function, and $u_0(x)$ is a positive $C^2(\bar{D})$ function. Under these assumptions, the classical parabolic equation theory [12] ensures that there exists a unique classical solution $u(x, t)$ with some $T > 0$ for the problem (1), and the solution is positive over $\bar{D} \times [0, T)$. Moreover, by the regularity theorem [13], $u(x, t) \in C^3(D \times (0, T)) \cap C^2(\bar{D} \times [0, T))$.

The problems of the global and blow-up solutions for nonlinear parabolic equations have been investigated extensively by many authors and have got a lot of meaningful results. Some special cases of problem (1) have been treated already. Ding [14] deals with the following problem:

$$\begin{aligned} (b(u))_t &= \nabla \cdot (a(u)\nabla u) + f(u), \quad \text{in } D \times (0, T), \\ \frac{\partial u}{\partial n} + \gamma u &= 0, \quad \text{on } \partial D \times (0, T), \\ u(x, 0) &= h(x) > 0, \quad \text{in } \bar{D}, \end{aligned} \quad (2)$$

where D is a bounded domain of $\mathbb{R}^N$ ($N \geq 2$) with smooth boundary ∂D . By constructing auxiliary functions and using a first-order differential inequality technique, Ding derives conditions on the data, which guarantee the existence of blow-up or global solution. The following problem is investigated by Enache in [15]:

$$\begin{aligned} u_t &= \nabla \cdot (a(u) \nabla u) + f(u), \quad \text{in } D \times (0, T), \\ \frac{\partial u}{\partial n} + \gamma u &= 0, \quad \text{on } \partial D \times (0, T), \\ u(x, 0) &= h(x) > 0, \quad \text{in } \bar{D}, \end{aligned} \quad (3)$$

where D is a bounded domain of $\mathbb{R}^N$ ($N \geq 2$) with smooth boundary ∂D . By constructing auxiliary functions and first-order differential inequality technique, Enache establishes some conditions on nonlinearities and the initial data to guarantee that $u(x, t)$ exists for all times $t > 0$ or blows up at some finite time T . Besides, the following problem is investigated by Zhang in [16]:

$$\begin{aligned} (b(u))_t &= \Delta u + f(u), \quad \text{in } D \times (0, T), \\ \frac{\partial u}{\partial n} + \gamma u &= 0, \quad \text{on } \partial D \times (0, T), \\ u(x, 0) &= h(x) > 0, \quad \text{in } \bar{D}, \end{aligned} \quad (4)$$

where D is a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) with smooth boundary. Under appropriate assumptions on the functions f , b , and h , Zhang obtains the conditions under which the solutions may exist globally or blow up in a finite time. Moreover, upper estimates of the “blow-up time,” blow-up rate, and global solutions are obtained also.

In this paper, we obtain the existence theorem of global and blow-up solution by constructing completely different auxiliary functions and technically using maximum principles. As a result, the sufficient conditions for the existence of a global solution and an upper estimate of the global solution and the sufficient conditions for the existence of a blow-up solution, an upper bound for “blow-up time,” and an upper estimate of “blow-up rate” are specified under some appropriate assumption on the functions f , h , k , b , and a and initial value u_0 . Our results extend and supplement those obtained in [14–16].

The content of this paper is organized as follows. In Section 2, we study the existence of the global solution of (1). In Section 3, we investigate the blow-up solution of (1). In Section 4, we will give a few examples to explain our results.

2. Global Solution

Our main result for the global solution is the following Theorem 1.

Theorem 1. *Let u be a solution of (1). Suppose that the following conditions (i)–(iv) are satisfied.*

(i) *For any $s \in \mathbb{R}^+$,*

$$\begin{aligned} (sb'(s))' &\geq 0, \\ sb'(s) - (sb'(s))' &\leq 0, \\ \left(\frac{a(s)}{b'(s)}\right)' &\leq 0, \\ \left[\frac{1}{a(s)}\left(\frac{a(s)}{b'(s)}\right)' + \frac{1}{b'(s)}\right]' + \frac{1}{a(s)}\left(\frac{a(s)}{b'(s)}\right)' + \frac{1}{b'(s)} &\leq 0. \end{aligned} \quad (5)$$

(ii) *For any $(x, s, d, t) \in D \times \mathbb{R}^+ \times \overline{\mathbb{R}^+} \times \mathbb{R}^+$,*

$$\begin{aligned} \left(\frac{f(x, s, d, t)}{h(t)}\right)_t &\leq 0, \\ f_d(x, s, d, t) \left[\left(\frac{1}{b'(s)}\right)' + \frac{1}{b'(s)}\right] &\leq 0, \\ \left(\frac{f(x, s, d, t)b'(s)}{a(s)}\right)_s - \frac{f(x, s, d, t)b'(s)}{a(s)} &+ \frac{h'(t)(b'(s))^2}{a(s)h(t)} \leq 0. \end{aligned} \quad (6)$$

