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Research Article

The Fixed Points of Solutions of Some q-Difference Equations

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The purpose of this paper is to investigate the fixed points of solutions $f(z)$ of some q-difference equations and obtain some results about the exponents of convergence of fixed points of $f(z)$ and $f(q^j z)$ $(j \in \mathbb{N}_+)$, q-differences $\Delta_q f(z) = f(qz) - f(z)$, and q-divided differences $\Delta_q f(z)/f(z)$.

1. Introduction and Main Results

Throughout this paper, we will assume that the readers are familiar with basic notations such as $m(r, f)$, $N(r, f)$, and $T(r, f)$ of Nevanlinna theory (see Hayman [1], Yang [2], and Yang and Yi [3]). We use $\rho(f)$, $\lambda(f)$, and $\lambda(1/f)$ to denote the order, the exponent of convergence of zeros, and the exponent of convergence of poles of $f(z)$, respectively, and we also use the notation $\tau(f)$ to denote the exponent of convergence of fixed points of $f(z)$, which is defined as

$$\tau(f) = \limsup_{r\to\infty} \frac{\log N(r, 1/(f(z) - z))}{\log r}, \tag{1}$$

and $S(r, f)$ to denote any quantity satisfying $S(r, f) = o(T(r, f))$ for all r on a set F of logarithmic density 1, where the logarithmic density of a set F is defined by

$$\lim_{r\to\infty} \frac{1}{\log r} \int_{[1,r]\cap F} \frac{1}{t} dt. \tag{2}$$

Throughout this paper, the set F of logarithmic density 1 will be not necessarily the same at each occurrence.

Recently, a number of papers (including [4–9]) focused on complex difference equations, system of complex difference equations, and difference analogues of Nevanlinna theory. Correspondingly, there are many papers focusing on the q-difference (or q-shift difference) equations, such as [10–16].

In 2013, Zhang [17] investigated the growth of meromorphic solutions of some complex q-difference equations and the exponents of convergence of fixed points and zeros of transcendental meromorphic solutions of the second order q-difference equation and obtained the following theorem.

Theorem 1 (see [17]). *Suppose that $f(z)$ is a transcendental meromorphic solution of the equation*

$$f(q^2 z) + \gamma_1 f(qz) = \frac{\alpha_0 + \alpha_1 f(z) + \alpha_2 f^2(z)}{\beta_0 + \beta_1 f(z) + \beta_2 f^2(z)}, \tag{3}$$

where $|q| < 1$, coefficients γ_1, α_0, α_1, α_2, β_0, β_1, and β_2 are constants, and at least one of α_2, β_2 is nonzero. Then, $\rho(f) = 0$ and (i) $f(z)$ has infinitely many fixed points, and (ii) $f(z)$ has infinitely many zeros, whenever $\alpha_0 \neq 0$.

Our first result of this paper is about the exponents of convergence of fixed points and zeros of transcendental meromorphic solutions of the higher order q-difference equation as follows.

Theorem 2. *Suppose that $f(z)$ is a transcendental meromorphic solution of the equation*

$$f(q^n z) + \sum_{t=1}^{n-1} \gamma_t f(q^t z) = \frac{\sum_{j=0}^{n} \alpha_j f^j(z)}{\sum_{j=0}^{n} \beta_j f^j(z)}, \tag{4}$$

where $q \in \mathbb{C}$, $|q| < 1$, coefficients γ_t ($t = 1, \ldots, n-1$), α_j, β_j, ($j = 0, \ldots, n$), are constants, and at least one of α_n, β_n is nonzero. Then, $\rho(f) = 0$ and (i) $f(z)$ has infinitely many fixed points, and (ii) $f(z)$ has infinitely many zeros, whenever $\alpha_0 \neq 0$.

From Theorem 2, it is a natural question to ask, What will happen if the right-hand side of (4) is a rational function in both arguments?

