Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 250870, 7 pages
http://dx.doi.org/10.1155/2014/250870

Research Article

Periodic Solutions for Second Order Hamiltonian Systems with Impulses via the Local Linking Theorem

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Received 19 April 2014; Accepted 29 June 2014; Published 10 July 2014

Academic Editor: Leszek Gasinski

A class of second order impulsive Hamiltonian systems are considered. By applying a local linking theorem, we establish the new criterion to guarantee that this impulsive Hamiltonian system has at least one nontrivial T-periodic solution under local superquadratic condition. This result generalizes and improves some existing results in the known literature.

1. Introduction and Main Results

Consider the second order Hamiltonian systems with impulsive effects

$$\begin{gathered}\ddot{u}(t) - Au(t) + \nabla F(t, u(t)) = 0, \quad \text{a.e. } t \in [0, T], \\ \Delta\left(\dot{u}^i\left(t_j\right)\right) = I_{ij}\left(u^i\left(t_j\right)\right), \quad i = 1, 2, \ldots, N,\ j = 1, 2, \ldots, l, \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0,\end{gathered} \tag{1}$$

where $u(t) = (u^1(t), u^2(t), \ldots, u^N(t))$, $t_0 = 0 < t_1 < t_2 < \cdots < t_l < t_{l+1} = T$, $T > 0$, $\Delta(\dot{u}^i(t_j)) = \dot{u}^i(t_j^+) - \dot{u}^i(t_j^-)$, where $\dot{u}^i(t_j^+)$ and $\dot{u}^i(t_j^-)$ denote the right and left limits of $\dot{u}^i(t)$ at $t = t_j$, respectively, $I_{ij} : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2, \ldots, N,\ j = 1, 2, \ldots, l)$ are continuous, and $F \in C^1([0, T] \times \mathbb{R}^N, \mathbb{R})$, $\nabla F(t, u) = \partial F(t, u)/\partial u$. $A = [a_{ij}]$ is a symmetric constant matrix.

Impulsive differential equations serve as basic models to study the dynamics of processes that are subject to sudden changes in their states. The theory of impulsive differential systems has been developed by numerous mathematicians (see [1–4]). These kinds of processes naturally occur in control theory, biology, optimization theory, medicine, and so on (see [5–9]).

In recent years, many existence results are obtained for impulsive differential systems by critical point theory, such as [10–23] and their references. In most superquadratic cases, there is so-called Ambrosetti-Rabinowitz condition (see [18–23]):

$$0 < \mu F(t, u) \le (\nabla F(t, u), u), \quad \forall t \in \mathbb{R},\ u \in \mathbb{R}^N \setminus \{0\}, \tag{2}$$

where $\mu > 2$ is a constant, which implies that $F(t, u)$ is of superquadratic growth as $|u| \to \infty$; that is,

$$\lim_{|u| \to \infty} \frac{F(t, u)}{|u|^2} = +\infty, \quad \text{uniformly for all } t. \tag{3}$$

Moreover, Wu and Zhang [24] study the homoclinic solutions without any periodicity assumption under the local Ambrosetti-Rabinowitz type condition. Two key conditions of the main results of [24] are listed as follows.

(A1) There exist $\mu > 2$ and $L_1 > 0$ such that

$$\mu F(t, u) \le (\nabla F(t, u), u), \quad \forall t \in \mathbb{R},\ |u| \ge L_1. \tag{4}$$

(A2) There exists $2 < \alpha < +\infty$ such that $\liminf_{|u| \to +\infty}(F(t, u)/|u|^\alpha) > 0$, uniformly in $t \in \mathbb{R}$.

In recent paper [25], Zhang and Tang had obtained some results of the nontrivial T-periodic solutions under much weaker assumptions instead of (A1) and (A2).

(B1) There exist constants $\mu > 2$, $0 < \beta_0 < 2$, and $L_2 > 0$ and a function $a_0(t) \in L^1(0,T;\mathbb{R}^+)$ such that

$$\mu F(t,u) \le \nabla(F(t,u),u) + a_0(t)|u|^{\beta_0}, \quad \forall |u| \ge L_2, \quad \text{a.e. } t \in [0,T]. \tag{5}$$

(B2) There exists a subset E_0 of $[0,T]$ with $\text{meas}(E_0) > 0$ such that

$$\liminf_{|u|\to\infty} \frac{F(t,u)}{|u|^2} > 0, \quad \text{a.e. } t \in E_0. \tag{6}$$

Remark 1. Condition (B2) is weaker than (A2) because condition (A2) implies $\lim_{|u|\to\infty}(F(t,u)/|u|^2) = +\infty$.

