Research Article

Lower Estimates for Certain Harmonic Functions in the Half Space

Gang Xu¹ and Xiaoyu Zhou²

¹ College of Computer and Information Engineering, Henan University of Economics and Law, Zhengzhou 450000, China

² College of Mathematics and Information Science, Henan University of Economics and Law, Zhengzhou 450000, China

Correspondence should be addressed to Gang Xu; 83041557@qq.com

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We will give the growth properties of harmonic functions of order greater than one in a half space, which generalize the result obtained by B. Levin in a half plane.

1. Introduction and Main Theorem

Let $\mathbf{R}$ and $\mathbf{R}_+$ be the sets of all real numbers and of all positive real numbers, respectively. Let $\mathbf{R}^n$ ($n \geq 3$) denote the n -dimensional Euclidean space with points $x = (x', x_n)$, where $x' = (x_1, x_2, \dots, x_{n-1}) \in \mathbf{R}^{n-1}$ and $x_n \in \mathbf{R}$. The boundary and closure of an open set D of $\mathbf{R}^n$ are denoted by ∂D and $\bar{D}$, respectively. The upper half space is the set $H = \{(x', x_n) \in \mathbf{R}^n : x_n > 0\}$, whose boundary is ∂H .

For a set E , $E \subset \mathbf{R}_+ \cup \{0\}$, we denote $\{x \in H : |x| \in E\}$ and $\{x \in \partial H : |x| \in E\}$ by HE and ∂HE , respectively. We identify $\mathbf{R}^n$ with $\mathbf{R}^{n-1} \times \mathbf{R}$ and $\mathbf{R}^{n-1}$ with $\mathbf{R}^{n-1} \times \{0\}$, writing typical points $x, y \in \mathbf{R}^n$ as $x = (x', x_n)$, $y = (y', y_n)$, where $y' = (y_1, y_2, \dots, y_{n-1}) \in \mathbf{R}^{n-1}$, and putting

$$x \cdot y = \sum_{j=1}^n x_j y_j = x' \cdot y' + x_n y_n,$$

$$|x| = \sqrt{x \cdot x}, \quad |x'| = \sqrt{x' \cdot x'}, \quad (1)$$

$$|x'| = |x| \cos \theta, \quad x_n = |x| \sin \theta \quad \left(0 < \theta \leq \frac{\pi}{2}\right).$$

Let B_r denote the open ball with center at the origin and radius r (> 0) in $\mathbf{R}^n$. We use the standard notations $u^+ = \max(u, 0)$ and $u^- = -\min(u, 0)$. In the sense of Lebesgue measure $dy' = dy_1 \cdots dy_{n-1}$ and $dy = dy' dy_n$. Let σ denote

$(n-1)$ -dimensional surface area measure and let $\partial/\partial n$ denote differentiation along the inward normal into H .

The estimate we deal with has a long history which can be traced back to Levin's estimate of harmonic functions from below (see, e.g., [1, page 209]).

Theorem A. Let A_1 be a constant and let, $u(z)$ be harmonic in the upper half space $\mathbf{C}_+$ and continuous on $\partial \mathbf{C}_+$. Suppose that

$$u(z) \leq A_1 R^\rho, \quad z \in \mathbf{C}_+, \quad R = |z| > 1, \quad \rho > 1, \quad (2)$$

$$|u(z)| \leq A_1, \quad |z| \leq 1, \quad \operatorname{Im} z \geq 0.$$

Then

$$u(\operatorname{Re}^{i\varphi}) \geq -A_2 A_1 (1 + R^\rho) \sin^{-1} \varphi, \quad \operatorname{Re}^{i\varphi} \in \mathbf{C}_+, \quad (3)$$

where A_2 is a constant independent of A_1 , R , φ , and the function $u(z)$.

Further versions and refinements of Theorem 1 may be found in [2, Chapter 1], [3, 4] and in the paper of Krasichkov-Ternovskii [5].

In this paper, we will consider functions $u(x)$ harmonic in H and continuous on $\bar{H}$. In what follows we shall denote by M various values which do not depend on K , R ($= |x|$), θ , and the function $u(x)$.

We prove in this note analogous estimates for $u(x)$ in H .

Theorem 1. Suppose that

$$u(x) \leq KR^{\rho(R)}, \quad x \in H, \quad R = |x| > 1, \quad \rho(R) > 1, \quad (4)$$

$$u(x) \geq -K, \quad |x| \leq 1, \quad x_n \geq 0. \quad (5)$$

Then

$$u(x) \geq -MK \left(1 + \rho(R) R^{\rho(R)}\right) \sin^{1-n}\theta, \quad (6)$$

where $x \in H$ and $\rho(R)$ is nondecreasing on $[1, +\infty)$.