Regarding the above question, we will investigate the exponents of convergence of fixed points of meromorphic solutions of the q-difference equation

$$f(qz) = \frac{R(z) f(z)}{Q(z) + P(z) f(z)}, \tag{5}$$

where $P(z)$, $Q(z)$, and $R(z)$ are nonzero polynomials, $q \in \mathbb{C}$, and $|q| \neq 0, 1$. Similar to [18, Page 99], we can call (5) a q-Pielou logistic equation, which is a special form of nonautonomous Schröder equations.

Theorem 3. *Let $P(z)$, $Q(z)$, and $R(z)$ be nonzero polynomials such that*

$$\deg P(z) \geq \max\{\deg R(z), \deg Q(z), 1\}. \tag{6}$$

Set $\Delta_q f(z) = f(qz) - f(z)$, where $q \in \mathbb{C}$ and $|q| \neq 0, 1$. Then every transcendental meromorphic solution $f(z)$ of (5) satisfies the following statements:

(i) *$f(q^j z)$ has infinitely many fixed points and $\tau(f(q^j z)) = \rho(f)$, ($j = 0, 1, 2, \ldots$);*

(ii) *if $R(z) - (z+1)Q(z) \not\equiv 0$, then $\Delta_q f(z)/f(z)$ has infinitely many fixed points and $\tau(\Delta_q f/f) = \rho(f)$.*

We also study fixed points of transcendental meromorphic solutions of the following q-difference equations:

$$a_n(z) f(q^n z) + \cdots + a_1(z) f(qz) + a_0(z) f(z) = 0, \tag{7}$$

$$a_n(z) f(q^n z) + \cdots + a_1(z) f(qz) + a_0(z) f(z) = F(z), \tag{8}$$

where $0 < |q| < 1$, $a_j(z)$ ($j = 0, 1, \ldots, n$), and $F(z)$ are polynomials and $a_n(z) a_0(z) \not\equiv 0$, and obtain the following results.

Theorem 4. *Let $q \in \mathbb{C}$, $0 < |q| < 1$, let $a_j(z)$ ($j = 0, 1, \ldots, n$) be polynomials, and let $a_n(z) a_0(z) \not\equiv 0$. If $a_0(z), a_1(z), \ldots, a_n(z)$ satisfy one of the following conditions:*

(i) *there exists an integer s ($0 \leq s \leq n$) such that*

$$\deg a_s(z) > \max\{\deg a_j(z), j = 0, 1, \ldots, n, j \neq s\}; \tag{9}$$

(ii)

$$q^n a_n(z) + \cdots + q a_1(z) + a_0(z) \not\equiv 0, \tag{10}$$

then every transcendental meromorphic solution $f(z)$ of (7) satisfies that $f(q^j z)$ has infinitely many fixed points and $\tau(f(q^j z)) = \rho(f)$ for $j \in \mathbb{N}$.

By using the same argument as that in Theorem 4, we can easily obtain the following theorem.

Theorem 5. *Let $q \in \mathbb{C}$, $0 < |q| < 1$, $a_j(z)$ ($j = 0, 1, \ldots, n$), and $F(z)$ be polynomials and let $a_n(z) a_0(z) \not\equiv 0$. If $a_0(z), a_1(z), \ldots, a_n(z), F(z)$ satisfy one of the following conditions:*

(i) *$a_0(z), a_1(z), \ldots, a_n(z)$ and $F(z)$ contain just one term of maximal total degree;*

(ii)

$$q^n a_n(z) + \cdots + q a_1(z) + a_0(z) - F(z) \not\equiv 0, \tag{11}$$

then every transcendental meromorphic solution $f(z)$ of (8) satisfies that $f(q^j z)$ has infinitely many fixed points and $\tau(f(q^j z)) = \rho(f)$ for $j \in \mathbb{N}$.

2. Some Lemmas

The following result is a difference counterpart to the standard result due to A. A. Mohon'ko and V. D. Mohon'ko [19].