Recently, applying the local linking theorem (see [26]), the works in [27–30] obtained the existence of periodic solutions or homoclinic solutions with (3) superquadratic condition under different systems. As shown in [25], condition (B2) is a local superquadratic condition; this situation has been considered only by a few authors.

Motivated by papers [24, 25, 31], in this paper, we aim to consider problem (1) under local superquadratic condition via a version of the local linking theorem (see [26]). In particular, the impulsive function I_{ij} satisfies a kind of new superquadratic condition which is different from that in the known literature. Our main results are the following theorems.

Theorem 2. *Suppose that $F \in C^1([0,T]\times\mathbb{R}^N,\mathbb{R})$ and $I_{ij} \in C(\mathbb{R},\mathbb{R})$, $i = 1,2,\ldots,N$, $j = 1,2,\ldots,l$, satisfies (B2) and consider the following.*

(H1) *There exists a positive constant θ such that*

$$(Au,u) \ge \theta|u|^2, \quad \forall u \in \mathbb{R}^N. \tag{7}$$

(H2) $\limsup_{|u|\to 0}(|\nabla F(t,u)|/|u|) = 0$ *uniformly for $t \in [0,T]$.*

(H3) *There exist constants $d > 1$, $c_1 > 0$, and $L_3 > 0$ such that, for every $t \in [0,T]$ and $u \in \mathbb{R}^N$ with $|u| \ge L_3$,*

$$|\nabla F(t,u)| \le c_1\left(|u|^d + 1\right). \tag{8}$$

(H4) *There exist constants $\mu > 2$, $L_4 > 0$, and $b_1 \in (0,(\mu/2-1)\theta)$ such that, for every $t \in [0,T]$ and $u \in \mathbb{R}^N$ with $|u| \ge L_4$,*

$$\mu F(t,u) \le (\nabla F(t,u),u) + b_1|u|^2. \tag{9}$$

(I1) *There exist constants $b_{ij} > 0$ and $r_{ij} \in (1,+\infty)$ such that*

$$\left|I_{ij}(u)\right| \le b_{ij}|u|^{r_{ij}}, \quad \forall u \in \mathbb{R}. \tag{10}$$

(I2) *There are two constants $b_2 > 0$ and $\gamma \in [0,2)$ such that*

$$I_{ij}(u)u \le \mu\int_0^u I_{ij}(s)\,ds + b_2|u|^\gamma, \quad \forall u \in \mathbb{R}. \tag{11}$$

(I3) *I_{ij} satisfies $\int_0^u I_{ij}(s)ds \le 0$, for all $u \in \mathbb{R}$.*

Then problem (1) *has at least one nontrivial T-periodic solution.*

Remark 3. Noting (3), obviously, conditions (B2) and (H4) are weaker than those of (2). From (B2), we only need $\lim_{|u|\to\infty}(F(t,u)/|u|^2) > 0$ to hold in a subset E_0 of $[0,T]$. What is more, F in (2) is asked to be positive globally. Here F need not be nonnegative globally; we also generalized Theorems 1.3 and 1.4 in [25]. For example, let

$$F(t,u) = \frac{1}{8}g(t)|u|^4 - 1, \quad \forall (t,u) \in [0,T]\times\mathbb{R}^N, \tag{12}$$

where

$$g(t) = \begin{cases} \sin\dfrac{2\pi t}{T}, & t \in \left[0,\dfrac{T}{2}\right], \\ 0, & t \in \left[\dfrac{T}{2},T\right]. \end{cases} \tag{13}$$

Let $E_0 = [T/8, T/4]$; then F satisfies our Theorem 2 but does not satisfy (2) and (3) and does not satisfy the corresponding conditions in [25].

