Research Article

Approximation by $q$-Bernstein Polynomials in the Case $q \to 1^+$

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Abstract and Applied Analysis

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Let $q > 0$. For any nonnegative integer $k$, the $q$-integer $[k]_q$ is defined by

$$[k]_q := 1 + q + \cdots + q^{k-1}, \quad (k = 1, 2, \ldots), \quad [0]_q := 0,$$

and the $q$-factorial $[k]_q!$ by

$$[k]_q! := [1]_q[2]_q \cdots [k]_q, \quad (k = 1, 2, \ldots), \quad [0]_q! := 1.$$  (2)

For integers $k, n$ with $0 \leq k \leq n$, the $q$-binomial coefficient is defined by

$$\binom{n}{k}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$  (3)

In [1], Phillips proposed the $q$-Bernstein polynomials: for each positive integer $n$ and $f \in C[0, 1]$, the $q$-Bernstein polynomial of $f$ is

$$B_{nq}(f; x) := \sum_{k=0}^{n} f(\frac{k}{n}) \binom{n}{k}_q p_{nk}(q; x),$$  (4)

where

$$p_{nk}(q; x) = \binom{n}{k}_q x^k \prod_{s=0}^{n-k-1} (1 - q^s x).$$  (5)

Note that, for $q = 1$, $B_{nq}(f; x)$ is the classical Bernstein polynomial $B_n(f; x)$:

$$B_n(f; x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}.$$  (6)

In recent years, the $q$-Bernstein polynomials have been investigated intensively and a great number of interesting results related to the $q$-Bernstein polynomials have been obtained. Reviews of the results on $q$-Bernstein polynomials are given in [2, Chapter 7] and [3, 4].

The $q$-Bernstein polynomials inherit some of the properties of the classical Bernstein polynomials, for example, the end-point interpolation property and the shape-preserving properties in the case $0 < q < 1$, representation via divided differences. We can also define the generalized Bézier curve and de Casteljau algorithm, which can be used for evaluating $q$-Bernstein polynomials iteratively. These properties stipulate the importance of $q$-Bernstein polynomials for the computer-aided geometric design. Like the classical Bernstein polynomials, the $q$-Bernstein polynomials reproduce linear functions and are degree reducing on the set of polynomials. Apart from that, the basic $q$-Bernstein polynomials $p_{nk}(q; x)$ admit a probabilistic interpretation via the stochastic process and the $q$-binomial distribution in the case $0 < q < 1$; see [5].
On the other hand, when passing from $q = 1$ to $q \neq 1$ convergence properties of the $q$-Bernstein polynomials dramatically change. More specially, in the case $0 < q < 1$, $B_{n,q}$ are positive linear operators on $C[0,1]$, and the convergence properties of the $q$-Bernstein polynomials have been investigated intensively (see, e.g., [6–11]). In the case $q > 1$, $B_{n,q}$ are not positive linear operators on $C[0,1]$, and the lack of positivity makes the investigation of convergence in the case $q > 1$ essentially more difficult. There are many unexpected results concerning convergence of $q$-Bernstein polynomials in the case $q > 1$ (see [2, 12–17]). For example, in [2, 15], strong asymptotic estimates for the norm $\|B_{n,q}\|$ as $n \to \infty$ for fixed $q > 1$ and as $q \to \infty$ are obtained. It was shown in [2] that $\|B_{n,q}\| \to +\infty$ faster than any geometric progression $n \to \infty$ for fixed $q > 1$. This fact provides an explanation for the unpredictable behavior of $q$-Bernstein polynomials $(q > 1)$ with respect to convergence.

This paper is devoted to studying approximation properties of $q$-Bernstein polynomials for $q$ taking varying values that tend to 1. We note that, from the very first papers (see [1]), there was interest in such approximation properties. In the case $0 < q_n < 1$, many interesting results including the convergence, the rate of convergence, Voronovskaya-type theorems, and the direct and converse theorem are obtained (see [1, 6, 8–11]). It was shown in [1, 8] that, in the case $q_n \leq 1$, the condition $q_n \to 1$ is necessary and sufficient for the sequence $(B_{n,q_n}(f))$ to be approximating for any $f \in C[0,1]$.