Lemma 6 (see [20], Theorem 2.2). *Let $f(z)$ be a nonconstant zero-order meromorphic solution of $P(z, f) = 0$, where $P(z, f)$ is a q-difference polynomial in $f(z)$. If $P(z, a) \not\equiv 0$ for a slowly moving target $a(z)$, then*

$$m\left(r, \frac{1}{f - a}\right) = S(r, f), \tag{12}$$

on a set of logarithmic density 1.

Lemma 7 (see [21, 22]). *Let $a_j(z)$, $j = 0, 1, \ldots, n$, and $Q(z)$ be rational functions, and let $a_0(z) \not\equiv 0$, $a_n(z) \equiv 1$, and q ($0 < |q| < 1$). Then*

(i) *all meromorphic solutions of the equation*

$$\sum_{j=0}^{n} a_j(z) f(q^j z) = Q(z) \tag{13}$$

satisfy $T(r, f) = O((\log r)^2)$;

(ii) *all transcendental meromorphic solutions of (13) satisfy $(\log r)^2 = O(T(r, f))$.*

Lemma 8 (see [17], Theorem 2). *Suppose that $f(z)$ is a nonconstant meromorphic solution of the equation*

$$\sum_{j=1}^{n} \gamma_j(z) f(q^j z) = R(z, f(z)) = \frac{\sum_{i=0}^{s} \alpha_i(z) f^i(z)}{\sum_{i=0}^{t} \beta_i(z) f^i(z)}, \tag{14}$$

where q ($0 < |q| < 1$) is a complex number, $\alpha_j(z)$ ($j = 0, 1, \ldots, s$), $\alpha_s(z) \not\equiv 0$, $\beta_j(z)$ ($j = 0, 1, \ldots, t$), $\beta_t(z) \not\equiv 0$, $\gamma_n(z) \equiv 1$, and $\gamma_j(z)$ ($j = 0, 1, \ldots, n$) are small functions of $f(z)$, and $R(z, f)$ is irreducible in $f(z)$. Then, $d = \max\{s, t\} \leq n$ and $\rho(f) \leq (\log n - \log d) / -\log|q|$.

Lemma 9 (see [21, page 249] or [23, Theorem 1.1]). *Let $f(z)$ be a transcendental meromorphic function of zero-order and let q be a nonzero complex constant. Then*

$$T(r, f(qz)) = T(r, f) + S(r, f) \tag{15}$$

on a set of logarithmic density 1.

3. Proof of Theorem 2

Suppose that $f(z)$ is a transcendental meromorphic solution of (4). From the assumptions of Theorem 2, it follows from Lemma 8 that $\rho(f) \le 0 = (\log n - \log n)/-\log|q|$. Thus, $\rho(f) = 0$. Clearly, we have $\lambda(f) = \tau(f) = \rho(f) = 0$.

(i) Firstly, we prove that $f(z)$ has infinitely many fixed points. Set $g(z) = f(z) - z$. Then $g(z)$ is transcendental, $T(r, g) = T(r, f) + O(\log r)$, and $S(r, f) = S(r, g)$. So, $g(z)$ is of zero-order. Then substituting $f(z) = g(z) + z$ into (4), we get that

$$\begin{aligned} &g(q^n z) + \sum_{t=1}^{n-1} \gamma_t g(q^t z) + q^n z + \sum_{t=1}^{n-1} \gamma_t q^t z \\ &\quad = \frac{\sum_{j=0}^{n} \alpha_j (g(z) + z)^j}{\sum_{j=0}^{n} \beta_j (g(z) + z)^j}. \end{aligned} \tag{16}$$

Set $A(z) = g(q^n z) + \sum_{t=1}^{n-1} \gamma_t g(q^t z) + q^n z + \sum_{t=1}^{n-1} \gamma_t q^t z$ and

$$P_1(z, g(z)) := A(z) \sum_{j=0}^{n} \beta_j (g(z) + z)^j - \sum_{j=0}^{n} \alpha_j (g(z) + z)^j. \tag{17}$$