Theorem 4. *Suppose that $F \in C^1([0,T]\times\mathbb{R}^N,\mathbb{R})$ and $I_{ij} \in C(\mathbb{R},\mathbb{R})$, $i = 1,2,\ldots,N$, $j = 1,2,\ldots,l$, satisfies (H1), (H2), (H3), (I1), (I2), and (I3) and the following condition holds.*

(H5) *There exist constants $\mu > 2$ and $b_3 \in (0,\min\{m_1(\mu-2),(\mu/2-1)\theta\})$ such that*

$$\mu F(t,u) \le (\nabla F(t,u),u) + b_3|u|^2, \quad \forall (t,u) \in [0,T]\times\mathbb{R}^N, \tag{14}$$

where

$$m_1 = \min\left\{F(t,u) \mid t \in [0,T], u \in \mathbb{R}^N, |u| = 1\right\}. \tag{15}$$

Then problem (1) *has at least one nontrivial T-periodic solution.*

Theorem 5. *Suppose that $F \in C^1([0,T]\times\mathbb{R}^N,\mathbb{R})$ and $I_{ij} \in C(\mathbb{R},\mathbb{R})$, $i = 1,2,\ldots,N$, $j = 1,2,\ldots,l$, satisfies (B2), (H1), (H2), (H3), (I1), (I2), and (I3) and the following condition holds.*

(H6) *There exist constants $\mu > 2$, $\beta \in [0,2)$, and $L_5 > 0$ and a function $a(t) \in L^1(0,T)$ and $a(t) \ge 0$ such that, for every $t \in [0,T]$ and $u \in \mathbb{R}^N$ with $|u| \ge L_5$,*

$$\mu F(t,u) \le \nabla(F(t,u),u) + a(t)|u|^\beta. \tag{16}$$

Then problem (1) *has at least one nontrivial T-periodic solution.*

The remaining of this paper is organized as follows. In Section 2, some fundamental facts are given. In Section 3, the main results of this paper are presented.

2. Preliminaries

Let X be a real Banach space with direct sum decomposition $X = X^1 \oplus X^2$. Consider two sequences of subspaces $X_0^1 \subset X_1^1 \subset \cdots \subset X^1$, $X_0^2 \subset X_1^2 \subset \cdots \subset X^2$ such that $X^j = \overline{\bigcup_{n\in N} X_n^j}$, $j = 1, 2$. For every multi-index $\alpha = (\alpha_1, \alpha_2) \in N^2$, let $X_\alpha = X_{\alpha_1} \oplus X_{\alpha_2}$; we define $\alpha \le \beta \Leftrightarrow \alpha_1 \le \beta_1,\ \alpha_2 \le \beta_2$. A sequence $\{\alpha_n\} \subset N^2$ is admissible if for every $\alpha \in N^2$ there is $m \in N$ such that $n \ge m \Rightarrow \alpha_n \ge \alpha$. For every $\varphi : X \to \mathbb{R}$, we define by φ_α the function φ restricted to X_α.

Definition 6 (see [26]). Let $\varphi \in C^1(X, \mathbb{R})$. The function φ satisfies the $(PS)^*$ condition if every sequence $\{u_{\alpha_n}\}$, such that $\{\alpha_n\}$ is admissible and

$$u_{\alpha_n} \in X_{\alpha_n}, \qquad \sup_n \varphi\left(u_{\alpha_n}\right) < \infty, \qquad \varphi'_{\alpha_n}\left(u_{\alpha_n}\right) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{17}$$

possesses a subsequence which converges to a critical point of φ.

Definition 7 (see [26]). Let X be a Banach space with direct sum decomposition $X = X^1 \oplus X^2$. The function $\varphi \in C^1(X, \mathbb{R})$ has local linking at 0 with respect to (X^1, X^2), if there exists $r > 0$ such that

$$\begin{aligned} \varphi(u) &\ge 0, \quad \forall u \in X^1 \text{ with } \|u\| \le r, \\ \varphi(u) &\le 0, \quad \forall u \in X^2 \text{ with } \|u\| \ge r. \end{aligned} \tag{18}$$

Theorem 8 (see [26]). *Suppose that $\varphi \in C^1(X, \mathbb{R})$ satisfies the following assumptions:*

(A1) *φ has local linking at 0 and $X^1 \ne \{0\}$,*

(A2) *φ satisfies $(PS)^*$ condition,*

(A3) *φ maps bounded sets into bounded sets,*

(A4) *for every $m \in N$ and $u \in X_m^1 \oplus X^2$, $\varphi(u) \to -\infty$ as $\|u\| \to \infty$.*

Then φ has at least three critical points.