Remark 2. If $n = 2$ and $\rho(R) \equiv \rho$, Theorem 1 is just the result of Theorem A.

Theorem 3. If (4) and (5) hold, then

$$u(x) \geq -MK \left(1 + \rho\left(\frac{N+1}{N}R\right) R^{\rho((N+1)/N R)}\right) \sin^{1-n}\theta, \quad (7)$$

where $x \in H$, $N (\geq 1)$ is a sufficiently large number, and $\rho(R)$ is defined in Theorem 1.

2. Main Lemmas

Carleman's formula [6] connects the modulus and the zeros of a function analytic in C_+ (see, e.g., [7, page 224]). Nevanlinna's formula (see [1, page 193]) refers to a harmonic function in a half disk. Ren obtained a generalized Nevanlinna-type formula in a half space and Poisson integral formula for half balls, respectively, which play important roles in our discussions.

Lemma 4 (see [8]). If $R > 1$, then one has

$$\begin{aligned} & \int_{\{x \in H: |x|=R\}} u(x) \frac{nx_n}{R^{n+1}} d\sigma(x) \\ & + \int_{\partial H(1,R)} u(x') \left(\frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx' = c_1 + \frac{c_2}{R^n}, \end{aligned} \quad (8)$$

where

$$\begin{aligned} c_1 &= \int_{\{x \in H: |x|=1\}} \left((n-1)x_n u(x) + x_n \frac{\partial u(x)}{\partial n} \right) d\sigma(x), \\ c_2 &= \int_{\{x \in H: |x|=1\}} \left(x_n u(x) - x_n \frac{\partial u(x)}{\partial n} \right) d\sigma(x). \end{aligned} \quad (9)$$

Lemma 5 (see [8]). Let $R > 1$ and let $u(x)$ be a function in $B_R^+ = B_R \cap H$ and continuous in $\bar{B}_R^+$. Then

$$\begin{aligned} u(x) &= \int_{\{y \in H: |y|=R\}} \frac{R^2 - |x|^2}{\omega_n R} \\ & \times \left(\frac{1}{|y-x|^n} - \frac{1}{|y-x^*|^n} \right) u(y) d\sigma(y) \\ & + \frac{2x_n}{\omega_n} \int_{\partial H(0,R)} \left(\frac{1}{|y'-x|^n} - \frac{R^n}{|x|^n} \frac{1}{|y'-\tilde{x}|^n} \right) \\ & \times u(y') dy', \end{aligned} \quad (10)$$

where $x \in B_R^+$, $\tilde{x} = R^2 x/|x|^2$, $x^* = (x', -x_n)$, and $\omega_n = \pi^{n/2} / \Gamma(1 + (n/2))$ is the volume of the unit n -ball in $\mathbf{R}^n$.

3. Proof of Theorem 1

By applying Lemma 4 to $u(x)$, we have

$$\begin{aligned} & \int_{\{x \in H: |x|=R\}} u^+(x) \frac{nx_n}{R^{n+1}} d\sigma(x) \\ & + \int_{\partial H(1,R)} u^+(x') \left(\frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx' \\ & = \int_{\{x \in H: |x|=R\}} u^-(x) \frac{nx_n}{R^{n+1}} d\sigma(x) \\ & + \int_{\partial H(1,R)} u^-(x') \left(\frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx' \\ & + c_1 + \frac{c_2}{R^n}. \end{aligned} \quad (11)$$

It immediately follows from (4) that

$$\begin{aligned} & \int_{\{x \in H: |x|=R\}} u^+(x) \frac{nx_n}{R^{n+1}} d\sigma(x) \leq MKR^{\rho(R)-1}, \\ & \int_{\partial H(1,R)} \int u^+(x') \left(\frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx' \leq MKR^{\rho(R)-1}. \end{aligned} \quad (12)$$

Hence from (11) and (12) we have

$$\int_{\{x \in H: |x|=R\}} u^-(x) \frac{nx_n}{R^{n+1}} d\sigma(x) \leq MKR^{\rho(R)-1}, \quad (13)$$

$$\int_{\partial H(1,R)} u^-(x') \left(\frac{1}{|x'|^n} - \frac{1}{R^n} \right) dx' \leq MKR^{\rho(R)-1}. \quad (14)$$

And (14) gives

$$\begin{aligned} & \int_{\partial H(1,R)} \frac{u^-(x')}{|x'|^n} dx' \\ & \leq \frac{2^n}{2^n - 1} \int_{\partial H(1,R)} u^-(x') \left(\frac{1}{|x'|^n} - \frac{1}{(2R)^n} \right) dx' \\ & \leq MK\rho(R) (2R)^{\rho(2R)-1}. \end{aligned} \quad (15)$$