Naturally, the question arises as to whether the sequence $(B_{n,q_n}(f))$ to be approximating for any $f \in C[0,1]$ as $q_n$ tends to 1 from above. It turns out that, in general, the answer is negative. Indeed, Ostrovskaya showed in [13] that if $q_n - 1 \not\to 0$ slower than $(\ln n)/n$, then the sequence $(B_{n,q_n}(f))$ may not be approximating for some $f \in C[0,1]$ (e.g., $f(x) = \sqrt{x}$). However, in [14] Ostrovskaya showed that if $q_n \to 1$ fast enough, the sequence $(B_{n,q_n}(f))$ is approximating for any $f \in C[0,1]$: a sufficient condition is $q_n = 1 + o(n^{-1/3})$.

In this paper, we continue to study the convergence of the sequence $(B_{n,q_n}(f))$ as $q_n$ tends to 1 from above. Clearly, the convergence of the sequence $(B_{n,q_n})$ depends heavily on the norm $\|B_{n,q_n}\|$. We remark that for $\|B_{n,q_n}\| = 1$ for all $0 < q_n < 1$. In contrast to this, $\|B_{n,q_n}\|$ vary with $q_n > 1$. By the delicate analysis of $\|B_{n,q_n}\|$, we obtain the sufficient and necessary condition under which $(B_{n,q_n}(f; \cdot))$ $(q_n > 1)$ approximates $f$ for any $f \in C[0,1]$. Based on this condition we get that if $(B_{n,q_n}(f; \cdot))$ can approximate $f$ for any $f \in C[0,1]$, then the sequence $(q_n)$ satisfies $\lim_{n \to \infty} n(q_n - 1) \leq \ln 2$. On the other hand, if $1 < q_n \leq 1 + \ln 2/n$ for sufficient large $n$, then $(B_{n,q_n}(f; \cdot))$ approximates $f$ for any $f \in C[0,1]$.

2. Statement of Results

From here on we assume that $q_n > 1$. The following theorem gives the sufficient and necessary condition for convergence of the sequence $(B_{n,q_n}(f))$ for any $f \in C[0,1]$.

**Theorem 1.** Let $q_n > 1$. Then the sequence $(B_{n,q_n}(f))$ converges to $f$ in $C[0,1]$ if and only if

$$\sup_{n \in \mathbb{N}} \left\{ \frac{\max_{x \in [q_n^{-1}, q_n^1]} |f(x)|}{n|\ln n|} \right\} < 1. \tag{7}$$

Based on Theorem 1, we obtain the following necessary condition for convergence of the sequence $(B_{n,q_n}(f))$.

Indeed, we show that if $\lim_{n \to \infty} n(q_n - 1) > \ln 2$, then

$$\sup_{n \in \mathbb{N}} \left\{ \frac{\max_{x \in [q_n^{-1}, q_n^1]} |f(x)|}{n|\ln n|} \right\} = \infty$$

with $q_n = (1 + q_n)/2q_n$.

**Theorem 2.** Let $q_n > 1$. If the sequence $(B_{n,q_n}(f))$ converges to $f$ in $C[0,1]$, for any $f \in C[0,1]$, then

$$\lim_{n \to \infty} n(q_n - 1) \leq \ln 2. \tag{8}$$

Finally, we give the sufficient condition for convergence of the sequence $(B_{n,q_n}(f))$.

**Theorem 3.** Let $q_n > 1$. If the sequence $(q_n)$ satisfies $q_n \leq 1 + \ln 2/n$ for sufficiently large $n$, then, for any $f \in C[0,1]$, $(B_{n,q_n}(f; x))$ converges to $f(x)$ uniformly on $[0,1]$.

The following corollary follows immediately for Theorem 3.

**Corollary 4.** Let $q_n > 1$. If the sequence $(q_n)$ satisfies

$$\lim_{n \to \infty} n(q_n - 1) < \ln 2,$$

then, for any $f \in C[0,1]$, $(B_{n,q_n}(f; x))$ converges to $f(x)$ uniformly on $[0,1]$.

**Remark 5.** Using the same technique as in the proof of Theorem 3, we can prove a slightly stronger conclusion: if

$$1 < q_n \leq 1 + \frac{\ln 2}{n} + \frac{C}{n^r} \tag{10}$$

for some positive constant $C$ and sufficiently large $n$, then, for any $f \in C[0,1]$, $(B_{n,q_n}(f; x))$ converges to $f(x)$ uniformly on $[0,1]$.