Let us recall some basic notation. In the Sobolev space $X := H_0^1(0, T)$, consider the inner product

$$(u, v) = \int_0^T (u(t), v(t))\, dt + \int_0^T (\dot u(t), \dot v(t))\, dt, \tag{19}$$

for any $u, v \in X$. The corresponding norm is defined by

$$\|u\| = \left(\int_0^T |u(t)|^2 dt + \int_0^T |\dot u(t)|^2 dt\right)^{1/2}, \tag{20}$$

for any $u \in X$. Moreover, it is well known that X is compactly embedded in $C[0, T]$, which implies that

$$\|u\|_\infty \le C\|u\|, \tag{21}$$

for some constant $C > 0$, where $\|u\|_\infty = \max_{t\in[0,T]} |u(t)|$.

Define the corresponding functional φ on X by

$$\begin{aligned} \varphi(u) = {} & \frac{1}{2}\int_0^T |\dot u(t)|^2 dt + \frac{1}{2}\int_0^T (Au(t), u(t))\, dt \\ & + \sum_{j=1}^{l}\sum_{i=1}^{N} \int_0^{u^i(t_j)} I_{ij}(s)\, ds - \int_0^T F(t, u(t))\, dt. \end{aligned} \tag{22}$$

By the conditions of F and I_{ij}, $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, l$, we get that functional φ is a continuously Gáteaux differential functional whose Gáteaux derivative is the functional $\varphi'(u)$, given by

$$\begin{aligned} \left(\varphi'(u), v\right) = {} & \int_0^T (\dot u(t), \dot v(t))\, dt + \int_0^T (Au(t), v(t))\, dt \\ & + \sum_{j=1}^{l}\sum_{i=1}^{N} I_{ij}\left(u^i\left(t_j\right)\right) v^i\left(t_j\right) \\ & - \int_0^T (\nabla F(t, u(t)), v(t))\, dt. \end{aligned} \tag{23}$$

If $u \in H_0^1(0, T)$, then u is absolutely continuous and $\dot u \in L^2(0, T)$. In this case, $\Delta \dot u(t) = \dot u(t^+) - \dot u(t^-) = 0$ may not hold for some $t \in (0, T)$; this leads to impulsive effects.

Following the ideas of [11, 12], we can prove that the critical points of φ are the weak solutions of problem (1).

To prove our main results, we have the following facts (see [32]).

Letting

$$\phi(u) = \int_0^T \frac{1}{2}\left[|\dot u(t)|^2 + (Au(t), u(t))\right] dt, \tag{24}$$

we see that

$$\begin{aligned} \phi(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_0^T ((I - A)u(t), u(t))\, dt \\ &= \frac{1}{2}((I - K)u, u), \end{aligned} \tag{25}$$

where $K : H_T^1 \to H_T^1$ is the liner self-adjoint operator defined and I is the $N \times N$ identity matrix. By the Riesz representation theorem, we have

$$\int_0^T ((I - A)u(t), v(t))\, dt = (Ku, v). \tag{26}$$

The compact imbedding of H_T^1 into $C([0, T], R^N)$ implies that K is compact. Summing up the above discussion, $\varphi(u)$ can be rewritten as

$$\begin{aligned} \varphi(u) = {} & \frac{1}{2}((I - K)u, u) + \sum_{j=1}^{l}\sum_{i=1}^{N} \int_0^{u^i(t_j)} I_{ij}(s)\, ds \\ & + \int_0^T F(t, u(t))\, dt. \end{aligned} \tag{27}$$

By classical spectral theory, we can decompose X into the orthogonal sum of invariant subspace for $I-K$

$$X = H^- \oplus H^0 \oplus H^+, \tag{28}$$

where $H^0 = \ker(I-K)$ and H^- and H^+ are such that, for some $\delta > 0$,

$$\begin{aligned} ((I-K)u,u) &\le -\delta\|u\|^2, \quad \text{if } u \in H^-, \\ ((I-K)u,u) &\ge \delta\|u\|^2, \quad \text{if } u \in H^+. \end{aligned} \tag{29}$$

Notice that H^- is finite dimensional.

In this paper, we set $\overline{a} := \max_{1\le i,j\le N}|a_{ij}|$.

3. Proof of Main Results

3.1. The Proof of Theorem 2. Let $X^1 = H^+$ and $X^2 = H^0 \oplus H^-$; then $X = X^1 \oplus X^2$. Suppose $\{e_n\}_{n\ge1}$ is an orthogonal basis of H^+. Correspondingly, let

$$X_n^1 = \operatorname{span}\{e_1, e_2, \ldots, e_n\}, \qquad X_n^2 = X^2, \quad n \in N; \tag{30}$$

then $X^j = \overline{\bigcup_{n\in N} X_n^j}$, $j = 1, 2$. We divide our proof into four steps.

