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Research Article

Steepest-Descent Approach to Triple Hierarchical Constrained Optimization Problems

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We introduce and analyze a hybrid steepest-descent algorithm by combining Korpelevich's extragradient method, the steepest-descent method, and the averaged mapping approach to the gradient-projection algorithm. It is proven that under appropriate assumptions, the proposed algorithm converges strongly to the unique solution of a triple hierarchical constrained optimization problem (THCOP) over the common fixed point set of finitely many nonexpansive mappings, with constraints of finitely many generalized mixed equilibrium problems (GMEPs), finitely many variational inclusions, and a convex minimization problem (CMP) in a real Hilbert space.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$; let C be a nonempty closed convex subset of H and let P_C be the metric projection of H onto C. Let $S : C \to H$ be a nonlinear mapping on C. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S and by $\mathbf{R}$ the set of all real numbers. A mapping $S : C \to H$ is called L-Lipschitz continuous if there exists a constant $L > 0$ such that

$$\|Sx - Sy\| \le L\|x - y\|, \quad \forall x, y \in C. \tag{1}$$

In particular, if $L = 1$ then S is called a nonexpansive mapping; if $L \in (0,1)$ then S is called a contraction.

Let $A : C \to H$ be a nonlinear mapping on C. The classical variational inequality problem (VIP) [1] is to find a point $x \in C$ such that

$$\langle Ax, y - x\rangle \ge 0, \quad \forall y \in C. \tag{2}$$

The solution set of VIP (2) is denoted by $\mathrm{VI}(C, A)$.

In 1976, Korpelevich [2] proposed an iterative algorithm for solving the VIP (2) in Euclidean space $\mathbf{R}^n$:

$$\begin{aligned} y_n &= P_C(x_n - \tau Ax_n), \\ x_{n+1} &= P_C(x_n - \tau Ay_n), \\ &\forall n \ge 0, \end{aligned} \tag{3}$$

with $\tau > 0$ a given number, which is known as the extragradient method. See, for example, [3–7] and the references therein.

Let $\varphi : C \to \mathbf{R}$ be a real-valued function; let $A : H \to H$ be a nonlinear mapping and let $\Theta : C \times C \to \mathbf{R}$ be a bifunction. In 2008, Peng and Yao [8] introduced the following generalized mixed equilibrium problem (GMEP) of finding $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x\rangle \ge 0, \quad \forall y \in C. \tag{4}$$

We denote the set of solutions of GMEP (4) by $\mathrm{GMEP}(\Theta, \varphi, A)$.

In [8], Peng and Yao assumed that $\Theta : C \times C \to \mathbf{R}$ is a bifunction satisfying conditions (A1)–(A4) and $\varphi : C \to \mathbf{R}$ is a lower semicontinuous and convex function with restriction (B1) or (B2), where

(A1) $\Theta(x, x) = 0$ for all $x \in C$;

(A2) Θ is monotone; that is, $\Theta(x, y) + \Theta(y, x) \leq 0$ for any $x, y \in C$;

(A3) Θ is upper-hemicontinuous; that is, for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} \Theta(tz + (1-t)x, y) \leq \Theta(x, y); \tag{5}$$

(A4) $\Theta(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$;

(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \backslash D_x$,

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0; \tag{6}$$

(B2) C is a bounded set.

Given a positive number $r > 0$. Let $T_r^{(\Theta,\varphi)} : H \to C$ be the solution set of the auxiliary mixed equilibrium problem; that is, for each $x \in H$,

$$T_r^{(\Theta,\varphi)}(x) := \left\{ y \in C : \Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle y - x, z - y\rangle \geq 0, \forall z \in C \right\}. \tag{7}$$

Let $f : C \to \mathbf{R}$ be a convex and continuously Fréchet differentiable functional. Consider the convex minimization problem (CMP) of minimizing f over the constraint set C:

$$\min_{x \in C} f(x) \tag{8}$$

(assuming the existence of minimizers). We denote by Γ the set of minimizers of CMP (8).

On the other hand, let B be a single-valued mapping of C into H and R be a set-valued mapping with $D(R) = C$. Considering the following variational inclusion, find a point $x \in C$ such that

$$0 \in Bx + Rx. \tag{9}$$

We denote by $I(B, R)$ the solution set of the variational inclusion (9). Let a set-valued mapping $R : D(R) \subset H \to 2^H$ be maximal monotone. We define the resolvent operator $J_{R,\lambda} : H \to \overline{D(R)}$ associated with R and λ as follows:

$$J_{R,\lambda} = (I + \lambda R)^{-1}, \quad \forall x \in H, \tag{10}$$

where λ is a positive number.

Let S and T be two nonexpansive mappings. In 2009, Yao et al. [9] considered the following hierarchical VIP: find hierarchically a fixed point of T, which is a solution to the VIP for monotone mapping $I - S$; namely, find $\tilde{x} \in \mathrm{Fix}(T)$ such that

$$\langle (I - S)\tilde{x}, p - \tilde{x}\rangle \geq 0, \quad \forall p \in \mathrm{Fix}(T). \tag{11}$$

The solution set of the hierarchical VIP (11) is denoted by Λ. It is not hard to check that solving the hierarchical VIP (11) is equivalent to the fixed point problem of the composite mapping $P_{\mathrm{Fix}(T)}S$; that is, find $\tilde{x} \in C$ such that $\tilde{x} = P_{\mathrm{Fix}(T)}S\tilde{x}$. The authors [9] introduced and analyzed the following iterative algorithm for solving the hierarchical VIP (11):

$$\begin{aligned} y_n &= \beta_n S x_n + (1 - \beta_n) x_n, \\ x_{n+1} &= \alpha_n V x_n + (1 - \alpha_n) T y_n, \\ &\qquad \forall n \geq 0. \end{aligned} \tag{12}$$

In this paper, we introduce and study the following triple hierarchical constrained optimization problem (THCOP) with constraints of the CMP (8), finitely many GMEPs and finitely many variational inclusions.

Problem I. Let M, N, and K be three positive integers. Assume that

(i) $f : C \to \mathbf{R}$ is a convex and continuously Fréchet differentiable functional with L-Lipschitz continuous gradient ∇f, $S_i : H \to H$ is a nonexpansive mapping, and $A_j : H \to H$ is ζ_j-inverse-strongly monotone for $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, K$;

(ii) $\widetilde{A}_1 : H \to H$ is α-inverse strongly monotone and $\widetilde{A}_2 : H \to H$ is β-strongly monotone and κ-Lipschitz continuous;

(iii) Θ_j is a bifunctions from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4), and $\varphi_j : C \to \mathbf{R}$ is a lower semicontinuous and convex functional with restriction (B1) or (B2) for $j = 1, 2, \ldots, K$;

(iv) $R_k : C \to 2^H$ is a maximal monotone mapping and $B_k : C \to H$ is η_k-inverse strongly monotone for $k = 1, 2, \ldots, M$;

(v) $\mathrm{VI}(\cap_{i=i}^N \mathrm{Fix}(S_i), \widetilde{A}_1) \neq \emptyset$ with $(\cap_{i=i}^N \mathrm{Fix}(S_i)) \subset (\cap_{j=1}^K \mathrm{GMEP}(\Theta_j, \varphi_j, A_j)) \cap (\cap_{k=1}^M I(B_k, R_k)) \cap \Gamma$.