It follows from (17) that

$$\begin{aligned} &P_1(z, 0) \\ &\quad = \left(q^n + \sum_{t=1}^{n-1} \gamma_t q^t\right) \sum_{j=0}^{n} \beta_j z^{j+1} - \sum_{j=0}^{n} \alpha_j z^j \\ &\quad = \left(q^n + \sum_{t=1}^{n-1} \gamma_t q^t\right) \beta_n z^{n+1} \\ &\qquad + \sum_{j=0}^{n-1} \left[\left(q^n + \sum_{t=1}^{n-1} \gamma_t q^t\right) \beta_j - \alpha_{j+1}\right] z^{j+1} - \alpha_0. \end{aligned} \tag{18}$$

Suppose that $P_1(z, 0) \equiv 0$. If $q^n + \sum_{t=1}^{n-1} \gamma_t q^t = 0$, then it follows from (18) that $\alpha_0 = \alpha_1 = \cdots = \alpha_n = 0$. Thus, the right-hand side of (4) is 0, which is in contradiction with the assumption of Theorem 2. If $q^n + \sum_{t=1}^{n-1} \gamma_t q^t \ne 0$, it follows from (18) that $\beta_n = \alpha_0 = 0$ and

$$\frac{\alpha_{j+1}}{\beta_j} = q^n + \sum_{t=1}^{n-1} \gamma_t q^t, \quad j = 0, 1, \ldots, n-1. \tag{19}$$

Thus, we have from (4) and (19) that

$$f(q^n z) + \sum_{t=1}^{n-1} \gamma_t f(q^t z) = \left(q^n + \sum_{t=1}^{n-1} \gamma_t q^t\right) f(z), \tag{20}$$

which is in contradiction with the assumption of Theorem 2. Hence, we have $P_1(z, 0) \not\equiv 0$. By Lemma 6, we get that

$$m\left(r, \frac{1}{g}\right) = S(r, g) = S(r, f) \tag{21}$$

on a set of logarithmic density 1. Thus, it follows from (21) that

$$N\left(r, \frac{1}{f - z}\right) = N\left(r, \frac{1}{g}\right) = T(r, f) + S(r, f) \tag{22}$$

on a set of logarithmic density 1. Since $f(z)$ is a transcendental meromorphic solution of (4), then it follows from (22) that $f(z)$ has infinitely many fixed points.

(ii) From (4), we have

$$\begin{aligned} P_2(z, f(z)) &:= \left[f(q^n z) + \sum_{t=1}^{n-1} \gamma_t f(q^t z)\right] \sum_{j=0}^{n} \beta_j f^j(z) \\ &\quad - \sum_{j=0}^{n} \alpha_j f^j(z). \end{aligned} \tag{23}$$

Since $\alpha_0 \ne 0$ and from (23), we derive that

$$P_2(z, 0) = \alpha_0 \not\equiv 0. \tag{24}$$

Thus, it follows from Lemma 6 that

$$m\left(r, \frac{1}{f}\right) = S(r, f) \tag{25}$$

on a set of logarithmic density 1; that is,

$$N\left(r, \frac{1}{f}\right) = T(r, f) + S(r, f) \tag{26}$$

on a set of logarithmic density 1. Since $f(z)$ is a transcendental solution of (4), then it follows from (26) that $f(z)$ has infinitely many zeros.

Thus, this completes the proof of Theorem 2.

4. Proof of Theorem 3

Suppose that $f(z)$ is a transcendental meromorphic solution of (5). Since $q \in \mathbb{C}$, $|q| \ne 0, 1$, and $P(z)$, $Q(z)$, and $R(z)$ are polynomials, it follows from Lemma 8 and [11] that $f(z)$ is of zero-order.