Step 1. φ has local linking at 0.

In view of (I1), we obtain

$$\left|\int_0^u I_{ij}(s)\,ds\right| \le \frac{b_{ij}}{r_{ij}+1}|u|^{r_{ij}+1}. \tag{31}$$

Combining this inequality, we have

$$\frac{\left|\int_0^u I_{ij}(s)\,ds\right|}{|u|^2} \le \frac{b_{ij}}{r_{ij}+1}|u|^{r_{ij}-1} \longrightarrow 0, \quad \text{as } u \longrightarrow 0. \tag{32}$$

Since $r_{ij} > 1$, this implies

$$\frac{\left|\int_0^u I_{ij}(s)\,ds\right|}{|u|^2} \longrightarrow 0, \quad \text{as } u \longrightarrow 0. \tag{33}$$

Applying (H2) and (33), for any $\varepsilon > 0$, there exists $r_1 > 0$ such that

$$\begin{aligned} |\nabla F(t,u)| \le 2\varepsilon|u|, \qquad \left|\int_0^u I_{ij}(s)\,ds\right| \le \varepsilon|u|^2, \\ \forall |u| \le r_1, \quad t \in [0,T], \end{aligned} \tag{34}$$

which implies that

$$\begin{aligned} |F(t,u)| &= \left|\int_0^1 (\nabla F(t,su),u)\,ds\right| \le \int_0^1 |\nabla F(t,su)|\,|u|\,ds \\ &\le \int_0^1 2s\varepsilon|u|^2 ds = \varepsilon|u|^2. \end{aligned} \tag{35}$$

On one hand, by (34) and (35), for all $u \in X^1 = H^+$ with $\|u\| \le r_2 := r_1/C$. Choose $\varepsilon = \delta/4(lN+T)C^2$; then one has

$$\begin{aligned} \varphi(u) &= \frac{1}{2}((I-K)u,u) + \sum_{j=1}^{l}\sum_{i=1}^{N}\int_0^{u^i(t_j)} I_{ij}(s)\,ds \\ &\quad - \int_0^T F(t,u(t))\,dt \\ &\ge \frac{\delta}{2}\|u\|^2 - \sum_{j=1}^{l}\sum_{i=1}^{N}\left(\varepsilon\left|u^i(t_j)\right|^2\right) - \varepsilon\int_0^T |u|^2 dt \\ &\ge \frac{\delta}{2}\|u\|^2 - lNC^2\varepsilon\|u\|^2 - TC^2\varepsilon\|u\|^2 \\ &= \frac{\delta}{4}\|u\|^2 \ge 0. \end{aligned} \tag{36}$$

On the other hand, since $\dim X^2$ is finite, there exists a constant $K_1 > 0$ such that

$$\|u\| \le K_1\|u^-\|, \quad \forall u \in X^2. \tag{37}$$

For all $u \in X^2 = H^- \oplus H^0$ with $\|u\| \le r_3 := r_1/C$. Choose $\varepsilon = \delta/4(lN+T)C^2K_1^2$; by (34), (35), and (37), we obtain

$$\begin{aligned} \varphi(u) &= \frac{1}{2}((I-K)u^-,u^-) + \sum_{j=1}^{l}\sum_{i=1}^{N}\int_0^{u^i(t_j)} I_{ij}(s)\,ds \\ &\quad - \int_0^T F(t,u(t))\,dt \\ &\le -\frac{\delta}{2}\|u^-\|^2 + \sum_{j=1}^{l}\sum_{i=1}^{N}\left(\varepsilon\left|u^i(t_j)\right|^2\right) + \varepsilon\int_0^T |u|^2 dt \\ &\le -\frac{\delta}{2}\|u^-\|^2 + (lN+T)C^2\varepsilon\|u\|^2 \\ &\le -\frac{\delta}{2}\|u^-\|^2 + (lN+T)C^2K_1^2\varepsilon\|u^-\|^2 \\ &= -\frac{\delta}{4}\|u^-\|^2 \le 0. \end{aligned} \tag{38}$$

Let $r = \min\{r_2, r_3\}$; one has

$$\begin{aligned} \varphi(u) &\ge 0, \quad u \in X^1, \ \|u\| \le r, \\ \varphi(u) &\le 0, \quad u \in X^2, \ \|u\| \ge r. \end{aligned} \tag{39}$$

Step 2. φ maps bounded sets into bounded sets.