(iii) *Consider the integration*

$$\int_{m_0}^{+\infty} \frac{b'(s)}{e^s} ds = +\infty, \quad m_0 = \min_{\bar{D}} u_0(x). \quad (7)$$

(iv) *Consider*

$$\alpha = \max_{\bar{D}} \left\{ \frac{\nabla \cdot (h(0)k(x)a(u_0)\nabla u_0) + f(x, u_0, q_0, 0)}{e^{u_0}} \right\} > 0, \quad q_0 = |\nabla u_0|^2. \quad (8)$$

Then the solution u to problem (1) must be a global solution and

$$u(x, t) \leq H^{-1}(\alpha t + H(u_0(x, t))), \quad (x, t) \in \bar{D} \times \overline{\mathbb{R}^+}, \quad (9)$$

where

$$H(z) = \int_{m_0}^z \frac{b'(s)}{e^s} ds, \quad z \geq m_0, \quad (10)$$

and H^{-1} is the inverse function of H .

Proof. Consider the auxiliary function

$$P(x, t) = b'(u)u_t - \alpha e^u. \quad (11)$$

Then, we have

$$\nabla P = b'' u_t \nabla u + b' \nabla u_t - \alpha e^u \nabla u, \quad (12)$$

$$\begin{aligned} \Delta P &= b''' u_t |\nabla u|^2 + 2b'' \nabla u \cdot \nabla u_t \\ &+ b'' u_t \Delta u + b' \Delta u_t - \alpha e^u |\nabla u|^2 - \alpha e^u \Delta u. \end{aligned} \quad (13)$$

By (1),

$$\begin{aligned} (b(u))_t &= b' u_t = \nabla \cdot (h(t) k(x) a(u) \nabla u) + f \\ &= h(t) k(x) a(u) \Delta u + h(t) k(x) a'(u) |\nabla u|^2 \\ &+ h(t) a(u) (\nabla k \cdot \nabla u) + f. \end{aligned} \quad (14)$$

We have

$$\begin{aligned} u_t &= \frac{akh}{b'} \Delta u + \frac{a' kh}{b'} |\nabla u|^2 + \frac{ah}{b'} (\nabla k \cdot \nabla u) + \frac{f}{b'}, \\ (u_t)_t &= h' \left(\frac{ak}{b'} \Delta u + \frac{a' k}{b'} |\nabla u|^2 + \frac{a}{b'} (\nabla k \cdot \nabla u) \right) \\ &+ h \left(\frac{ak}{b'} \Delta u + \frac{a' k}{b'} |\nabla u|^2 + \frac{a}{b'} (\nabla k \cdot \nabla u) \right)_t \\ &+ \left(\frac{f}{b'} \right)_t \\ &= \left(\frac{a''}{b'} - \frac{a' b''}{(b')^2} \right) h k u_t |\nabla u|^2 + \frac{2a' kh}{b'} (\nabla u \cdot \nabla u_t) \\ &+ \left(\frac{a'}{b'} - \frac{ab''}{(b')^2} \right) h u_t (\nabla k \cdot \nabla u) \\ &+ \frac{ah}{b'} (\nabla k \cdot \nabla u_t) + \left(\frac{a'}{b'} - \frac{ab''}{(b')^2} \right) h k u_t \Delta u \\ &+ \frac{akh}{b'} \Delta u_t + \frac{a' kh'}{b'} |\nabla u|^2 + \frac{ah'}{b'} (\nabla k \cdot \nabla u) \\ &+ \frac{akh'}{b'} \Delta u \\ &+ \frac{f_u u_t + 2f_q (\nabla u \cdot \nabla u_t) + f_t}{b'} - \frac{fb'' u_t}{(b')^2}. \end{aligned} \quad (15)$$

Then

$$\begin{aligned} P_t &= b'' (u_t)^2 + b' (u_t)_t - \alpha e^u u_t \\ &= b'' (u_t)^2 + \left(a' - \frac{ab''}{b'} \right) k h u_t \Delta u + a k h \Delta u_t \\ &+ a k h' \Delta u + \left(a'' - \frac{a' b''}{b'} \right) k h u_t |\nabla u|^2 + a' k h' |\nabla u|^2 \\ &+ (2a' k h + 2f_q) (\nabla u \cdot \nabla u_t) \end{aligned}$$

$$\begin{aligned} &+ \left(a' - \frac{ab''}{b'} \right) h u_t (\nabla k \cdot \nabla u) + a h (\nabla k \cdot \nabla u_t) \\ &+ a h' (\nabla k \cdot \nabla u) \\ &+ \left(f_u - \frac{fb''}{b'} - \alpha e^u \right) u_t + f_t. \end{aligned}$$

(16)