3. Proofs of Theorems 1–3

For $f \in C[0,1]$, we set

$$\|f\| = \max_{x \in [0,1]} |f(x)|,$$

$$\|f\|_\infty := \max_{x \in [q_n^{-1}, q_n^1]} |f(x)|. \tag{11}$$

Let $F_n(x) := \sum_{k=0}^{n} p_{n,k}(q_n; x), x \in [0,1]$. Clearly,

$$\|B_{n,q_n}\| = \|F_n\| = \max_{x \in [0,1]} \left( \sum_{k=0}^{n} |p_{n,n-k}(q_n; x)| \right). \tag{12}$$
Note that $\sum_{k=0}^{n} p_{nk}(q_n^s; x) = 1$ for $x \in [0, 1]$ and $p_{nk}(q_n^s; x) \geq 0$ for $x \in [q_n^{s-1}, q_n^{-1}]$ and $k = 0, 1, \ldots, n$. This means that

$$F_n(x) = 1, \quad x \in [0, q_n^{-1}],$$

$$F_n(x) \geq 1, \quad x \in [q_n^{s-1}, 1].$$

(13)

It follows that

$$\|B_{n, q_n}\| = \|F_n\| = \max_{0 \leq s \leq n-2} \|F_n\|.$$  

(14)

**Proof of Theorem 1.** From Corollary 7 in [12] we know that, for any polynomial $P(x)$, we have

$$B_{n, q_n}(P; x) \to P(x)$$

(15)

uniformly in $[0, 1]$ as $n \to \infty$. It follows from the well-known Ba fish - Steinhaus theorem that $(B_{n, q_n}(f))$ $(q_n > 1)$ approximates $f$ for any $f \in C[0, 1]$ if and only if

$$\sup_{n \in \mathbb{N}} \|B_{n, q_n}\| = \sup_{n \in \mathbb{N}} \sup_{x \in [0, 1]} \left( \sum_{k=0}^{n} \|p_{n-k}(q_n^s; x)\| \right) < +\infty.$$  

(16)

We set

$$G_{s, n}(x) = \sum_{k=s+1}^{n} |p_{n-k}(q_n^s; x)|, \quad s = 0, 1, \ldots, n-2.$$  

(17)

Since $p_{n-k}(q_n^s; x) \geq 0$ for $x \in [q_n^{s-1}, q_n^{-1}]$ and $k = 0, 1, \ldots, s+1$, we get, for $x \in [q_n^{s-1}, q_n^{-1}]$,

$$\sum_{k=0}^{s+1} |p_{n-k}(q_n^s; x)| = \sum_{k=0}^{s+1} p_{n-k}(q_n^s; x)$$

$$= 1 - \sum_{k=s+2}^{n} p_{n-k}(q_n^s; x) \leq 1 + G_{s, n}(x),$$

and, therefore,

$$\|F_n\| \leq \|1 + 2G_{s, n}\| = 1 + 2\|G_{s, n}\|, \quad s = 0, 1, \ldots, n-2.$$  

(19)

Next we will show that

$$\|G_{s, n}\|_\infty \leq \|G_{s-1, n}\|_{s-1}, \quad s = 1, 2, \ldots, n-2.$$  

(20)

Note that, for $x \in [q_n^{s-1}, q_n^{-1}]$,

$$G_{s, n}(x) = \sum_{k=s+1}^{n} |p_{n-k}(q_n^s; x)|,$$

$$G_{s-1, n}(q_n^s) = \sum_{k=s+1}^{n} |p_{n-k}(q_n^s; q_n^s)|.$$  

(21)

If we show that, for $x \in [q_n^{s-1}, q_n^{-1}]$ and $k = s + 1, \ldots, n-1$,

$$|p_{n-k}(q_n^s; x)| \leq |p_{n-k}(q_n^s; q_n^s)|,$$

then

$$G_{s, n}(x) \leq G_{s-1, n}(q_n^s), \quad x \in [q_n^{s-1}, q_n^{-1}],$$

and (20) follows. Indeed, for $x \in [q_n^{s-1}, q_n^{-1}]$ and $k = s + 1, \ldots, n-1$,

$$|p_{n-k}(q_n^s; q_n^s)| \leq \frac{(n-1) \min \left\{ \frac{(q_n - 1)}{2}, \ln 2 \right\}}{n-k}.$$  

(32)
We have
\[ \|G_{0,n}\|_0 \geq \|p_{n-2}(q_n;\cdot)\|_0 \geq \|p_{n-2}(q_n; \frac{q_n + 1}{2q_n})\|_0 \]
\[ = \frac{(q_n^{n-1} - 1)}{(q_n^{n-1} - 1)} \left( \frac{1 + q_n}{2q_n} \right)^{n-2} \]
\[ \times \left( 1 - \frac{1 + q_n}{2q_n} \right) \left( \frac{q_n + 1}{2q_n} - 1 \right) \]
\[ = \frac{1 - q_n^{-n+1}}{8} \left( \frac{q_n^{n-1} - 1}{2} \right)^{n-3} \]
\[ \geq \frac{1 - q_n^{-n+1}}{8} (q_n^{n-1} - 1) \rightarrow +\infty, \quad (n \rightarrow \infty). \]  
(33)

This leads to a contradiction by Theorem 1. Hence, (30) holds.