By (H3) and $F \in C^1([0,T]\times\mathbb{R}^N,\mathbb{R})$, there exists $c_2 > 0$ such that

$$|\nabla F(t,u)| \le c_2 + c_1|u|^d, \qquad \forall (t,u) \in [0,T]\times\mathbb{R}^N, \tag{40}$$

which implies that

$$\begin{aligned} |F(t,u)| &\le \int_0^1 |\nabla F(t,su)|\,|u|\,ds \le \int_0^1 \left(c_2|u| + c_1|u|^{d+1}\right) ds \\ &= c_2|u| + c_1|u|^{d+1}, \quad \forall (t,u) \in [0,T]\times\mathbb{R}^N. \end{aligned} \tag{41}$$

Note that

$$\begin{aligned}
|\varphi(u)| &= \left|\frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt + \frac{1}{2}\int_0^T (Au(t), u(t))\, dt \right.\\
&\quad \left. + \sum_{j=1}^{l}\sum_{i=1}^{N}\int_0^{u^i(t_j)} I_{ij}(s)\, ds - \int_0^T F(t, u(t))\, dt\right| \\
&\le \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt + \frac{1}{2}N\overline{a}\int_0^T |u(t)|^2 dt \\
&\quad + \sum_{j=1}^{l}\sum_{i=1}^{N}\left(\frac{b_{ij}}{r_{ij}+1}\left|u^i(t_j)\right|^{r_{ij}+1}\right) \\
&\quad + c_2\int_0^T |u|\, dt + c_1\int_0^T |u|^{d+1} dt \\
&\le \left(\frac{1}{2}+\frac{1}{2}N\overline{a}\right)\|u\|^2 + \sum_{j=1}^{l}\sum_{i=1}^{N}\left(\frac{b_{ij}}{r_{ij}+1}C^{r_{ij}+1}\|u\|^{r_{ij}+1}\right) \\
&\quad + c_2 CT\|u\| + c_1 C^{d_1+1}T\|u\|^{d+1}.
\end{aligned} \tag{42}$$

It implies that φ maps bounded sets into bounded sets.

Step 3. φ satisfies the $(PS)^*$ condition.

Consider a $(PS)^*$ sequence $\{u_{\alpha_n}\}$ such that $\{\alpha_n\}$ is admissible. Then there exists a constant $M_1 > 0$ such that

$$\left|\varphi\left(u_{\alpha_n}\right)\right| \le M_1, \qquad \left\|\varphi'_{\alpha_n}\left(u_{\alpha_n}\right)\right\| \le M_1. \tag{43}$$

By (41), for $|u| \le L_4$, one has

$$|F(t,u)| \le c_2|u| + c_1|u|^{d+1} \le c_2 L_4 + c_1 {L_4}^{d+1}; \tag{44}$$

together with (H4), one has

$$\mu F(t,u) \le (\nabla F(t,u), u) + b_1|u|^2 + \mu\left(c_2 L_4 + c_1 {L_4}^{d+1}\right), \tag{45}$$

for all $(t,u) \in [0,T]\times\mathbb{R}^N$.

It follows from (21), (43), (45), (H1), and (I2) that

$$\begin{aligned}
&M_1 + \frac{M_1}{\mu}\left\|u_{\alpha_n}\right\| \\
&\quad \ge \varphi\left(u_{\alpha_n}\right) - \frac{1}{\mu}\left(\varphi'\left(u_{\alpha_n}\right), u_{\alpha_n}\right) \\
&\quad = \left(\frac{1}{2}-\frac{1}{\mu}\right)\int_0^T \left|\dot{u}_{\alpha_n}\right|^2 dt + \left(\frac{1}{2}-\frac{1}{\mu}\right)\int_0^T \left(Au_{\alpha_n}, u_{\alpha_n}\right) dt \\
&\qquad + \frac{1}{\mu}\int_0^T \left(\left(\nabla F\left(t, u_{\alpha_n}\right), u_{\alpha_n}\right) - \mu F\left(t, u_{\alpha_n}\right)\right) dt \\
&\qquad + \frac{1}{\mu}\sum_{j=1}^{l}\sum_{i=1}^{N}\left(\mu\int_0^{u^i_{\alpha_n}(t_j)} I_{ij}(s)\, ds - I_{ij}\left(u^i_{\alpha_n}\left(t_j\right)\right)u^i_{\alpha_n}\left(t_j\right)\right) \\
&\quad \ge \left(\frac{1}{2}-\frac{1}{\mu}\right)\int_0^T \left|\dot{u}_{\alpha_n}\right|^2 dt + \left(\frac{\theta}{2}-\frac{\theta}{\mu}-\frac{b_1}{\mu}\right)\int_0^T \left|u_{\alpha_n}\right|^2 dt \\
&\qquad - \mu T\left(c_2 L_4 + c_1 {L_4}^{d+1}\right) - \frac{b_2}{\mu}\sum_{j=1}^{l}\sum_{i=1}^{N}\left\|u_{\alpha_n}\right\|_\infty^\gamma \\
&\quad \ge \min\left\{\left(\frac{1}{2}-\frac{1}{\mu}\right), \left(\frac{1}{2}-\frac{1}{\mu}\right)\theta - \frac{b_1}{\mu}\right\}\left\|u_{\alpha_n}\right\|^2 \\
&\qquad - \frac{b_2 lN}{\mu}C^\gamma\left\|u_{\alpha_n}\right\|^\gamma - \mu T\left(c_2 L_4 + c_1 {L_4}^{d+1}\right).
\end{aligned} \tag{46}$$