By (13) and (16), it follows that

$$\begin{aligned} \frac{akh}{b'} \Delta P - P_t &= \left(\frac{akhb'''}{b'} + \frac{a' khb''}{b'} - a'' kh \right) u_t |\nabla u|^2 \\ &+ \left(\frac{2akhb''}{b'} - 2a' kh - 2f_q \right) (\nabla u \cdot \nabla u_t) \\ &+ \left(\frac{2akhb''}{b'} - a' kh \right) u_t \Delta u \\ &- \left(\frac{akh}{b'} \alpha e^u + a' h' k \right) |\nabla u|^2 - \left(\frac{akh}{b'} \alpha e^u + a k h' \right) \Delta u \\ &- b'' (u_t)^2 + \left(\frac{fb''}{b'} + \alpha e^u - f_u \right) u_t \\ &+ \left(\frac{ahb''}{b'} - a' h \right) u_t (\nabla k \cdot \nabla u) \\ &- a h (\nabla k \cdot \nabla u_t) - a h' (\nabla k \cdot \nabla u) - f_t. \end{aligned} \quad (17)$$

By (14), we have

$$\Delta u = \frac{b'}{akh} u_t - \frac{a'}{a} |\nabla u|^2 - \frac{1}{k} (\nabla k \cdot \nabla u) - \frac{f}{akh}. \quad (18)$$

Substitute (18) into (17) to get

$$\begin{aligned} \frac{akh}{b'} \Delta P - P_t &= \left(\frac{akhb'''}{b'} + \frac{(a')^2 kh}{a} - \frac{a' khb''}{b'} - a'' kh \right) u_t |\nabla u|^2 \\ &+ \left(\frac{2akhb''}{b'} - 2a' kh - 2f_q \right) (\nabla u \cdot \nabla u_t) \\ &+ \left(b'' - \frac{a' b'}{a} \right) u_t^2 + \left(\frac{a' f}{a} - \frac{fb''}{b'} - f_u - \frac{b' h'}{h} \right) u_t \\ &+ \left(\frac{a' kh}{b'} \alpha e^u - \frac{akh}{b'} \alpha e^u \right) |\nabla u|^2 \end{aligned}$$

$$\begin{aligned}
& -\frac{ahb''}{b'}u_t(\nabla k \cdot \nabla u) - ah(\nabla k \cdot \nabla u_t) \\
& + \frac{ah}{b'}\alpha e^u(\nabla k \cdot \nabla u) + \frac{f}{b'}\alpha e^u + \frac{fh'}{h} - f_t.
\end{aligned} \tag{19}$$

By (12), we have

$$\nabla u_t = \frac{1}{b'}\nabla P - \frac{b''}{b'}u_t\nabla u + \alpha \frac{e^u}{b'}\nabla u. \tag{20}$$

Next, we substitute (20) into (19) to obtain

$$\begin{aligned}
& \frac{akh}{b'}\Delta P - P_t \\
& = \left(\frac{2akhb''}{(b')^2} - \frac{2a'kh}{b'} - \frac{2f_q}{b'} \right) (\nabla u \cdot \nabla P) - \frac{ah}{b'}(\nabla k \cdot \nabla P) \\
& + \left(\frac{akhb'''}{b'} - \frac{2akh(b'')^2}{(b')^2} + \frac{a'khb''}{b'} - a''kh \right. \\
& \quad \left. + \frac{(a')^2kh}{a} + \frac{2b''f_q}{b'} \right) u_t |\nabla u|^2 \\
& + \left(\frac{2akhb''}{(b')^2}\alpha e^u - \frac{a'kh}{b'}\alpha e^u - \frac{akh}{b'}\alpha e^u - \frac{2f_q}{b'}\alpha e^u \right) |\nabla u|^2 \\
& + \left(b'' - \frac{a'b'}{a} \right) u_t^2 + \left(\frac{a'f}{a} - \frac{fb''}{b'} - f_u - \frac{b'h'}{h} \right) u_t \\
& + \frac{f}{b'}\alpha e^u + \frac{fh'}{h} - f_t.
\end{aligned} \tag{21}$$

So we have

$$\begin{aligned}
& \frac{akh}{b'}\Delta P + \left(\frac{2a'kh}{b'} + \frac{2f_q}{b'} - \frac{2akhb''}{(b')^2} \right) \\
& \times (\nabla u \cdot \nabla P) + \frac{ah}{b'}(\nabla k \cdot \nabla P) - P_t \\
& = \left(\frac{akhb'''}{b'} - \frac{2akh(b'')^2}{(b')^2} + \frac{a'khb''}{b'} - a''kh + \frac{(a')^2kh}{a} \right. \\
& \quad \left. + \frac{2b''f_q}{b'} \right) u_t |\nabla u|^2 \\
& + \left(\frac{2akhb''}{(b')^2}\alpha e^u - \frac{a'kh}{b'}\alpha e^u - \frac{akh}{b'}\alpha e^u - \frac{2f_q}{b'}\alpha e^u \right) |\nabla u|^2 \\
& + \left(b'' - \frac{a'b'}{a} \right) u_t^2 + \left(\frac{a'f}{a} - \frac{fb''}{b'} - f_u - \frac{b'h'}{h} \right) u_t \\
& + \frac{f}{b'}\alpha e^u + \frac{fh'}{h} - f_t.
\end{aligned} \tag{22}$$