(i) We first prove that $f(z)$ has infinitely many fixed points and $\tau(f) = \rho(f)$. Set $g(z) = f(z) - z$. Then $g(z)$ is transcendental, $T(r, g) = T(r, f) + O(\log r)$, and $S(r, g) = S(r, f)$. Then it follows that $g(z)$ is of zero-order. Set

$$\begin{aligned} P_3(z, f(z)) &:= P(z) f(z) f(qz) \\ &\quad + f(qz) Q(z) - R(z) f(z) \equiv 0. \end{aligned} \tag{27}$$

Then substituting $f(z) = g(z) + z$ into (27), we have

$$\begin{aligned} P_4(z, g(z)) &= P(z)(g(z) + z)(g(qz) + qz) \\ &\quad + Q(z)(g(qz) + qz) - R(z)(g(z) + z) = 0. \end{aligned} \tag{28}$$

It follows from (28) that

$$P_4(z,0) = qz^2 P(z) + qzQ(z) - zR(z). \tag{29}$$

Thus, we derive by (6) and (29) that $P_4(z,0) \not\equiv 0$. Thus, by Lemma 6 and $P_4(z,0) \not\equiv 0$, we have

$$m\left(r, \frac{1}{g}\right) = S(r,g) = S(r,f) \tag{30}$$

on a set of logarithmic density 1; that is,

$$N\left(r, \frac{1}{f-z}\right) = N\left(r, \frac{1}{g}\right) = T(r,f) + S(r,f) \tag{31}$$

on a set of logarithmic density 1.

Since $f(z)$ is a transcendental meromorphic solution of (5), then it follows from (31) that $f(z)$ has infinitely many fixed points.

Next, we prove that $f(qz)$ has infinitely many fixed points and $\tau(f(qz)) = \rho(f)$. From (5), we have

$$\begin{aligned} f(qz) - z &= \frac{(R(z) - zP(z)) f(z) - zQ(z)}{Q(z) + P(z) f(z)} \\ &= \frac{(R(z) - zP(z)) \left[f(z) - zQ(z)/(R(z) - zP(z))\right]}{Q(z) + P(z) f(z)}. \end{aligned} \tag{32}$$

By (6), we have $R(z) - zP(z) \not\equiv 0$. Since $f(z)$ is transcendental and $P(z)$, $Q(z)$, and $R(z)$ are polynomials, we have by (32) the fact that $f(z) - zQ(z)/(R(z) - zP(z))$ and $Q(z) + P(z)f(z)$ have the same poles, except possibly finitely many poles. Moreover, we can get that $(R(z) - zP(z))f(z) - zQ(z)$ and $Q(z) + P(z)f(z)$ have at most finitely many common zeros. In fact, suppose that z_0 is a common zero of $(R(z) - zP(z))f(z) - zQ(z)$ and $Q(z) + P(z)f(z)$. Then $(R(z_0) - z_0P(z_0))f(z_0) - z_0Q(z_0) = 0$; that is, $f(z_0) = z_0Q(z_0)/(R(z_0) - z_0P(z_0))$. Substituting it into $Q(z_0) + P(z_0)f(z_0)$, we have

$$\frac{z_0 Q(z_0)}{R(z_0) - z_0 P(z_0)} P(z_0) + Q(z_0) = \frac{R(z_0) Q(z_0)}{R(z_0) - z_0 P(z_0)} = 0. \tag{33}$$

Thus, this shows that z_0 must be the zeros of $R(z)Q(z)/(R(z) - zP(z))$. Since $P(z)$, $Q(z)$, and $R(z)$ are polynomials, then $R(z)Q(z)/(R(z) - zP(z))$ has only finitely many zeros. So, $f(z) - zQ(z)/(R(z) - zP(z))$ and $Q(z) + P(z)f(z)$ have at most finitely many common zeros. Then it follows from (32) that

$$\tau(f(qz)) = \lambda(f(qz) - z) = \lambda\left(f(z) - \frac{zQ(z)}{R(z) - zP(z)}\right). \tag{34}$$