Since $\mu > 2$, $\gamma \in [0,2)$, and $b_1 \in (0, (\mu/2-1)\theta)$, (46) shows that $\{u_{\alpha_n}\}$ is bounded in H_T^1. By a method similar to that of [10], we can prove that $\{u_{\alpha_n}\}$ has a convergent subsequence. Thus φ satisfies $(PS)^*$ condition.

Step 4. For every $m \in N$ and $u \in X_m^1 \oplus X^2$, $\varphi(u) \to -\infty$ as $\|u\| \to \infty$.

Since $\dim(X_m^1 \oplus X^2)$ is finite, there exists $M_2 > 0$ such that

$$\|u\| \le M_2\left(\int_{E_0}|u|^2 dt\right)^{1/2}, \quad \forall u \in X_m^1 \oplus X^2. \tag{47}$$

In fact, for $M_3 = M_2^2(\max\{1/2, (1/2)N\overline{a}\} + \delta/2) > 0$, by (B2), there exists $M_4 > 0$ such that

$$F(t,u) \ge M_3|u|^2 - M_4, \quad \forall u \in \mathbb{R}^N, \text{ a.e. } t \in E_0. \tag{48}$$

Hence, it follows from (47), (48), and (I3) that

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}((I-K)u, u) + \sum_{j=1}^{l}\sum_{i=1}^{N}\int_0^{u^i(t_j)} I_{ij}(s)\, ds \\
&\quad - \int_0^T F(t,u)\, dt \\
&= \frac{1}{2}\left((I-K)u^-, u^-\right) + \frac{1}{2}\left((I-K)u^+, u^+\right) \\
&\quad + \sum_{j=1}^{l}\sum_{i=1}^{N}\int_0^{u^i(t_j)} I_{ij}(s)\, ds - \int_0^T F(t,u)\, dt \\
&\le -\frac{\delta}{2}\left\|u^-\right\|^2 + \frac{1}{2}\int_0^T \left|\dot{u}^+\right|^2 dt + \frac{1}{2}\int_0^T \left(Au^+, u^+\right) dt \\
&\quad - \int_{E_0} F(t,u)\, dt \\
&\le -\frac{\delta}{2}\left\|u^-\right\|^2 + \max\left\{\frac{1}{2}, \frac{1}{2}N\overline{a}\right\}\left\|u^+\right\|^2 \\
&\quad - M_3\int_{E_0}|u|^2 dt + M_4 T
\end{aligned}$$

$$\begin{aligned}
&\le -\frac{\delta}{2}\|u^-\|^2 + \max\left\{\frac{1}{2}, \frac{1}{2}N\overline{a}\right\}\|u^+\|^2 \\
&\quad -\left(\max\left\{\frac{1}{2}, \frac{1}{2}N\overline{a}\right\} + \frac{\delta}{2}\right)\|u\|^2 + M_4T \\
&\le -\frac{\delta}{2}\|u^-\|^2 + \max\left\{\frac{1}{2}, \frac{1}{2}N\overline{a}\right\}\|u^+\|^2 \\
&\quad -\left(\max\left\{\frac{1}{2}, \frac{1}{2}N\overline{a}\right\} + \frac{\delta}{2}\right)\left(\|u^+\|^2 + \|u^0\|^2\right) + M_4T \\
&\le -\frac{\delta}{2}\|u\|^2 + M_4T,
\end{aligned} \tag{49}$$

for $u \in X_m^1 \oplus X^2$ and a.e. $t \in E_0$, which implies that

$$\varphi(u) \longrightarrow -\infty, \quad \text{as } \|u\| \longrightarrow \infty \text{ on } u \in X_m^1 \oplus X^2. \tag{50}$$

Therefore, all the assumptions of Theorem 8 are verified. Then, the proof of Theorem 2 is completed.