According to (11), we have

$$u_t = \frac{1}{b'}P + \alpha \frac{e^u}{b'}. \tag{23}$$

Substituting (23) into (22), we have

$$\begin{aligned}
& \frac{akh}{b'}\Delta P + \left[2kh\left(\frac{a}{b'}\right)' + \frac{2f_q}{b'} \right] (\nabla u \cdot \nabla P) + \frac{ah}{b'}(\nabla k \cdot \nabla P) \\
& + \left[\frac{a}{(b')^2}\left(\frac{fb'}{a}\right)_u + \frac{h'}{h} \right] P \\
& + \left[akh\left(\frac{1}{a}\left(\frac{a}{b'}\right)'\right)' + 2f_q\left(\frac{1}{b'}\right)' \right] |\nabla u|^2 P - P_t \\
& = (-\alpha e^u) \left(\frac{2f_q}{b'} - \frac{2b''f_q}{(b')^2} \right) |\nabla u|^2 \\
& + (-\alpha e^u) kh \left(\frac{a'}{b'} - \frac{ab''}{(b')^2} \right) |\nabla u|^2 \\
& + \alpha e^u \left(\frac{akhb'''}{(b')^2} - \frac{2akh(b'')^2}{(b')^3} + \frac{a'khb''}{(b')^2} - \frac{a''kh}{b'} \right. \\
& \quad \left. + \frac{(a')^2kh}{ab'} \right) |\nabla u|^2 \\
& - \alpha e^u akh \left(\frac{1}{b'} - \frac{b''}{(b')^2} \right) |\nabla u|^2 + \left(b'' - \frac{a'b'}{a} \right) u_t^2 \\
& + \alpha e^u \left(\frac{a'f}{ab'} - \frac{fb''}{(b')^2} - \frac{f_u}{b'} + \frac{f}{b'} - \frac{h'}{h} \right) + \frac{h'}{h}f - f_t.
\end{aligned} \tag{24}$$

Namely,

$$\begin{aligned}
& \frac{akh}{b'}\Delta P + \left[\left(2kh\left(\frac{a}{b'}\right)' + \frac{2f_q}{b'} \right) \nabla u + \frac{ah}{b'}\nabla k \right] \cdot \nabla P \\
& + \left\{ \frac{a}{(b')^2}\left(\frac{fb'}{a}\right)_u + \frac{h'}{h} \right. \\
& \quad \left. + \left[akh\left(\frac{1}{a}\left(\frac{a}{b'}\right)'\right)' + 2f_q\left(\frac{1}{b'}\right)' \right] |\nabla u|^2 \right\} P - P_t \\
& = -\alpha e^u \left\{ 2f_q \left[\left(\frac{1}{b'}\right)' + \frac{1}{b'} \right] + akh \right. \\
& \quad \left. \times \left[\left(\frac{1}{a}\left(\frac{a}{b'}\right)'\right)' + \frac{1}{b'} \right] + \frac{1}{a}\left(\frac{a}{b'}\right)' + \frac{1}{b'} \right\} |\nabla u|^2
\end{aligned}$$

$$\begin{aligned}
& -\frac{(b')^2}{a} \left(\frac{a}{b'} \right)' u_t^2 - \alpha e^u \frac{a}{(b')^2} \\
& \times \left[\left(\frac{fb'}{a} \right)_u - \frac{fb'}{a} + \frac{h'(b')^2}{ah} \right] - h \left(\frac{f}{h} \right)_t.
\end{aligned} \quad (25)$$

The assumptions (5) and (6) guarantee that the right-hand side of (25) is nonnegative; that is,

$$\begin{aligned}
& \frac{akh}{b'} \Delta P + \left[\left(2kh \left(\frac{a}{b'} \right)' + \frac{2f_q}{b'} \right) \nabla u + \frac{ah}{b'} \nabla k \right] \cdot \nabla P \\
& + \left\{ \frac{a}{(b')^2} \left(\frac{fb'}{a} \right)_u + \frac{h'}{h} \right. \\
& \quad \left. + \left[akh \left(\frac{1}{a} \left(\frac{a}{b'} \right)' \right)' + 2f_q \left(\frac{1}{b'} \right)' \right] |\nabla u|^2 \right\} P - P_t \\
& \geq 0, \quad \text{in } D \times (0, T).
\end{aligned} \quad (26)$$

By applying maximum principle (see [17]), it follows from (26) that P can attain its nonnegative maximum only for $\bar{D} \times \{0\}$ or $\partial D \times (0, T)$.