Then the objective is to

$$\begin{aligned} \text{find } x^* \in \mathrm{VI}&\left(\mathrm{VI}\left(\bigcap_{i=i}^N \mathrm{Fix}(S_i), \widetilde{A}_1\right), \widetilde{A}_2\right) \\ &:= \left\{ x^* \in \mathrm{VI}\left(\bigcap_{i=i}^N \mathrm{Fix}(S_i), \widetilde{A}_1\right) : \langle \widetilde{A}_2 x^*, v - x^*\rangle \right. \\ &\qquad \left. \geq 0, \forall v \in \mathrm{VI}\left(\bigcap_{i=i}^N \mathrm{Fix}(S_i), \widetilde{A}_1\right) \right\}. \end{aligned} \tag{13}$$

Motivated and inspired by the above facts, we introduce and analyze a hybrid iterative algorithm via Korpelevich's extragradient method, the steepest-descent method, and the gradient-projection algorithm obtained by the averaged mapping approach. It is proven that under mild conditions, the proposed algorithm converges strongly to a unique element of $\mathrm{VI}(\mathrm{VI}(\cap_{i=1}^{N}\mathrm{Fix}(S_i),\widetilde{A}_1),\widetilde{A}_2)$ with $(\cap_{i=1}^{N}\mathrm{Fix}(S_i)) \subset (\cap_{j=1}^{K}\mathrm{GMEP}(\Theta_j,\varphi_j,A_j)) \cap (\cap_{k=1}^{M}I(B_k,R_k)) \cap \Gamma$, that is, the unique solution of the THCOP (13). In this paper, the results we acquired improve and extend the existing results found in this field.

2. Preliminaries

Throughout this paper, we assume that H is a real Hilbert space of which inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$; that is,

$$\omega_w(x_n) := \left\{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \left\{x_{n_i}\right\} \text{ of } \{x_n\}\right\}. \tag{14}$$

Definition 1. A mapping $A : C \to H$ is called

(i) monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C; \tag{15}$$

(ii) η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C; \tag{16}$$

(iii) ζ-inverse-strongly monotone if there exists a constant $\zeta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \zeta\|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{17}$$

It is obvious that if A is ζ-inverse-strongly monotone, then A is monotone and $1/\zeta$-Lipschitz continuous. Moreover, we also have that, for all $u, v \in C$ and $\lambda > 0$,

$$\begin{aligned}&\|(I - \lambda A)u - (I - \lambda A)v\|^2 \\ &\quad \leq \|u - v\|^2 + \lambda(\lambda - 2\zeta)\|Au - Av\|^2.\end{aligned} \tag{18}$$

So, if $\lambda \leq 2\zeta$, then $I - \lambda A$ is a nonexpansive mapping from C to H.

The metric projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$, the unique point $P_C x \in C$, satisfying the property

$$\|x - P_C x\| = \inf_{y\in C}\|x - y\| =: d(x, C). \tag{19}$$

Some important properties of projections are gathered in the following proposition.

Proposition 2. *For given $x \in H$ and $z \in C$:*

(i) $z = P_C x \Leftrightarrow \langle x - z, y - z\rangle \leq 0, \ \forall y \in C$;

(ii) $z = P_C x \Leftrightarrow \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2, \ \forall y \in C$;

(iii) $\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2, \ \forall y \in H$. *(This implies that P_C is nonexpansive and monotone.)*

Next we list some elementary conclusions for the mixed equilibrium problem where $\mathrm{MEP}(\Theta,\varphi)$ is the solution set.

Proposition 3 (see [10]). *Assume that $\Theta : C \times C \to \mathbf{R}$ satisfies (A1)–(A4) and let $\varphi : C \to \mathbf{R}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r^{(\Theta,\varphi)} : H \to C$ as follows:*

$$\begin{aligned}T_r^{(\Theta,\varphi)}(x) = \Big\{&z \in C : \Theta(z, y) + \varphi(y) - \varphi(z) \\ &+ \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\Big\}\end{aligned} \tag{20}$$

for all $x \in H$. Then the following hold:

(i) *for each $x \in H$, $T_r^{(\Theta,\varphi)}(x)$ is nonempty and single-valued;*

(ii) *$T_r^{(\Theta,\varphi)}$ is firmly nonexpansive; that is, for any $x, y \in H$,*

$$\left\|T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y\right\|^2 \leq \left\langle T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y, x - y\right\rangle; \tag{21}$$

(iii) $\mathrm{Fix}(T_r^{(\Theta,\varphi)}) = \mathrm{MEP}(\Theta,\varphi)$;

(iv) *$\mathrm{MEP}(\Theta,\varphi)$ is closed and convex;*

(v) *$\|T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x\|^2 \leq (s - t)/s\langle T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x, T_s^{(\Theta,\varphi)}x - x\rangle$ for all $s, t > 0$ and $x \in H$.*

In the following, we recall some facts and tools in a real Hilbert space H.

Lemma 4. *Let X be a real inner product space. Then there holds the following inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in X. \tag{22}$$

Lemma 5. *Let H be a real Hilbert space. Then the following hold:*

(a) *$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$ for all $x, y \in H$;*

(b) *$\|\lambda x + \mu y\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 - \lambda\mu\|x - y\|^2$ for all $x, y \in H$ and $\lambda, \mu \in [0, 1]$ with $\lambda + \mu = 1$;*

(c) *if $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup x$, it follows that*

$$\begin{aligned}&\limsup_{n\to\infty}\|x_n - y\|^2 \\ &\quad = \limsup_{n\to\infty}\|x_n - x\|^2 + \|x - y\|^2, \quad \forall y \in H.\end{aligned} \tag{23}$$

Definition 6. A mapping $T : H \to H$ is said to be an averaged mapping if it can be written as the average of the identity I and a nonexpansive mapping; that is,

$$T \equiv (1-\alpha) I + \alpha S, \tag{24}$$

where $\alpha \in (0,1)$ and $S : H \to H$ is nonexpansive. More precisely, when the last equality holds, we say that T is α-averaged. Thus firmly nonexpansive mappings (particularly, projections) are 1/2-averaged mappings.