From (27), we have

$$\begin{aligned} P_3&\left(z, \frac{zQ(z)}{R(z) - zP(z)}\right) \\ &= P(z) \frac{zQ(z)}{R(z) - zP(z)} \frac{qzQ(qz)}{R(qz) - qzP(qz)} \\ &\quad + \frac{qzQ(qz)}{R(qz) - qzP(qz)} Q(z) - R(z) \frac{zQ(z)}{R(z) - zP(z)} \\ &= \left(qz^2 P(qz) Q(z) R(z) + qzQ(qz) Q(z) R(z) \right. \\ &\quad \left. - zQ(z) R(z) R(qz)\right) \\ &\quad \times \left((R(z) - zP(z))(R(qz) - qzP(qz))\right)^{-1}. \end{aligned} \tag{35}$$

Since $\deg P(z) \geq \max\{\deg R(z), \deg Q(z)\}$ and $\deg P(qz) = \deg P(z)$, then we have $\deg\{qz^2P(qz)Q(z)R(z) + qzQ(qz)Q(z)R(z) - zQ(z)R(z)R(qz)\} \geq 1$. Thus, it follows from (35) that $P_3(z, zQ(z)/(R(z) - zP(z))) \not\equiv 0$. Since $f(z)$ is transcendental function of zero-order and $zQ(z)/(R(z) - zP(z))$ is a rational function, then we have by Lemma 6 the fact that

$$m\left(r, \frac{1}{f(z) - zQ(z)/(R(z) - zP(z))}\right) = S(r,f) \tag{36}$$

on a set of logarithmic density 1; that is,

$$\begin{aligned} N&\left(r, \frac{1}{f(z) - zQ(z)/(R(z) - zP(z))}\right) \\ &= T(r,f) + S(r,f) \end{aligned} \tag{37}$$

on a set of logarithmic density 1. Since $f(z)$ is transcendental, we can derive from (34) and (37) that $f(qz)$ has infinitely many fixed points and $\tau(f(qz)) = \rho(f)$.

Now, we prove that $f(q^2z)$ has infinitely many fixed points and $\tau(f(q^2z)) = \rho(f)$. From (5), we have

$$f_1(qz) = \frac{R(qz) f_1(z)}{Q(qz) + P(qz) f_1(z)}, \tag{38}$$

where $f_1(z) = f(qz)$. By Lemma 9, we have $\rho(f_1) = \rho(f) = 0$. Obviously, $\deg P(qz) = \deg P(z) \geq 1$, $\deg R(qz) = \deg R(z)$, and $\deg Q(qz) = \deg Q(z)$. Thus, by using the same argument as in the proof of $\tau(f(qz)) = \rho(f)$, we can prove that $f_1(qz) = f(q^2z)$ has infinitely many fixed points and $\tau(f(q^2z)) = \tau(f_1(qz)) = \rho(f_1) = \rho(f)$.

Thus, by using the same method as above, we can obtain that $f(q^jz)$ has infinitely many fixed points and $\tau(f(q^jz)) = \rho(f)$ for $j = 0, 1, \ldots$.

(ii) Now, we prove that $\Delta_q f(z)/f(z)$ has infinitely many fixed points and

$$\tau\left(\frac{\Delta_q f}{f}\right) = \rho(f). \tag{39}$$

By (5) and from $R(z)-(z+1)Q(z) \not\equiv 0$, we have

$$\begin{aligned}
&\frac{\Delta_q f(z)}{f(z)} - z \\
&\quad = \frac{f(qz)-f(z)}{f(z)} - z \\
&\quad = \frac{R(z)-(z+1)Q(z)-(z+1)P(z)f(z)}{Q(z)+P(z)f(z)} \\
&\quad = -(z+1)P(z) \\
&\qquad \times \left(f(z) - \frac{R(z)-(z+1)Q(z)}{(z+1)P(z)}\right) \\
&\qquad \times (Q(z)+P(z)f(z))^{-1}.
\end{aligned} \tag{40}$$