Since $-u(x) \leq u^-(x)$, by applying Lemma 5 to $-u(x)$, we have

$$-u(x) \leq I_1(x) + I_2(x), \quad (16)$$

where

$$\begin{aligned} I_1(x) &= \int_{\{y \in H: |y|=R\}} \frac{R^2 - |x|^2}{\omega_n R} \\ &\quad \times \left(\frac{1}{|y-x|^n} - \frac{1}{|y-x^*|^n} \right) u^-(y) d\sigma(y), \\ I_2(x) &= \frac{2x_n}{\omega_n} \int_{\partial H[0,R]} \left(\frac{1}{|y'-x|^n} - \frac{R^n}{|x|^n} \frac{1}{|y'-\tilde{x}|^n} \right) \\ &\quad \times u^-(y') dy'. \end{aligned} \quad (17)$$

We remark that

$$\begin{aligned} \frac{1}{|y-x|^n} - \frac{1}{|y-x^*|^n} &\leq \frac{2nx_n y_n}{|y-x|^{n+2}}, \\ |y-x|^n &\geq x_n^n = |x|^n \sin^n \theta, \quad x \in H, \quad y_n = 0. \end{aligned} \quad (18)$$

If we put $|x| = r > 1/2$ and $R = 2r$ in (16), then we finally have from (13) and (18)

$$\begin{aligned} I_1(x) &\leq \int_{\{y \in H: |y|=R\}} \frac{R^2 - r^2}{\omega_n R} \frac{2nx_n y_n}{\omega_n |y-x|^{n+2}} u^-(y) d\sigma(y) \\ &\leq MK\rho(R) R^{\rho(R)}, \\ I_2(x) &\leq I_{21}(x) + I_{22}(x), \end{aligned} \quad (19)$$

where

$$\begin{aligned} I_{21}(x) &= \frac{2}{\omega_n x_n^{n-1}} \int_{\partial H(1,R)} u^-(y') dy', \\ I_{22}(x) &= \frac{2}{\omega_n x_n^{n-1}} \int_{\partial H[0,1]} u^-(y') dy'. \end{aligned} \quad (20)$$

We obtain that

$$\begin{aligned} I_{21}(x) &\leq \frac{2R^n}{\omega_n x_n^{n-1}} \int_{\partial H(1,R)} \frac{u^-(y')}{|y'|^n} dy' \\ &\leq MK\rho(R) R^{\rho(R)} \sin^{1-n} \theta, \\ I_{22}(x) &\leq \frac{2K}{\omega_n x_n^{n-1}} \int_{\partial H[0,1]} dy' \\ &\leq MK\rho(R) \sin^{1-n} \theta, \end{aligned} \quad (21)$$

from (15) and (5), respectively.

From (16), (19), and (21), we have for $|x| > 1/2$

$$-u(x) \leq MK\rho(R) \left(1 + \rho(R) R^{\rho(R)}\right) \sin^{1-n} \theta. \quad (22)$$

For $|x| \leq 1/2$, we have from (5)

$$-u(x) \leq K \leq K \left(1 + \rho(R) R^{\rho(R)}\right) \sin^{1-n} \theta. \quad (23)$$

Thus the conclusion immediately follows from (22) and (23).

4. Proof of Theorem 3

By modifying (15), we have

$$\begin{aligned} &\int_{\partial H(1,R)} \frac{u^-(x')}{|x'|^n} dx' \\ &\leq \frac{(N+1)^n}{(N+1)^n - N^n} \int_{\partial H(1,R)} u^-(x') \\ &\quad \times \left(\frac{1}{|x'|^n} - \frac{1}{((N+1)/N)R^n} \right) dx' \\ &\leq MK\rho \left(\frac{N+1}{N} R \right) \left(\frac{N+1}{N} R \right)^{\rho(((N+1)/N)R)-1}. \end{aligned} \quad (24)$$

Then (21), (22), and (23) are replaced accordingly by the following estimates:

$$\begin{aligned} I_{21}(x) &\leq MK\rho \left(\frac{N+1}{N} R \right) \left(\frac{N+1}{N} R \right)^{\rho(((N+1)/N)R)-1} \sin^{1-n} \theta, \\ -u(x) &\leq MK \left(1 + \rho \left(\frac{N+1}{N} R \right) R^{\rho(((N+1)/N)R)} \right) \sin^{1-n} \theta, \\ -u(x) &\leq K \leq MK \left(1 + \rho \left(\frac{N+1}{N} R \right) R^{\rho(((N+1)/N)R)} \right) \sin^{1-n} \theta. \end{aligned} \quad (25)$$

All (16), (19), (25), and (21) give

$$u(x) \geq -MK \left(1 + \rho \left(\frac{N+1}{N} R \right) R^{\rho(((N+1)/N)R)} \right) \sin^{1-n} \theta, \quad (26)$$

from which the conclusion immediately follows.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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