Lemma 7 (see [11]). *Let $T : H \to H$ be a given mapping.*

(i) *T is nonexpansive if and only if the complement $I - T$ is 1/2-ism.*

(ii) *If T is ν-ism, then for $\gamma > 0$, γT is ν/γ-ism.*

(iii) *T is averaged if and only if the complement $I-T$ is ν-ism for some $\nu > 1/2$. Indeed, for $\alpha \in (0,1)$, T is α-averaged if and only if $I - T$ is $1/2\alpha$-ism.*

Lemma 8 (see [11]). *Let $S, T, V : H \to H$ be given operators.*

(i) *If $T = (1-\alpha)S + \alpha V$ for some $\alpha \in (0,1)$ and if S is averaged and V is nonexpansive, then T is averaged.*

(ii) *T is firmly nonexpansive if and only if the complement $I - T$ is firmly nonexpansive.*

(iii) *If $T = (1-\alpha)S + \alpha V$ for some $\alpha \in (0,1)$ and if S is firmly nonexpansive and V is nonexpansive, then T is averaged.*

(iv) *The composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, then so is the composite $T_1 \cdots T_N$. In particular, if T_1 is α_1-averaged and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0,1)$, then the composite T_1T_2 is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.*

(v) *If the mappings $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then*

$$\bigcap_{i=1}^{N} \operatorname{Fix}(T_i) = \operatorname{Fix}(T_1T_2\cdots T_N). \tag{25}$$

The notation $\operatorname{Fix}(T)$ *denotes the set of all fixed points of the mapping T; that is,* $\operatorname{Fix}(T) = \{x \in H : Tx = x\}$.

Let $f : C \to \mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f. It is well known that the gradient-projection algorithm (GPA) generates a sequence $\{x_n\}$ determined by the gradient ∇f and the metric projection P_C:

$$x_{n+1} := P_C(x_n - \lambda \nabla f(x_n)), \quad \forall n \geq 0, \tag{26}$$

or more generally,

$$x_{n+1} := P_C(x_n - \lambda_n \nabla f(x_n)), \quad \forall n \geq 0, \tag{27}$$

where, in both (26) and (27), the initial guess x_0 is taken from C arbitrarily, and the parameters λ or λ_n are positive real numbers. The convergence of algorithms (26) and (27) depends on the behavior of the gradient ∇f.

Lemma 9 (see [12, Demiclosedness principle]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a nonexpansive self-mapping on C. Then $I - T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-T)x_n\}$ strongly converges to some y, it follows that $(I-T)x = y$. Here I is the identity operator of H.*

Lemma 10. *Let $A : C \to H$ be a monotone mapping. In the context of the variational inequality problem the characterization of the projection (see* Proposition 2(i)*) implies*

$$u \in \operatorname{VI}(C, A) \Longleftrightarrow u = P_C(u - \lambda A u), \quad \lambda > 0. \tag{28}$$

Let C be a nonempty closed convex subset of a real Hilbert space H. We introduce some notations. Let λ be a number in $(0,1]$ and let $\mu > 0$. Associating with a nonexpansive mapping $T : C \to H$, we define the mapping $T^\lambda : C \to H$ by

$$T^\lambda x := Tx - \lambda\mu F(Tx), \quad \forall x \in C, \tag{29}$$

where $F : H \to H$ is an operator such that, for some positive constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone on H; that is, F satisfies the conditions:

$$\|Fx - Fy\| \leq \kappa \|x - y\|, \qquad \langle Fx - Fy, x - y\rangle \geq \eta \|x - y\|^2 \tag{30}$$

for all $x, y \in H$.

Lemma 11 (see [13, Lemma 3.1]). *T^λ is a contraction provided by $0 < \mu < 2\eta/\kappa^2$; that is,*

$$\|T^\lambda x - T^\lambda y\| \leq (1 - \lambda\tau)\|x - y\|, \quad \forall x, y \in C, \tag{31}$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0,1]$.

Lemma 12 (see [13]). *Let $\{s_n\}$ be a sequence of nonnegative numbers satisfying the conditions*

$$s_{n+1} \leq (1 - \alpha_n) s_n + \alpha_n \beta_n, \quad \forall n \geq 1, \tag{32}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers such that

(i) *$\{\alpha_n\} \subset [0,1]$ and $\sum_{n=1}^\infty \alpha_n = \infty$, or equivalently,*

$$\prod_{n=1}^{\infty}(1 - \alpha_n) := \lim_{n\to\infty}\prod_{k=1}^{n}(1 - \alpha_k) = 0; \tag{33}$$

(ii) *$\limsup_{n\to\infty}\beta_n \leq 0$, or $\sum_{n=1}^\infty |\alpha_n\beta_n| < \infty$.*

Then $\lim_{n\to\infty} s_n = 0$.

Recall that a Banach space X is said to satisfy Opial's property [12] if, for any given sequence $\{x_n\} \subset X$ which converges weakly to an element $x \in X$, there holds the inequality

$$\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\|, \quad \forall y \in X,\ y \neq x. \tag{34}$$

It is well known that every Hilbert space H satisfies Opial's property in [12].

Finally, recall that a set-valued mapping $T : D(T) \subset H \to 2^H$ is called monotone if for all $x, y \in D(T)$, $f \in Tx$, and $g \in Ty$ imply

$$\langle f - g, x - y \rangle \geq 0. \tag{35}$$

A set-valued mapping T is called maximal monotone if T is monotone and $(I + \lambda T)D(T) = H$ for each $\lambda > 0$, where I is the identity mapping of H. We denote by $G(T)$ the graph of T. It is known that a monotone mapping T is maximal if and only if, for $(x, f) \in H \times H$, $\langle f - g, x - y \rangle \geq 0$, for every $(y, g) \in G(T)$, implies $f \in Tx$. Let $A : C \to H$ be a monotone, k-Lipschitz-continuous mapping and let $N_C v$ be the normal cone to C at $v \in C$; that is,

$$N_C v = \{u \in H : \langle v - p, u \rangle \geq 0, \ \forall p \in C\}. \tag{36}$$

Define

$$\widetilde{T} v = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases} \tag{37}$$

Then, $\widetilde{T}$ is maximal monotone such that

$$0 \in \widetilde{T} v \Longleftrightarrow v \in \mathrm{VI}(C, A). \tag{38}$$

Let $R : D(R) \subset H \to 2^H$ be a maximal monotone mapping. Let $\lambda, \mu > 0$ be two positive numbers.

Lemma 13 (see [14]). *There holds the resolvent identity*

$$J_{R,\lambda} x = J_{R,\mu}\left(\frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda}\right) J_{R,\lambda} x\right), \quad \forall x \in H. \tag{39}$$

For $\lambda, \mu > 0$, there holds the following relation that

$$\begin{aligned} \left\| J_{R,\lambda} x - J_{R,\mu} y \right\| &\leq \|x - y\| + |\lambda - \mu| \\ &\quad \times \left(\frac{1}{\lambda} \left\| J_{R,\lambda} x - y \right\| + \frac{1}{\mu} \left\| x - J_{R,\mu} y \right\| \right), \\ &\quad \forall x, y \in H. \end{aligned} \tag{40}$$

Based on Huang [15], there holds the following property for the resolvent operator $J_{R,\lambda} : H \to \overline{D(R)}$.

Lemma 14. *$J_{R,\lambda}$ is single-valued and firmly nonexpansive; that is,*

$$\langle J_{R,\lambda} x - J_{R,\lambda} y, x - y \rangle \geq \left\| J_{R,\lambda} x - J_{R,\lambda} y \right\|^2, \quad \forall x, y \in H. \tag{41}$$

Consequently, $J_{R,\lambda}$ is nonexpansive and monotone.