For $\bar{D} \times \{0\}$, by (8), we have

$$\begin{aligned}
& \max_{\bar{D}} P(x, 0) \\
& = \max_{\bar{D}} \{b'(u_0)(u_0)_t - \alpha e^{u_0}\} \\
& = \max_{\bar{D}} \{[\nabla \cdot (h(0)k(x)a(u_0)\nabla u_0) \\
& \quad + f(x, u_0, q_0, 0)] - \alpha e^{u_0}\} \\
& = \max_{\bar{D}} \left\{ e^{u_0} \left[\frac{\nabla \cdot (h(0)k(x)a(u_0)\nabla u_0) + f(x, u_0, q_0, 0)}{e^{u_0}} \right. \right. \\
& \quad \left. \left. - \alpha \right] \right\} = 0.
\end{aligned} \quad (27)$$

For $\partial D \times (0, T)$, we claim that P cannot take a positive maximum at any point (x, t) . In fact, suppose that P can take a positive maximum at one point $(x_0, t_0) \in \partial D \times (0, T)$; then

$$P(x_0, t_0) > 0, \quad \left. \frac{\partial P}{\partial n} \right|_{(x_0, t_0)} > 0. \quad (28)$$

Combine (1) and (11) with (23); we have

$$\begin{aligned}
\frac{\partial P}{\partial n} & = b'' u_t \frac{\partial u}{\partial n} + b' \frac{\partial u_t}{\partial n} - \alpha e^u \frac{\partial u}{\partial n} \\
& = -\gamma b'' u u_t + b' \left(\frac{\partial u}{\partial n} \right)_t + \gamma \alpha u e^u \\
& = -\gamma b'' u u_t + b'(-\gamma u)_t + \gamma \alpha u e^u
\end{aligned}$$

$$\begin{aligned}
& = -\gamma (ub')' u_t + \gamma \alpha u e^u \\
& = -\gamma (ub')' \left(\frac{1}{b'} P + \alpha \frac{1}{b'} e^u \right) + \gamma \alpha u e^u \\
& = -\gamma \frac{(ub')'}{b'} P + \gamma \alpha e^u \frac{ub' - (ub')'}{b'}, \\
& \quad \text{on } \partial D \times (0, T).
\end{aligned} \quad (29)$$

Next, by using a part condition of (5) $(sb'(s))' \geq 0$, $sb'(s) - (sb'(s))' \leq 0$ for any $s \in \mathbb{R}^+$, we can obtain

$$\left. \frac{\partial P}{\partial n} \right|_{(x_0, t_0)} \leq 0, \quad (30)$$

which contradicts with inequality (28). Thus, we know that the maximum of P in $\bar{D} \times [0, T)$ is zero; that is,

$$P \leq 0, \quad \text{in } \bar{D} \times [0, T). \quad (31)$$

With (11), we know

$$\frac{b'(u)}{e^u} u_t \leq \alpha. \quad (32)$$

For each fixed $x \in \bar{D}$, we integrate (32) from 0 to t :

$$\int_0^t \frac{b'(u)}{e^u} u_t dt = \int_{u_0(x)}^{u(x,t)} \frac{b'(s)}{e^s} ds \leq \alpha t, \quad (33)$$

which implies that u must be a global solution of (1). In fact, suppose that u blows up at finite time T ; then

$$\lim_{t \rightarrow T^-} u(x, t) = +\infty. \quad (34)$$

Passing to the limit as $t \rightarrow T^-$ in (33) yields

$$\begin{aligned}
& \int_{u_0(x)}^{+\infty} \frac{b'(s)}{e^s} ds \leq \alpha T, \\
& \int_{m_0}^{+\infty} \frac{b'(s)}{e^s} ds = \int_{m_0}^{u_0(x)} \frac{b'(s)}{e^s} ds + \int_{u_0(x)}^{+\infty} \frac{b'(s)}{e^s} ds \\
& \leq \int_{m_0}^{u_0(x)} \frac{b'(s)}{e^s} ds + \alpha T < +\infty,
\end{aligned} \quad (35)$$

which contradicts with the condition (iii). This shows that u is global solution. Moreover, it follows from (33) that

$$\begin{aligned}
& \int_{u_0(x)}^{u(x,t)} \frac{b'(s)}{e^s} ds = \int_{m_0}^{u(x,t)} \frac{b'(s)}{e^s} ds - \int_{m_0}^{u_0(x)} \frac{b'(s)}{e^s} ds \\
& = H(u(x, t)) - H(u_0(x)) \leq \alpha t.
\end{aligned} \quad (36)$$

Since H is an increasing function, we have

$$u(x, t) \leq H^{-1}(\alpha t + H(u_0(x))). \quad (37)$$

The proof is completed. $\square$

3. Blow-Up Solution

The following theorem is the main result for the blow-up solution of (1).