Since $R(z)-(z+1)Q(z) \not\equiv 0$, $f(z)$ is transcendental, and $P(z)$, $Q(z)$, and $R(z)$ are polynomials, we have by (40) the fact that $f(z)-(R(z)-(z+1)Q(z))/(z+1)P(z)$ and $Q(z)+P(z)f(z)$ have the same poles, except possibly finitely many poles. Moreover, by using the same argument as in (i), we can get that $R(z)-(z+1)Q(z)-(z+1)P(z)f(z)$ and $Q(z)+P(z)f(z)$ have at most finitely many common zeros. Then it follows from (40) that

$$\begin{aligned}
\tau\left(\frac{\Delta_q f}{f}\right) &= \lambda\left(\frac{\Delta_q f}{f} - z\right) \\
&= \lambda\left(f(z) - \frac{R(z)-(z+1)Q(z)}{(z+1)P(z)}\right).
\end{aligned} \tag{41}$$

From (27), we have

$$\begin{aligned}
&P_3\left(z, \frac{R(z)-(z+1)Q(z)}{(z+1)P(z)}\right) \\
&\quad = P(z)\frac{R(z)-(z+1)Q(z)}{(z+1)P(z)}\frac{R(qz)-(qz+1)Q(qz)}{(qz+1)P(qz)} \\
&\qquad + \frac{R(qz)-(qz+1)Q(qz)}{(qz+1)P(qz)}Q(z) \\
&\qquad - R(z)\frac{R(z)-(z+1)Q(z)}{(z+1)P(z)} \\
&\quad := \frac{B(z)}{(qz+1)(z+1)P(qz)P(z)},
\end{aligned} \tag{42}$$

where

$$\begin{aligned}
B(z) = {} & (qz+1)(z+1)P(qz)Q(z)R(z) \\
& - (qz+1)P(z)Q(qz)R(z) \\
& - (qz+1)P(qz)R^2(z) + P(z)R(z)R(qz).
\end{aligned} \tag{43}$$

Since $P(z)$, $Q(z)$, and $R(z)$ are polynomials satisfying (6), then it follows from (43) that $B(z)$ is a polynomial of degree $t \geq 1$. Thus, from (42) we have $P_3(z, (R(z)-(z+1)Q(z))/(z+1)P(z)) \not\equiv 0$. Since $f(z)$ is transcendental function of zero-order and $(R(z)-(z+1)Q(z))/(z+1)P(z)$ is a rational function, then we have by Lemma 6 the fact that

$$\begin{aligned}
&m\left(r, \frac{1}{f(z)-(R(z)-(z+1)Q(z))/(z+1)P(z)}\right) \\
&\quad = S(r,f)
\end{aligned} \tag{44}$$

on a set of logarithmic density 1; that is,

$$\begin{aligned}
&N\left(r, \frac{1}{f(z)-(R(z)-(z+1)Q(z))/(z+1)P(z)}\right) \\
&\quad = T(r,f) + S(r,f)
\end{aligned} \tag{45}$$

on a set of logarithmic density 1. Since $f(z)$ is transcendental, we can derive from (41) and (45) that $\Delta_q f(z)/f(z)$ has infinitely many fixed points and $\tau(\Delta_q f/f) = \rho(f)$.

Thus, this completes the proof of Theorem 3.

5. Proof of Theorem 4

Suppose that $f(z)$ is a transcendental meromorphic solution of (7). Since $q \in \mathbb{C}$, $0 < |q| < 1$, and $a_j(z)$, $j = 0, 1, \ldots, n$, are polynomials, by Lemma 7, we see that $f(z)$ is of zero-order. Set

$$\begin{aligned}
P_5(z, f(z)) := {} & a_n(z) f(q^n z) + \cdots + a_1(z) f(qz) \\
& + a_0(z) f(z) = 0.
\end{aligned} \tag{46}$$