Lemma 15 (see [16]). *Let R be a maximal monotone mapping with $D(R) = C$. Then for any given $\lambda > 0$, $u \in C$ is a solution of problem (10) if and only if $u \in C$ satisfies*

$$u = J_{R,\lambda}(u - \lambda B u). \tag{42}$$

Lemma 16 (see [17]). *Let R be a maximal monotone mapping with $D(R) = C$ and let $B : C \to H$ be a strongly monotone, continuous, and single-valued mapping. Then, for each $z \in H$, the equation $z \in (B + \lambda R)x$ has a unique solution x_λ for $\lambda > 0$.*

Lemma 17 (see [16]). *Let R be a maximal monotone mapping with $D(R) = C$ and let $B : C \to H$ be a monotone, continuous, and single-valued mapping. Then $(I + \lambda(R + B))C = H$ for each $\lambda > 0$. In this case, $R + B$ is maximal monotone.*

3. Main Results

In this section, we will introduce and analyze a hybrid steepest-descent algorithm for finding a solution of the THCOP (13) with constraints of several problems: the CMP (8), finitely many GMEPs, and finitely many variational inclusions in a real Hilbert space. This algorithm is based on Korpelevich's extragradient method, the steepest-descent method, and the averaged mapping approach to the gradient-projection algorithm. We prove the strong convergence of the proposed algorithm to a unique solution of THCOP (13) under suitable conditions. Throughout this paper, let $\{S_i\}_{i=1}^N$ be N nonexpansive mappings $S_i : H \to H$ with $N \geq 1$ an integer. We write $S_{[k]} := S_{k \bmod N}$, for integer $k \geq 1$, with the mod function taking values in the set $\{1, 2, \ldots, N\}$ (i.e., if $k = jN + q$ for some integers $j \geq 0$ and $0 \leq q < N$, then $T_{[k]} = N$ if $q = 0$ and $T_{[k]} = q$ if $1 \leq q < N$).

The following is to state and prove the main result in this paper.

Theorem 18. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $f : C \to \mathbf{R}$ be a convex and continuously Fréchet differentiable functional with L-Lipschitz continuous gradient ∇f. Let $M, N, K \geq 1$ be three integers. Let Θ_j be a bifunctions from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4), $\varphi_j : C \to \mathbf{R}$ a lower semicontinuous and convex functional with restriction (B1) or (B2), and $A_j : H \to H$ ζ_j-inverse-strongly monotone for $j = 1, 2, \ldots, K$. Let $R_k : C \to 2^H$ be a maximal monotone mapping and let $B_k : C \to H$ be η_k-inverse strongly monotone for $k = 1, 2, \ldots, M$. Let $\{S_i\}_{i=1}^N$ be a finite family of nonexpansive mappings on H. Let $\widetilde{A}_1 : H \to H$ be α-inverse strongly monotone and let $\widetilde{A}_2 : H \to H$ be β-strongly monotone and κ-Lipschitz continuous. Assume that $\mathrm{VI}(\cap_{i=1}^N \mathrm{Fix}(S_i), \widetilde{A}_1) \neq \emptyset$ with $(\cap_{i=1}^N \mathrm{Fix}(S_i)) \subset (\cap_{j=1}^K \mathrm{GMEP}(\Theta_j, \varphi_j, A_j)) \cap (\cap_{k=1}^M I(B_k, R_k)) \cap \Gamma$. Let $\mu \in (0, 2\beta/\kappa^2)$, $\{\alpha_n\}_{n=0}^\infty \subset (0, 1]$, $\{\rho_n\}_{n=0}^\infty \subset (0, 2\alpha]$, $\{\lambda_{k,n}\}_{n=0}^\infty \subset [a_k, b_k] \subset (0, 2\eta_k)$, and $\{r_{j,n}\}_{n=0}^\infty \subset [c_j, d_j] \subset (0, 2\zeta_j)$ where*

$j \in \{1,2,\ldots,K\}$ and $k \in \{1,2,\ldots,M\}$. For arbitrarily given $x_0 \in H$, let $\{x_n\}$ be a sequence generated by

$$\begin{aligned} u_n &= T_{r_{K,n}}^{(\Theta_K,\varphi_K)}\left(I - r_{K,n}A_K\right) T_{r_{K-1,n}}^{(\Theta_{K-1},\varphi_{K-1})}\left(I - r_{K-1,n}A_{K-1}\right) \\ &\quad \cdots T_{r_{1,n}}^{(\Theta_1,\varphi_1)}\left(I - r_{1,n}A_1\right) x_n, \\ v_n &= J_{R_M,\lambda_{M,n}}\left(I - \lambda_{M,n}B_M\right) J_{R_{M-1},\lambda_{M-1,n}}\left(I - \lambda_{M-1,n}B_{M-1}\right) \\ &\quad \cdots J_{R_1,\lambda_{1,n}}\left(I - \lambda_{1,n}B_1\right) u_n, \\ y_n &= S_{[n+1]}\left(I - \rho_n\widetilde{A}_1\right) T_n v_n, \\ x_{n+1} &= y_n - \mu\alpha_n\widetilde{A}_2 y_n, \quad \forall n \geq 0, \end{aligned} \tag{43}$$

where $P_C(I-\lambda_n\nabla f) = s_nI + (1-s_n)T_n$ (here T_n is nonexpansive and $s_n := s_n(\lambda_n) = (2-\lambda_nL)/4 \in (0,1/2)$ for each $\lambda_n \in (0,2/L)$). Assume that

$$\begin{aligned} \bigcap_{i=1}^{N} \operatorname{Fix}(S_i) &= \operatorname{Fix}(S_1S_2\cdots S_N) \\ &= \operatorname{Fix}(S_NS_1\cdots S_{N-1}) \\ &= \cdots = \operatorname{Fix}(S_2S_3\cdots S_NS_1) \end{aligned} \tag{44}$$

and that the following conditions are satisfied:

(i) $\lim_{n\to\infty}\alpha_n = 0, \sum_{n=0}^{\infty}\alpha_n = \infty$ *and* $\rho_n \leq \alpha_n$ *for all* $n \geq 0$;

(ii) $\lim_{n\to\infty}(|\alpha_n - \alpha_{n+N}|/\alpha_{n+N}) = 0$ *or* $\sum_{n=0}^{\infty}|\alpha_n - \alpha_{n+N}| < \infty$;

(iii) $\lim_{n\to\infty}(|s_n - s_{n+N}|/\alpha_{n+N}) = 0$ *or* $\sum_{n=0}^{\infty}|s_n - s_{n+N}| < \infty$;

(iv) $\lim_{n\to\infty}(|\rho_n - \rho_{n+N}|/\rho_{n+N}) = 0$ *or* $\sum_{n=0}^{\infty}|\rho_n - \rho_{n+N}| < \infty$;

(v) $\lim_{n\to\infty}(|\lambda_{k,n} - \lambda_{k,n+N}|/(\alpha_{n+N})) = 0$ *or* $\sum_{n=0}^{\infty}|\lambda_{k,n} - \lambda_{k,n+N}| < \infty$ *for* $k = 1,2,\ldots,M$;

(vi) $\lim_{n\to\infty}(|r_{j,n} - r_{j,n+N}|/(\alpha_{n+N})) = 0$ *or* $\sum_{n=0}^{\infty}|r_{j,n} - r_{j,n+N}| < \infty$ *for* $j = 1,2,\ldots,K$.