Theorem 2. *Let u be a solution of problem (1). Assume that the following conditions (i)–(iv) are satisfied.*

(i) For any $s \in \mathbb{R}^+$,

$$\begin{aligned} (sb'(s))' &\geq 0, \quad sb'(s) - (sb'(s))' \geq 0, \quad \left(\frac{a(s)}{b'(s)}\right)' \geq 0, \\ \left[\frac{1}{a(s)}\left(\frac{a(s)}{b'(s)}\right)' + \frac{1}{b'(s)}\right]' + \frac{1}{a(s)}\left(\frac{a(s)}{b'(s)}\right)' + \frac{1}{b'(s)} &\geq 0. \end{aligned} \quad (38)$$

(ii) For any $(x, s, d, t) \in D \times \mathbb{R}^+ \times \overline{\mathbb{R}^+} \times \mathbb{R}^+$,

$$\begin{aligned} \left(\frac{f(x, s, d, t)}{h(t)}\right)_t &\geq 0, \\ f_d(x, s, d, t) \left[\left(\frac{1}{b'(s)}\right)' + \frac{1}{b'(s)}\right] &\geq 0, \\ \left(\frac{f(x, s, d, t)b'(s)}{a(s)}\right)_s - \frac{f(x, s, d, t)b'(s)}{a(s)} & \\ + \frac{h'(t)(b'(s))^2}{a(s)h(t)} &\geq 0. \end{aligned} \quad (39)$$

(iii) Consider the integration

$$\int_{M_0}^{+\infty} \frac{b'(s)}{e^s} ds < +\infty, \quad M_0 = \max_{\overline{D}} u_0(x). \quad (40)$$

(iv) Consider

$$\beta = \min_{\overline{D}} \left\{ \frac{\nabla \cdot (h(0)k(x)a(u_0)\nabla u_0) + f(x, u_0, q_0, 0)}{e^{u_0}} \right\} > 0, \quad (41)$$

$$q_0 = |\nabla u_0|^2.$$

Then the solution u of problem (1) must blow up in finite time T , and

$$T \leq \frac{1}{\beta} \int_{M_0}^{+\infty} \frac{b'(s)}{e^s} ds, \quad (42)$$

$$u(x, t) \leq G^{-1}(\beta(T-t)), \quad (x, t) \in \overline{D} \times [0, T),$$

where

$$G(z) = \int_z^{+\infty} \frac{b'(s)}{e^s} ds, \quad z > 0, \quad (43)$$

and G^{-1} is the inverse function of G .

Proof. Construct the following auxiliary function:

$$Q(x, t) = b'(u)u_t - \beta e^u. \quad (44)$$

So we have

$$\begin{aligned} \nabla Q &= b''u_t \nabla u + b' \nabla u_t - \beta e^u \nabla u, \\ \Delta Q &= b'''u_t |\nabla u|^2 + 2b'' \nabla u \cdot \nabla u_t + b''u_t \Delta u + b' \Delta u_t \\ &\quad - \beta e^u |\nabla u|^2 - \beta e^u \Delta u. \end{aligned} \quad (45)$$

As the previous derivation from (14) to (25), we can obtain

$$\begin{aligned} \frac{akh}{b'} \Delta Q &+ \left[\left(2kh \left(\frac{a}{b'} \right)' + \frac{2f_q}{b'} \right) \nabla u + \frac{ah}{b'} \nabla k \right] \cdot \nabla Q \\ &+ \left\{ \frac{a}{(b')^2} \left(\frac{fb'}{a} \right)_u + \frac{h'}{h} \right. \\ &\quad \left. + \left[akh \left(\frac{1}{a} \left(\frac{a}{b'} \right)' \right)' + 2f_q \left(\frac{1}{b'} \right)' \right] |\nabla u|^2 \right\} Q - Q_t \\ &= -\beta e^u \left\{ 2f_q \left[\left(\frac{1}{b'} \right)' + \frac{1}{b'} \right] \right. \\ &\quad \left. + akh \left[\left(\frac{1}{a} \left(\frac{a}{b'} \right)' + \frac{1}{b'} \right)' + \frac{1}{a} \left(\frac{a}{b'} \right)' + \frac{1}{b'} \right] \right\} |\nabla u|^2 \\ &\quad - \frac{(b')^2}{a} \left(\frac{a}{b'} \right)' u_t^2 - \beta e^u \frac{a}{(b')^2} \\ &\quad \times \left[\left(\frac{fb'}{a} \right)_u - \frac{fb'}{a} + \frac{h'(b')^2}{ah} \right] - h \left(\frac{f}{h} \right)_t. \end{aligned} \quad (46)$$

It is seen from (38) and (39) that the right-hand side of (46) is nonpositive; that is,

$$\begin{aligned} \frac{akh}{b'} \Delta Q &+ \left[\left(2kh \left(\frac{a}{b'} \right)' + \frac{2f_q}{b'} \right) \nabla u + \frac{ah}{b'} \nabla k \right] \cdot \nabla Q \\ &+ \left\{ \frac{a}{(b')^2} \left(\frac{fb'}{a} \right)_u + \frac{h'}{h} \right. \\ &\quad \left. + \left[akh \left(\frac{1}{a} \left(\frac{a}{b'} \right)' \right)' + 2f_q \left(\frac{1}{b'} \right)' \right] |\nabla u|^2 \right\} Q - Q_t \\ &\leq 0, \quad \text{in } D \times (0, T). \end{aligned} \quad (47)$$

By applying maximum principle (see [17]), it follows from (47) that Q can attain its nonpositive minimum only for $\overline{D} \times \{0\}$ or $\partial D \times (0, T)$.