Next, we show Theorem 2. Assume that \( \lim_{n \to -\infty} r(n(q_n - 1)) > \ln 2 \). Then by (30) we may suppose that, for some \( A, B, \)
\( \ln 2 < A < B < +\infty, \)
\[ 1 + \frac{A}{n} \leq q_n \leq 1 + \frac{B}{n}, \]
(34)

For \( 0 < a < b, \) we set \( h(x) = (x^a - 1)/(x^b - 1), \) \( x > 1. \)

Direct computation gives that
\[ h'(x) = \frac{bx^{a-1}((b-a)/b)x^b - a/b}{(x^b - 1)^2}. \]  
(35)

Since the function \( g(y) = x^y \) is convex on \( (-\infty, +\infty) \) for a
fixed \( x > 0, \) we get that
\[ x^{b-a} = x^y \leq \frac{b-a}{b} x^b - \frac{a}{b}. \]  
(36)

This means that \( h'(x) \leq 0 \) and \( h(x) \) is nonincreasing on
\( (1, +\infty). \) Hence, for \( x \in (1, x_0), \) \( x_0 > 1, \) we have
\[ h(x_0) \leq h(x) \leq \lim_{x \to +\infty} h(x) = \frac{a}{b}. \]  
(37)

Put \( x_0 = (1 + q_n)/2q_n \in (q_n^{-1} - 1). \) Then, for \( k_0 = [\ln n], \) we have
\[ \|G_{0,n}\|_0 \geq \|p_{n-k_0}(q_n;\cdot)\|_0 \geq \|p_{n-k_0}(q_n; x_0)\|_0 \]
\[ = \frac{(q_n^{n-1} - 1) \cdots (q_n^{n-k_0+1} - 1)}{(q_n^{n-1} - 1) \cdots (q_n^{n-k_0} - 1)} (x_0^{-k_0}) \]
\[ \times (1 - x_0) \prod_{s=1}^{k_0-1} (q_n^{s} - 1) \]
\[ \geq (q_n^{n-k_0} - 1)^{k_0} x_0^{n-k_0} (1 - x_0) \]
\[ \times \frac{(q_n^{k_0} x_0 - 1) \cdots (q_n x_0 - 1)}{(q_n^{k_0} - 1) \cdots (q_n - 1)}. \]  
(38)

Using (34), the inequalities
\[ \frac{q_n^{s+1} x_0 - 1}{q_n^s - 1} \geq 1, \quad s = 1, \ldots, k_0, \]
\[ x_0^{n-k_0} (1 - x_0) (q_n x_0 - 1) \geq q_n^{n+k_0-1} \frac{(q_n - 1)^2}{4} \]
\[ \geq (1 + \frac{B}{n})^n \frac{(q_n - 1)^2}{4} \]
\[ \geq (q_n - 1)^2 \exp(-B), \]
(39)

and the nonincreasing property of \( h(x), \) we continue to obtain that
\[ \|G_{0,n}\|_0 \]
\[ \geq \left( (1 + \frac{A}{n})^{n\ln n} - 1 \right)^{k_0} \frac{\exp(-B)}{4} \left( \frac{q_n - 1}{q_n^{k_0} - 1} \right) \]
\[ \geq \left( (1 + \frac{A}{n})^{n\ln n} - 1 \right)^{k_0} \frac{\exp(-B)}{4} \left( \frac{A/n}{(1 + B/n)^{k_0} - 1} \right) \]
\[ \exp(-B) \frac{(A/n)^2}{(1 + B/n)^{k_0} - 1}. \]  
(40)