Then the following hold:

(a) $\{x_n\}_{n=0}^{\infty}$ *is bounded;*

(b) $\lim_{n\to\infty}\|x_n - x_{n+N}\| = 0$;

(c) $\lim_{n\to\infty}\|x_n - S_{[n+N]}\cdots S_{[n+1]}x_n\| = 0$ *provided* $\lim_{n\to\infty}(\|x_n - y_n\| + \|T_nv_n - v_n\|) = 0$;

(d) $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the unique element of* $\mathrm{VI}(\mathrm{VI}(\cap_{i=1}^{N}\operatorname{Fix}(S_i),\widetilde{A}_1),\widetilde{A}_2)$ *provided* $\|x_n - y_n\| + \|T_nv_n - v_n\| = o(\rho_n)$.

Proof. Let $\{x^*\} = \mathrm{VI}(\mathrm{VI}(\Omega,\widetilde{A}_1),\widetilde{A}_2)$. Since ∇f is L-Lipschitzian, it follows that ∇f is $1/L$-ism. By Lemma 7(ii), we know that for $\lambda > 0$, $\lambda\nabla f$ is $1/\lambda L$-ism. So by Lemma 7(iii), we deduce that $I - \lambda\nabla f$ is $\lambda L/2$-averaged. Now since the projection P_C is $1/2$-averaged, it is easy to see from Lemma 8(iv) that the composite $P_C(I-\lambda\nabla f)$ is $(2+\lambda L)/4$-averaged for $\lambda \in (0,2/L)$. Hence we obtain that, for each $n \geq 0$, $P_C(I-\lambda_n\nabla f)$ is $(2+\lambda_nL)/4$-averaged for each $\lambda_n \in (0,2/L)$. Therefore, we can write

$$\begin{aligned} P_C\left(I - \lambda_n\nabla f\right) &= \frac{2-\lambda_nL}{4}I + \frac{2+\lambda_nL}{4}T_n \\ &= s_nI + \left(1 - s_n\right)T_n, \end{aligned} \tag{45}$$

where T_n is nonexpansive and $s_n := s_n(\lambda_n) = (2-\lambda_nL)/4 \in (0,1/2)$ for each $\lambda_n \in (0,2/L)$. Since $\widetilde{A}_2$ is κ-Lipschitz continuous, we get

$$\left\|\widetilde{A}_2y_n - \widetilde{A}_2x^*\right\| \leq \kappa\left\|y_n - x^*\right\|, \quad \forall n \geq 0. \tag{46}$$

Putting $z_n = (I - \rho_n\widetilde{A}_1)T_nv_n$, for all $n \geq 0$, we have

$$\begin{aligned} x_{n+1} &= y_n - \mu\alpha_n\widetilde{A}_2y_n \\ &= S_{[n+1]}z_n - \mu\alpha_n\widetilde{A}_2S_{[n+1]}z_n \\ &= S_{[n+1]}^{\alpha_n}z_n, \quad \forall n \geq 0. \end{aligned} \tag{47}$$

Put

$$\begin{aligned} \Delta_n^j &= T_{r_{j,n}}^{(\Theta_j,\varphi_j)}\left(I - r_{j,n}A_j\right) T_{r_{j-1,n}}^{(\Theta_{j-1},\varphi_{j-1})}\left(I - r_{j-1,n}A_{j-1}\right) \\ &\quad \cdots T_{r_{1,n}}^{(\Theta_1,\varphi_1)}\left(I - r_{1,n}A_1\right) x_n \end{aligned} \tag{48}$$

for all $j \in \{1,2,\ldots,K\}$ and $n \geq 0$,

$$\begin{aligned} \Lambda_n^k &= J_{R_k,\lambda_{k,n}}\left(I - \lambda_{k,n}B_k\right) J_{R_{k-1},\lambda_{k-1,n}}\left(I - \lambda_{k-1,n}B_{k-1}\right) \\ &\quad \cdots J_{R_1,\lambda_{1,n}}\left(I - \lambda_{1,n}B_1\right) \end{aligned} \tag{49}$$

for all $k \in \{1,2,\ldots,M\}$, $\Delta_n^0 = I$, and $\Lambda_n^0 = I$, where I is the identity mapping on H. Then we have that $u_n = \Delta_n^Kx_n$ and $v_n = \Lambda_n^Mu_n$.

We divide the rest of the proof into several steps.

Step 1. We prove that $\{x_n\}$ is bounded.

Indeed, utilizing (18) and Proposition 3(ii), we have

$$\begin{aligned} &\left\|u_n - x^*\right\| \\ &= \Big\|T_{r_{K,n}}^{(\Theta_K,\varphi_K)}\left(I - r_{K,n}B_K\right)\Delta_n^{K-1}x_n \\ &\qquad - T_{r_{K,n}}^{(\Theta_K,\varphi_K)}\left(I - r_{K,n}B_K\right)\Delta_n^{K-1}x^*\Big\| \\ &\leq \left\|\left(I - r_{K,n}B_K\right)\Delta_n^{K-1}x_n - \left(I - r_{K,n}B_K\right)\Delta_n^{K-1}x^*\right\| \\ &\leq \left\|\Delta_n^{K-1}x_n - \Delta_n^{K-1}x^*\right\| \\ &\quad \vdots \\ &\leq \left\|\Delta_n^0x_n - \Delta_n^0x^*\right\| \\ &= \left\|x_n - x^*\right\|. \end{aligned} \tag{50}$$

Utilizing (18) and Lemma 14 we have

$$\begin{aligned}
&\|v_n - x^*\| \\
&= \left\| J_{R_M,\lambda_{M,n}} (I - \lambda_{M,n} A_M) \Lambda_n^{M-1} u_n \right. \\
&\quad \left. - J_{R_M,\lambda_{M,n}} (I - \lambda_{M,n} A_M) \Lambda_n^{M-1} x^* \right\| \\
&\le \left\| (I - \lambda_{M,n} A_M) \Lambda_n^{M-1} u_n - (I - \lambda_{M,n} A_M) \Lambda_n^{M-1} x^* \right\| \\
&\le \left\| \Lambda_n^{M-1} u_n - \Lambda_n^{M-1} x^* \right\| \\
&\vdots \\
&\le \left\| \Lambda_n^{0} u_n - \Lambda_n^{0} x^* \right\| \\
&= \|u_n - x^*\|.
\end{aligned} \tag{51}$$