For $\overline{D} \times \{0\}$, with (41), we have

$$\begin{aligned} \min_{\overline{D}} Q(x, 0) &= \min_{\overline{D}} \{b'(u_0)(u_0)_t - \beta e^{u_0}\} \\ &= \min_{\overline{D}} \{\nabla \cdot (h(0)k(x)a(u_0)\nabla u_0) \\ &\quad + f(x, u_0, q_0, 0) - \beta e^{u_0}\} \\ &= \min_{\overline{D}} \left\{ e^{u_0} \left[\frac{\nabla \cdot (h(0)k(x)a(u_0)\nabla u_0) + f(x, u_0, q_0, 0)}{e^{u_0}} \right. \right. \\ &\quad \left. \left. - \beta \right] \right\} = 0. \end{aligned} \quad (48)$$

For $\partial D \times (0, T)$, substituting P and α with Q and β in (29), respectively, we have

$$\frac{\partial Q}{\partial n} = -\gamma \frac{(ub')'}{b'} Q + \gamma \beta e^u \frac{ub' - (ub')'}{b'}, \quad \text{on } \partial D \times (0, T). \quad (49)$$

Combining (47)–(49) with condition (i), we can apply the maximum principles again to obtain that the minimum of Q in $\overline{D} \times [0, T)$ is zero. Thus,

$$Q \geq 0, \quad \text{in } \overline{D} \times [0, T), \quad (50)$$

$$\frac{b'(u)}{e^u} u_t \geq \beta. \quad (51)$$

At the point $x^* \in \overline{D}$, where $u_0(x^*) = M_0$, we can integrate (51) from 0 to t to get

$$\int_0^t \frac{b'(u)}{e^u} u_t dt = \int_{M_0}^{u(x^*, t)} \frac{b'(s)}{e^s} ds \geq \beta t, \quad (52)$$

which implies that u must blow up in finite time. Actually, if u is a global solution of (1), then, for any $t > 0$, (52) shows

$$\int_{M_0}^{+\infty} \frac{b'(s)}{e^s} ds \geq \int_{M_0}^{u(x^*, t)} \frac{b'(s)}{e^s} ds \geq \beta t. \quad (53)$$

Letting $t \rightarrow +\infty$ in (53), we have

$$\int_{M_0}^{+\infty} \frac{b'(s)}{e^s} ds = +\infty, \quad (54)$$

which contradicts with assumption (40). This shows that u must blow up in finite time $t = T$. Moreover, letting $t \rightarrow T$ in (52), we have

$$T \leq \frac{1}{\beta} \int_{M_0}^{+\infty} \frac{b'(s)}{e^s} ds. \quad (55)$$

By integrating inequality (51) over $[t, s]$ ($0 < t < s < T$), for each fixed x , we obtain

$$\begin{aligned} G(u(x, t)) &\geq G(u(x, t)) - G(u(x, s)) \\ &= \int_{u(x, t)}^{+\infty} \frac{b'(s)}{e^s} ds - \int_{u(x, s)}^{+\infty} \frac{b'(s)}{e^s} ds \\ &= \int_{u(x, t)}^{u(x, s)} \frac{b'(s)}{e^s} ds = \int_t^s \frac{b'(u)}{e^u} u_t dt \geq \beta(s - t). \end{aligned} \quad (56)$$

Hence, by letting $s \rightarrow T$, we have

$$G(u(x, t)) \geq \beta(T - t). \quad (57)$$

Since G is a decreasing function, we obtain

$$u(x, t) \leq G^{-1}(\beta(T - t)). \quad (58)$$

The proof is completed. $\square$

4. Applications

When $h(t) \equiv 1$, $k(x) \equiv 1$, $f(x, u, q, t) = f(u)$ or $b(u) = u$, $h(t) \equiv 1$, $k(x) \equiv 1$, $f(x, u, q, t) = f(u)$, or $h(t) \equiv 1$, $k(x) \equiv 1$, $a(u) \equiv 1$, $f(x, u, q, t) = f(u)$, the conclusions of Theorems 1 and 2 still hold true. In this sense, our results extend and supplement the results in [14–16]. In what follows, we present several examples to demonstrate the applications of the abstract results.