3.2. The Proof of Theorem 4. Following the same procedures in the proof of Theorem 2, we can prove that φ satisfies (A1), (A2), and (A3) in Theorem 8.

To prove (A4), set $\phi(\xi) = F(t,\xi u)\xi^{-\mu} + (b_3/(\mu-2))(1-\xi^{2-\mu})|u|^2$, $\xi \in (0,+\infty)$; then by (H5), we have

$$\begin{aligned}
\phi'(\xi) &= (\nabla F(t,\xi u), u)\xi^{-\mu} - \mu\xi^{-\mu-1}F(t,\xi u) + b_3\xi^{1-\mu}|u|^2 \\
&= \left((\nabla F(t,\xi u), \xi u) - \mu F(t,\xi u)\right)\xi^{-\mu-1} + b_3\xi^{1-\mu}|u|^2 \\
&\ge 0.
\end{aligned} \tag{51}$$

When $0 < \xi \le 1$, it follows from (51) that

$$\begin{aligned}
\phi(1) = F(t,u) \ge \phi(\xi) &= F(t,\xi u)\xi^{-\mu} \\
&\quad + \frac{b_3}{\mu-2}\left(1-\xi^{2-\mu}\right)|u|^2;
\end{aligned} \tag{52}$$

this implies

$$F(t,u) \ge \left(F\left(t, \frac{u}{|u|}\right) - \frac{b_3}{\mu-2}\right)|u|^\mu + \frac{b_3}{\mu-2}|u|^2, \quad \text{if } |u| \ge 1. \tag{53}$$

Set $m_2 = \max\{F(t,u) \mid t \in [0,T],\ u \in \mathbb{R}^N,\ |u| \le 1\}$; then by (53), we have

$$F(t,u) \ge \left(m_1 - \frac{b_3}{\mu-2}\right)|u|^\mu + \frac{b_3}{\mu-2}|u|^2 - m_2, \tag{54}$$

$\forall (t,u) \in [0,T]\times\mathbb{R}^N$; since $\dim(X_m^1 \oplus X^2)$ is finite, there exists $M_5 > 0$ such that

$$\|u\|^\mu \le M_5\int_0^T |u|^\mu dt, \quad \forall u \in X_m^1 \oplus X^2. \tag{55}$$

By (54), (55), and (I3), we have

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_0^T |\dot{u}|^2 dt + \frac{1}{2}\int_0^T (Au,u)\,dt \\
&\quad + \sum_{j=1}^{l}\sum_{i=1}^{N}\int_0^{u^i(t_j)} I_{ij}(s)\,ds - \int_0^T F(t,u)\,dt \\
&\le \frac{1}{2}\int_0^T |\dot{u}|^2 dt + \frac{1}{2}N\overline{a}\int_0^T |u|^2 dt \\
&\quad - \left(m_1 - \frac{b_3}{\mu-2}\right)\int_0^T |u|^\mu dt + \frac{b_3}{\mu-2}\int_0^T |u|^2 dt - m_2T \\
&\le \max\left\{\frac{1}{2}, \frac{1}{2}N\overline{a}\right\}\|u\|^2 - \frac{1}{M_5}\left(m_1 - \frac{b_3}{\mu-2}\right)\|u\|^\mu \\
&\quad + \frac{b_3}{M_5(\mu-2)}\|u\|^2 - m_2T.
\end{aligned} \tag{56}$$

Since $\mu > 2$ and $m_1 - (b_3/(\mu-2)) > 0$, (56) implies

$$\varphi(u) \longrightarrow -\infty, \quad \text{as } \|u\| \longrightarrow \infty \text{ on } u \in X_m^1 \oplus X^2. \tag{57}$$

Consequently, the conclusion follows from Theorem 8. This completes the proof.

3.3. The Proof of Theorem 5. Similar to the proof of Theorem 2, φ satisfies all conditions of Theorem 8. Thus, problem (1) has at least one nontrivial T-periodic solution.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This work is supported by the National Natural Science Foundation of China (no. 11271371).

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