Combining (50) and (51), we have

$$\|v_n - x^*\| \le \|x_n - x^*\|. \tag{52}$$

Since $\widetilde{A}_1$ is α-inverse strongly monotone and $\{\rho_n\}_{n=0}^{\infty} \subset (0, 2\alpha]$, we have

$$\begin{aligned}
&\left\| T_n v_n - x^* - \rho_n (\widetilde{A}_1 T_n v_n - \widetilde{A}_1 x^*) \right\|^2 \\
&= \|T_n v_n - x^*\|^2 \\
&\quad - 2\rho_n \left\langle \widetilde{A}_1 T_n v_n - \widetilde{A}_1 x^*, T_n v_n - x^* \right\rangle \\
&\quad + \rho_n^2 \left\| \widetilde{A}_1 T_n v_n - \widetilde{A}_1 x^* \right\|^2 \\
&\le \|T_n v_n - x^*\|^2 - \rho_n (2\alpha - \rho_n) \left\| \widetilde{A}_1 T_n v_n - \widetilde{A}_1 x^* \right\|^2 \\
&\le \|T_n v_n - x^*\|^2 \\
&\le \|v_n - x^*\|^2.
\end{aligned} \tag{53}$$

Utilizing Lemma 11, we deduce from (52), $\rho_n \le \alpha_n$, and $S_{[n+1]}^{\alpha_n} x^* = x^* - \alpha_n \mu \widetilde{A}_2 x^*$ that for all $n \ge 0$

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&= \left\| S_{[n+1]}^{\alpha_n} z_n - x^* \right\| \\
&\le \left\| S_{[n+1]}^{\alpha_n} z_n - S_{[n+1]}^{\alpha_n} x^* \right\| + \left\| S_{[n+1]}^{\alpha_n} x^* - x^* \right\| \\
&\le (1 - \alpha_n \tau) \|z_n - x^*\| + \alpha_n \mu \left\| \widetilde{A}_2 x^* \right\| \\
&= (1 - \alpha_n \tau) \left\| \left(I - \rho_n \widetilde{A}_1\right) T_n v_n - x^* \right\| + \alpha_n \mu \left\| \widetilde{A}_2 x^* \right\| \\
&= (1 - \alpha_n \tau) \left\| T_n v_n - x^* - \rho_n \left(\widetilde{A}_1 T_n v_n - \widetilde{A}_1 x^*\right) - \rho_n \widetilde{A}_1 x^* \right\| \\
&\quad + \alpha_n \mu \left\| \widetilde{A}_2 x^* \right\| \\
&\le (1 - \alpha_n \tau) \left[\left\| T_n v_n - x^* - \rho_n \left(\widetilde{A}_1 T_n v_n - \widetilde{A}_1 x^*\right) \right\| \right. \\
&\qquad \left. + \rho_n \left\| \widetilde{A}_1 x^* \right\| \right] + \alpha_n \mu \left\| \widetilde{A}_2 x^* \right\| \\
&\le (1 - \alpha_n \tau) \left[\|v_n - x^*\| + \rho_n \left\| \widetilde{A}_1 x^* \right\| \right] + \alpha_n \mu \left\| \widetilde{A}_2 x^* \right\| \\
&\le (1 - \alpha_n \tau) \left[\|x_n - x^*\| + \rho_n \left\| \widetilde{A}_1 x^* \right\| \right] + \alpha_n \mu \left\| \widetilde{A}_2 x^* \right\| \\
&\le (1 - \alpha_n \tau) \|x_n - x^*\| + \rho_n \left\| \widetilde{A}_1 x^* \right\| + \alpha_n \mu \left\| \widetilde{A}_2 x^* \right\| \\
&\le (1 - \alpha_n \tau) \|x_n - x^*\| + \alpha_n \left\| \widetilde{A}_1 x^* \right\| \\
&\quad + \alpha_n \mu \left\| \widetilde{A}_2 x^* \right\| \\
&= (1 - \alpha_n \tau) \|x_n - x^*\| \\
&\quad + \alpha_n \tau \frac{\left\| \widetilde{A}_1 x^* \right\| + \mu \left\| \widetilde{A}_2 x^* \right\|}{\tau} \\
&\le \max \left\{ \|x_n - x^*\|, \frac{\left\| \widetilde{A}_1 x^* \right\| + \mu \left\| \widetilde{A}_2 x^* \right\|}{\tau} \right\},
\end{aligned} \tag{54}$$

where $\tau = 1 - \sqrt{1 - \mu(2\beta - \mu\kappa^2)}$. So, by induction we obtain

$$\begin{aligned}
&\|x_n - x^*\| \\
&\le \max \left\{ \|x_0 - x^*\|, \frac{\left\| \widetilde{A}_1 x^* \right\| + \mu \left\| \widetilde{A}_2 x^* \right\|}{\tau} \right\}, \quad \forall n \ge 0.
\end{aligned} \tag{55}$$

Hence $\{x_n\}_{n=0}^{\infty}$ is bounded. Since $\widetilde{A}_1 : H \to H$ is α-inverse strongly monotone, it is known that $\widetilde{A}_1$ is $1/\alpha$-Lipschitz continuous. Thus, from (52), we get

$$\begin{aligned}
\left\| \widetilde{A}_1 T_n v_n - \widetilde{A}_1 x^* \right\| &\le \frac{1}{\alpha} \|T_n v_n - x^*\| \le \frac{1}{\alpha} \|v_n - x^*\| \\
&\le \frac{1}{\alpha} \|x_n - x^*\|, \quad \forall n \ge 0.
\end{aligned} \tag{56}$$

Consequently, the boundedness of $\{x_n\}$ ensures the boundedness of $\{v_n\}$, $\{T_n v_n\}$, and $\{\widetilde{A}_1 T_n v_n\}$. From $y_n = S_{[n+1]}(I - \rho_n \widetilde{A}_1) T_n v_n$ and the nonexpansivity of $S_{[n+1]}$, it follows that $\{y_n\}$ is bounded. Since $\widetilde{A}_2$ is κ-Lipschitz continuous, $\{\widetilde{A}_2 y_n\}$ is also bounded.

Step 2. We prove that $\lim_{n\to\infty} \|x_n - x_{n+N}\| = 0$.