Example 1. Let u be a solution of the following problem:

$$\begin{aligned} (ue^u)_t &= \nabla \cdot \left(\frac{1}{1+t} e^{|x|^2} (1+u) e^u \nabla u \right) \\ &\quad + \frac{1}{1+t} (e^{-u} + e^q) (e^{-t} + |x|^2), \quad \text{in } D \times (0, T), \\ \frac{\partial u}{\partial n} + 2u &= 0, \quad \text{on } \partial D \times (0, T), \\ u(x, 0) &= 2 - |x|^2, \quad \text{in } \overline{D}, \end{aligned} \quad (59)$$

where $q = |\nabla u|^2$, $D = \{x = (x_1, x_2, x_3) \mid |x|^2 < 1\}$ is the unit ball of $\mathbb{R}^3$. Now we have

$$\begin{aligned} b(u) &= ue^u, \quad h(t) = \frac{1}{1+t}, \\ k(x) &= e^{|x|^2}, \quad a(u) = (1+u)e^u, \quad \gamma = 2, \\ f(x, u, q, t) &= \frac{1}{1+t} (e^{-u} + e^q) (e^{-t} + |x|^2), \\ u_0(x) &= 2 - |x|^2. \end{aligned} \quad (60)$$

In order to determine the constant α , we assume

$$s = |x|^2, \quad (61)$$

and then $0 \leq s \leq 1$ and

$$\begin{aligned}\alpha &= \max_{\overline{D}} \frac{\nabla \cdot (h(0) k(x) a(u_0) \nabla u_0) + f(x, u_0, q_0, 0)}{e^{u_0}} \\ &= \max_{\overline{D}} \left\{ e^{|x|^2} (10|x|^2 - 18) + (1 + |x|^2) \right. \\ &\quad \left. \times (e^{-4+2|x|^2} + e^{-2+5|x|^2}) \right\} \\ &= \max_{0 \leq s \leq 1} \left\{ e^s (10s - 18) + (1 + s) \right. \\ &\quad \left. \times [e^{-4+2s} + e^{-2+5s}] \right\} = 18.6955.\end{aligned}\quad (62)$$

It is easy to check that (5)–(7) hold. By Theorem 1, u must be a global solution, and

$$\begin{aligned}u(x, t) &\leq H^{-1}(\alpha t + H(u_0(x))) \\ &= -1 + \sqrt{18.6955t + (1 + u_0(x))^2} \\ &= -1 + \sqrt{18.6955t + (3 - |x|^2)^2}.\end{aligned}\quad (63)$$

Example 2. Let u be a solution of the following problem:

$$\begin{aligned}(u + \ln u)_t &= \nabla \cdot \left((1+t) e^{-|x|^2} \left(1 + \frac{1}{u} \right) \nabla u \right) \\ &\quad + (1+t) (e^u - e^{-q}) (6 + t|x|^2), \\ &\quad \text{in } D \times (0, T),\end{aligned}\quad (64)$$

$$\frac{\partial u}{\partial n} + 2u = 0, \quad \text{on } \partial D \times (0, T),$$

$$u(x, 0) = 2 - |x|^2, \quad \text{in } \overline{D},$$

where $q = |\nabla u|^2$, $D = \{x = (x_1, x_2, x_3) \mid |x|^2 < 1\}$ is the unit ball of $\mathbb{R}^3$. Now we have

$$\begin{aligned}b(u) &= u + \ln u, & h(t) &= 1 + t, \\ k(x) &= e^{-|x|^2}, & a(u) &= 1 + \frac{1}{u}, & \gamma &= 2, \\ f(x, u, q, t) &= (1+t) (e^u - e^{-q}) (6 + t|x|^2), \\ u_0(x) &= 2 - |x|^2.\end{aligned}\quad (65)$$

In order to determine the constant β , we assume

$$s = |x|^2, \quad (66)$$

and then $0 \leq s \leq 1$ and

$$\begin{aligned}\beta &= \min_{\overline{D}} \frac{\nabla \cdot (h(0) k(x) a(u_0) \nabla u_0) + f(x, u_0, q_0, 0)}{e^{u_0}} \\ &= \min_{\overline{D}} \left\{ \frac{4|x|^6 - 26|x|^4 + 50|x|^2 - 36}{(2 - |x|^2)^2 e^2} \right. \\ &\quad \left. + 6(1 - e^{-3|x|^2-2}) \right\} \\ &= \min_{0 \leq s \leq 1} \left\{ \frac{4s^3 - 26s^2 + 50s - 36}{(2-s)^2 e^2} \right. \\ &\quad \left. + 6(1 - e^{-3s-2}) \right\} = 3.96997.\end{aligned}\quad (67)$$

It is easy to check that (38)–(40) hold. By Theorem 2, u must blow up in finite time T , and

$$\begin{aligned}T &\leq \frac{1}{\beta} \int_{M_0}^{+\infty} \frac{b'(s)}{e^s} ds \\ &= \frac{1}{3.96997} \int_2^{+\infty} \left(1 + \frac{1}{s} \right) \frac{1}{e^s} ds = 0.04641,\end{aligned}\quad (68)$$

$$u(x, t) \leq G^{-1}(\beta(T-t)) = G^{-1}(3.96997(T-t)),$$

where

$$G(z) = \int_z^{+\infty} \frac{b'(s)}{e^s} ds = \int_z^{+\infty} \left(1 + \frac{1}{s} \right) \frac{1}{e^s} ds, \quad z \geq 0, \quad (69)$$

and G^{-1} is the inverse function of G .

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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