We observe that
\[ \lim_{n \to -\infty} \left( 1 + \frac{A}{n} \right)^{n\ln n} \]
\[ = \exp \left( \lim_{n \to -\infty} (n - \ln n) \ln \left( 1 + \frac{A}{n} \right) \right) \]
\[ = \exp \left( \lim_{n \to -\infty} \frac{A(n - \ln n)}{n} \right) = \exp(A) > 2, \]
(41)

and, for \( s = k_0, k_0 - 1, \)
\[ \lim_{n \to -\infty} \frac{(1 + B/n)^s - 1}{B \ln n/n} \]
\[ = \lim_{n \to -\infty} \frac{\exp(s \ln(1 + B/n)) - 1}{B \ln n/n} = \lim_{n \to -\infty} \frac{s \ln(1 + B/n)}{B \ln n/n} = 1. \]  
(42)

Thus, for some \( a \in (1, e^{A-1}) \) and sufficiently large \( n, \) we have
\[ \|G_{0,n}\|_0 \geq \frac{a^{n\ln n - 1}}{(\ln n)^2} \frac{\exp(-B) A^2}{4B^2} \rightarrow +\infty. \]  
(43)

By Theorem 1, we know that there exists a function \( f \in C[0,1] \) such that the sequence \( (B_{n,q_n}(f)) \) does not converge to \( f \) in \( C[0,1]. \) This leads to a contradiction. Hence, \( \lim_{n \to -\infty} r(n(q_n - 1)) \leq \ln 2. \) Theorem 2 is proved. \( \Box \)
Proof of Theorem 3. From Theorem 1, we know that it is sufficient to show that if $q_n \leq 1 + \ln 2/n$ for sufficiently large $n$, then

$$\sup_{n \in \mathbb{N}} \| G_{\alpha n} \|_0 < \infty. \quad (44)$$

For $x \in (q_n^{-1}, 1)$, we set $\alpha = -\log_{q_n} x$. Then $\alpha \in (0, 1)$ and $x = q_n^{-\alpha}$. Since, for $k = 2, \ldots, n-1,$

$$q_n^{-\alpha} \left( q_n^{n-k-1} - 1 \right) \leq q_n^{-k+\alpha} - 1 \leq q_n^{-k-1} \leq \left( 1 + \ln \frac{2}{n} \right)^n - 1 \leq 1,$$ \hspace{1cm} (45)

by (37) we get that

$$\left| \frac{\nu_{n-k} (q_n^{-\alpha} x)}{\nu_{n-k} (q_n^{-\alpha})} \right| = \frac{\left( q_n^{-k-1} - 1 \right) \left( q_n^{-\alpha} - 1 \right)}{\left( q_n^{-k} - 1 \right) q_n^{-\alpha}} \leq q_n^{-\alpha} - 1 \leq k - \alpha \leq k + 1.$$ \hspace{1cm} (46)

On the other hand, by (37) we have

$$| \nu_{n-k} (q_n^{-\alpha} x) | \leq \frac{\alpha (1 - \alpha) \cdots (k - 1 - \alpha)}{k!} \cdot (q_n - 1)^2 \leq \frac{\alpha (1 - \alpha) \cdots (k - 1 - \alpha)}{k!}.$$ \hspace{1cm} (47)

It follows from (46) and (47) that

$$| \nu_{n-k} (q_n^{-\alpha} x) | \leq \frac{\alpha (1 - \alpha) \cdots (k - 1 - \alpha)}{k!}. \quad (48)$$

Hence, for $x = q_n^{-\alpha}, \alpha \in (0, 1),$

$$G_{\alpha n} (x) = \sum_{k=2}^{\infty} \nu_{n-k} (q_n^{-\alpha} x) \leq \sum_{k=2}^{\infty} \frac{\alpha (1 - \alpha) \cdots (k - 1 - \alpha)}{k!}.$$ \hspace{1cm} (49)

Obviously (49) is satisfied for $x \in \{0, 1\}$. We note that, for $x \in [0, 1],$

$$\left( 1 - x \right)^\alpha = 1 - \alpha x - \sum_{k=2}^{\infty} \frac{\alpha (1 - \alpha) \cdots (k - 1 - \alpha)}{k!} x^k.$$ \hspace{1cm} (50)

The above formula with $x = 1$ means that

$$\sum_{k=2}^{\infty} \frac{\alpha (1 - \alpha) \cdots (k - 1 - \alpha)}{k!} = 1 - \alpha.$$ \hspace{1cm} (51)

Thus, by (49),

$$\| G_{\alpha n} \|_0 \leq \sup_{\alpha \in [0, 1]} \sum_{k=2}^{\infty} \frac{\alpha (1 - \alpha) \cdots (k - 1 - \alpha)}{k!} = 1.$$ \hspace{1cm} (52)

This completes the proof of Theorem 3. \hfill \Box

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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