Indeed, utilizing (18) and (40), we obtain that

$$\begin{aligned}
&\|v_{n+N}-v_n\| \\
&= \left\|\Lambda_{n+N}^{M}u_{n+N}-\Lambda_{n}^{M}u_n\right\| \\
&= \left\|J_{R_M,\lambda_{M,n+N}}\left(I-\lambda_{M,n+N}B_M\right)\Lambda_{n+N}^{M-1}u_{n+N}\right. \\
&\qquad \left.-J_{R_M,\lambda_{M,n}}\left(I-\lambda_{M,n}B_M\right)\Lambda_{n}^{M-1}u_n\right\| \\
&\le \left\|J_{R_M,\lambda_{M,n+N}}\left(I-\lambda_{M,n+N}B_M\right)\Lambda_{n+N}^{M-1}u_{n+N}\right. \\
&\qquad \left.-J_{R_M,\lambda_{M,n+N}}\left(I-\lambda_{M,n}B_M\right)\Lambda_{n+N}^{M-1}u_{n+N}\right\| \\
&\quad + \left\|J_{R_M,\lambda_{M,n+N}}\left(I-\lambda_{M,n}B_M\right)\Lambda_{n+N}^{M-1}u_{n+N}\right. \\
&\qquad \left.-J_{R_M,\lambda_{M,n}}\left(I-\lambda_{M,n}B_M\right)\Lambda_{n}^{M-1}u_n\right\| \\
&\le \left\|\left(I-\lambda_{M,n+N}B_M\right)\Lambda_{n+N}^{M-1}u_{n+N}-\left(I-\lambda_{M,n}B_M\right)\Lambda_{n+N}^{M-1}u_{n+N}\right\| \\
&\quad + \left\|\left(I-\lambda_{M,n}B_M\right)\Lambda_{n+N}^{M-1}u_{n+N}-\left(I-\lambda_{M,n}B_M\right)\Lambda_{n}^{M-1}u_n\right\| \\
&\quad + \left|\lambda_{M,n+N}-\lambda_{M,n}\right| \\
&\quad \times\left(\frac{1}{\lambda_{M,n+N}}\left\|J_{R_M,\lambda_{M,n+N}}\left(I-\lambda_{M,n}B_M\right)\Lambda_{n+N}^{M-1}u_{n+N}\right.\right. \\
&\qquad \left.-\left(I-\lambda_{M,n}B_M\right)\Lambda_{n}^{M-1}u_n\right\| \\
&\qquad + \frac{1}{\lambda_{M,n}}\left\|\left(I-\lambda_{M,n}B_M\right)\Lambda_{n+N}^{M-1}u_{n+N}\right. \\
&\qquad \left.\left.-J_{R_M,\lambda_{M,n}}\left(I-\lambda_{M,n}B_M\right)\Lambda_{n}^{M-1}u_n\right\|\right) \\
&\le \left|\lambda_{M,n+N}-\lambda_{M,n}\right|\left(\left\|B_M\Lambda_{n+N}^{M-1}u_{n+N}\right\|+\widetilde{M}\right) \\
&\quad + \left\|\Lambda_{n+N}^{M-1}u_{n+N}-\Lambda_{n}^{M-1}u_n\right\| \\
&\le \left|\lambda_{M,n+N}-\lambda_{M,n}\right|\left(\left\|B_M\Lambda_{n+N}^{M-1}u_{n+N}\right\|+\widetilde{M}\right) \\
&\quad + \left|\lambda_{M-1,n+N}-\lambda_{M-1,n}\right|\left(\left\|B_{M-1}\Lambda_{n+N}^{M-2}u_{n+N}\right\|+\widetilde{M}\right) \\
&\quad + \left\|\Lambda_{n+N}^{M-2}u_{n+N}-\Lambda_{n}^{M-2}u_n\right\| \\
&\quad \vdots \\
&\le \left|\lambda_{M,n+N}-\lambda_{M,n}\right|\left(\left\|B_M\Lambda_{n+N}^{M-1}u_{n+N}\right\|+\widetilde{M}\right) \\
&\quad + \left|\lambda_{M-1,n+N}-\lambda_{M-1,n}\right|\left(\left\|B_{M-1}\Lambda_{n+N}^{M-2}u_{n+N}\right\|+\widetilde{M}\right) \\
&\quad + \cdots + \left|\lambda_{1,n+N}-\lambda_{1,n}\right|\left(\left\|B_{1}\Lambda_{n+N}^{0}u_{n+N}\right\|+\widetilde{M}\right) \\
&\quad + \left\|\Lambda_{n+N}^{0}u_{n+N}-\Lambda_{n}^{0}u_n\right\| \\
&\le \widetilde{M}_0\sum_{k=1}^{M}\left|\lambda_{k,n+N}-\lambda_{k,n}\right|+\left\|u_{n+N}-u_n\right\|,
\end{aligned} \tag{57}$$

where

$$\begin{aligned}
&\sup_{n\ge 0,1\le i\le M}\left\{\frac{1}{\lambda_{i,n+N}}\left\|J_{R_i,\lambda_{i,n+N}}\left(I-\lambda_{i,n}B_i\right)\Lambda_{n+N}^{i-1}u_{n+N}\right.\right. \\
&\qquad \left.-\left(I-\lambda_{i,n}B_i\right)\Lambda_{n}^{i-1}u_n\right\| \\
&\qquad + \frac{1}{\lambda_{i,n}}\left\|\left(I-\lambda_{i,n}B_i\right)\Lambda_{n+N}^{i-1}u_{n+N}\right. \\
&\qquad \left.\left.-J_{R_i,\lambda_{i,n}}\left(I-\lambda_{i,n}B_i\right)\Lambda_{n}^{i-1}u_n\right\|\right\}\le\widetilde{M},
\end{aligned} \tag{58}$$

for some $\widetilde{M}>0$ and $\sup_{n\ge 0}\{\sum_{k=1}^{M}\|B_k\Lambda_{n+N}^{k-1}u_{n+N}\|+\widetilde{M}\}\le\widetilde{M}_0$ for some $\widetilde{M}_0>0$.

Furthermore, since ∇f is $1/L$-ism, $P_C(I-\lambda_n\nabla f)$ is nonexpansive for $\lambda_n\in(0,2/L)$. So, it follows that

$$\begin{aligned}
&\left\|P_C\left(I-\lambda_{n+N}\nabla f\right)v_n\right\| \\
&\quad \le \left\|P_C\left(I-\lambda_{n+N}\nabla f\right)v_n-x^*\right\|+\left\|x^*\right\| \\
&\quad = \left\|P_C\left(I-\lambda_{n+N}\nabla f\right)v_n-P_C\left(I-\lambda_{n+N}\nabla f\right)x^*\right\|+\left\|x^*\right\| \\
&\quad \le \left\|v_n-x^*\right\|+\left\|x^*\right\| \\
&\quad \le \left\|v_n\right\|+2\left\|x^*\right\|.
\end{aligned} \tag{59}$$

With the boundedness of $\{v_n\}$, this implies that $\{P_C(I-\lambda_{n+N}\nabla f)v_n\}$ is bounded. Also, observe that

$$\begin{aligned}
&\|T_{n+N}v_n-T_nv_n\| \\
&= \left\|\frac{4P_C\left(I-\lambda_{n+N}\nabla f\right)-\left(2-\lambda_{n+N}L\right)I}{2+\lambda_{n+N}L}v_n\right. \\
&\qquad \left.-\frac{4P_C\left(I-\lambda_{n}\nabla f\right)-\left(2-\lambda_{n}L\right)I}{2+\lambda_{n}L}v_n\right\| \\
&\le \left\|\frac{4P_C\left(I-\lambda_{n+N}\nabla f\right)}{2+\lambda_{n+N}L}v_n-\frac{4P_C\left(I-\lambda_{n}\nabla f\right)}{2+\lambda_{n}L}v_n\right\| \\
&\quad + \left\|\frac{2-\lambda_nL}{2+\lambda_nL}v_n-\frac{2-\lambda_{n+N}L}{2+\lambda_{n+N}L}v_n\right\| \\
&= \left\|\left(4\left(2+\lambda_nL\right)P_C\left(I-\lambda_{n+N}\nabla f\right)v_n\right.\right. \\
&\qquad -4\left(2+\lambda_{n+N}L\right)P_C\left(I-\lambda_n\nabla f\right)v_n) \\
&\qquad \left.\times\left((2+\lambda_{n+N}L)(2+\lambda_nL)\right)^{-1}\right\| \\
&\quad + \frac{4L\left|\lambda_{n+N}-\lambda_n\right|}{\left(2+\lambda_{n+N}L\right)\left(2+\lambda_nL\right)}\left\|v_n\right\|
\end{aligned}$$