Thus, it follows from (46) that

$$\begin{aligned}
P_5(z,z) &= a_n(z) q^n z + \cdots + a_1(z) qz + a_0(z) z \\
&= z\left[q^n a_n(z) + \cdots + q a_1(z) + a_0(z)\right].
\end{aligned} \tag{47}$$

(i) Suppose that $a_0(z), \ldots, a_n(z)$ satisfy condition (9). Then it follows that $P_5(z,z) \not\equiv 0$. Since $f(z)$ is a transcendental solution of zero-order, then it follows from Lemma 6 that

$$m\left(r, \frac{1}{f-z}\right) = S(r,f) \tag{48}$$

on a set of logarithmic density 1. So,

$$N\left(r, \frac{1}{f-z}\right) = T(r,f) + S(r,f) \tag{49}$$

on a set of logarithmic density 1. Thus, it follows that $f(z)$ has infinitely many fixed points and $\tau(f) = \rho(f)$.

Now, we prove that $f(qz)$ has infinitely many fixed points and $\tau(f(qz)) = \rho(f)$. By (7), we derive

$$a_n(qz) f_1(q^n z) + \cdots + a_1(qz) f_1(qz) + a_0(qz) f_1(z) = 0, \tag{50}$$

where $f_1(z) = f(qz)$. Since $f(z)$ is a transcendental meromorphic function of zero-order, then we have by Lemma 9 the fact that $f_1(z)$ is a transcendental and $\rho(f_1) = \rho(f)$. By $\deg a_j(qz) = \deg a_j(z)$, $j = 0, 1, \ldots, n$, and (9), we have

$$\begin{aligned}\deg a_s(qz) \\ = \deg a_s(z) > \max\left\{a_j(qz), j = 0, 1, \ldots, n, j \neq s\right\}.\end{aligned} \quad (51)$$

Thus, by the above proof of $\tau(f) = \rho(f)$, we see that $f_1(z) = f(qz)$ has infinitely many fixed points and $\tau(f_1) = \tau(f(qz)) = \rho(f_1) = \rho(f)$. Continuing to use the same method as the above, we can prove that $f(q^j z)$ has infinitely many fixed points and $\tau(f(q^j z)) = \rho(f)$ for $j = 0, 1, \ldots$.

(ii) Suppose that $a_0(z), \ldots, a_n(z)$ satisfy the condition (10).

By using the same argument as the one above, we can prove that $f(z)$ has infinitely many fixed points and $\tau(f) = \rho(f)$ easily.

Now, we prove that $f(qz)$ has infinitely many fixed points and $\tau(f(qz)) = \rho(f)$. Set

$$\begin{aligned}P_6(z, f_1(z)) := a_n(qz) f_1(q^n z) + \cdots + a_1(qz) f_1(qz) \\ + a_0(qz) f_1(z) = 0.\end{aligned} \quad (52)$$

Thus, it follows from (10) that

$$P_6(z, z) = z\left[q^n a_n(qz) + \cdots + q a_1(qz) + a_0(qz)\right] \not\equiv 0. \quad (53)$$

In fact, if $P_6(z, z) \equiv 0$, replacing z by z/q into (53), we have

$$P_6\left(\frac{z}{q}, \frac{z}{q}\right) = \frac{z}{q}\left[q^n a_n(z) + \cdots + q a_1(z) + a_0(z)\right] \equiv 0, \quad (54)$$

which is in contradiction with the condition (10). Since $f_1(z) = f(qz)$ and $f(z)$ is transcendental meromorphic of zero-order, then it follows from (53) and Lemma 6 that $f_1(z) = f(qz)$ has infinitely many fixed points and $\tau(f_1) = \tau(f(qz)) = \rho(f_1) = \rho(f)$. Continuing to use the same method as the one above, we can prove that $f(q^j z)$ has infinitely many fixed points and $\tau(f(q^j z)) = \rho(f)$ for $j = 0, 1, \ldots$.

Thus, this completes the proof of Theorem 4.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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