$$
\begin{aligned}
&= \Big\|\big(4L(\lambda_n-\lambda_{n+N})P_C(I-\lambda_{n+N}\nabla f)v_n + 4(2+\lambda_{n+N}L) \\
&\quad \times (P_C(I-\lambda_{n+N}\nabla f)v_n - P_C(I-\lambda_n\nabla f)v_n)) \\
&\quad \times ((2+\lambda_{n+N}L)(2+\lambda_n L))^{-1}\Big\| \\
&\quad + \frac{4L|\lambda_{n+N}-\lambda_n|}{(2+\lambda_{n+N}L)(2+\lambda_n L)}\|v_n\| \\
&\le \frac{4L|\lambda_n-\lambda_{n+N}|\,\|P_C(I-\lambda_{n+N}\nabla f)v_n\|}{(2+\lambda_{n+N}L)(2+\lambda_n L)} \\
&\quad + (4(2+\lambda_{n+N}L) \\
&\quad \times \|P_C(I-\lambda_{n+N}\nabla f)v_n - P_C(I-\lambda_n\nabla f)v_n\|) \\
&\quad \times ((2+\lambda_{n+N}L)(2+\lambda_n L))^{-1} \\
&\quad + \frac{4L|\lambda_{n+N}-\lambda_n|}{(2+\lambda_{n+N}L)(2+\lambda_n L)}\|v_n\| \\
&\le |\lambda_{n+N}-\lambda_n|\,[L\|P_C(I-\lambda_{n+N}\nabla f)v_n\| \\
&\qquad + 4\|\nabla f(v_n)\| + L\|v_n\|] \\
&\le \widetilde{M}_1|\lambda_{n+N}-\lambda_n|,
\end{aligned}
\tag{60}
$$

where $\sup_{n\ge 0}\{L\|P_C(I-\lambda_{n+N}\nabla f)v_n\|+4\|\nabla f(v_n)\|+L\|v_n\|\} \le \widetilde{M}_1$ for some $\widetilde{M}_1 > 0$. Thus, we conclude from (57) and (60) that

$$
\begin{aligned}
&\|T_{n+N}v_{n+N} - T_n v_n\| \\
&\quad \le \|T_{n+N}v_{n+N} - T_{n+N}v_n\| + \|T_{n+N}v_n - T_n v_n\| \\
&\quad \le \|v_{n+N} - v_n\| + \widetilde{M}_1|\lambda_{n+N}-\lambda_n| \\
&\quad \le \|v_{n+N} - v_n\| + \frac{4\widetilde{M}_1}{L}|s_{n+N}-s_n| \\
&\quad \le \widetilde{M}_0\sum_{k=1}^{M}|\lambda_{k,n+N}-\lambda_{k,n}| + \|u_{n+N}-u_n\| \\
&\qquad + \frac{4\widetilde{M}_1}{L}|s_{n+N}-s_n|.
\end{aligned}
\tag{61}
$$

Also, utilizing Proposition 3(ii), (v), we deduce that

$$
\begin{aligned}
&\|u_{n+N}-u_n\| \\
&= \|\Delta^K_{n+N}x_{n+N} - \Delta^K_n x_n\| \\
&= \Big\|T^{(\Theta_K,\varphi_K)}_{r_{K,n+N}}(I-r_{K,n+N}A_K)\Delta^{K-1}_{n+N}x_{n+N} \\
&\qquad - T^{(\Theta_K,\varphi_K)}_{r_{K,n}}(I-r_{K,n}A_K)\Delta^{K-1}_n x_n\Big\| \\
&\le \Big\|T^{(\Theta_K,\varphi_K)}_{r_{K,n+N}}(I-r_{K,n+N}A_K)\Delta^{K-1}_{n+N}x_{n+N} \\
&\qquad - T^{(\Theta_K,\varphi_K)}_{r_{K,n}}(I-r_{K,n}A_K)\Delta^{K-1}_{n+N}x_{n+N}\Big\| \\
&\qquad + \Big\|T^{(\Theta_K,\varphi_K)}_{r_{K,n}}(I-r_{K,n}A_K)\Delta^{K-1}_{n+N}x_{n+N} \\
&\qquad\quad - T^{(\Theta_K,\varphi_K)}_{r_{K,n}}(I-r_{K,n}A_K)\Delta^{K-1}_n x_n\Big\| \\
&\le \Big\|T^{(\Theta_K,\varphi_K)}_{r_{K,n+N}}(I-r_{K,n+N}A_K)\Delta^{K-1}_{n+N}x_{n+N} \\
&\qquad - T^{(\Theta_K,\varphi_K)}_{r_{K,n}}(I-r_{K,n+N}A_K)\Delta^{K-1}_{n+N}x_{n+N}\Big\| \\
&\qquad + \Big\|T^{(\Theta_K,\varphi_K)}_{r_{K,n}}(I-r_{K,n+N}A_K)\Delta^{K-1}_{n+N}x_{n+N} \\
&\qquad\quad - T^{(\Theta_K,\varphi_K)}_{r_{K,n}}(I-r_{K,n}A_K)\Delta^{K-1}_{n+N}x_{n+N}\Big\| \\
&\qquad + \|(I-r_{K,n}A_K)\Delta^{K-1}_{n+N}x_{n+N} - (I-r_{K,n}A_K)\Delta^{K-1}_n x_n\| \\
&\le \frac{|r_{K,n+N}-r_{K,n}|}{r_{K,n+N}} \\
&\qquad \times \Big\|T^{(\Theta_K,\varphi_K)}_{r_{K,n+N}}(I-r_{K,n+N}A_K)\Delta^{K-1}_{n+N}x_{n+N} \\
&\qquad\quad - (I-r_{K,n+N}A_K)\Delta^{K-1}_{n+N}x_{n+N}\Big\| \\
&\qquad + |r_{K,n+N}-r_{K,n}|\,\|A_K\Delta^{K-1}_{n+N}x_{n+N}\| \\
&\qquad + \|\Delta^{K-1}_{n+N}x_{n+N} - \Delta^{K-1}_n x_n\| \\
&= |r_{K,n+N}-r_{K,n}|\Big[\|A_K\Delta^{K-1}_{n+N}x_{n+N}\| + \frac{1}{r_{K,n+N}} \\
&\qquad\qquad \times \Big\|T^{(\Theta_K,\varphi_K)}_{r_{K,n+N}}(I-r_{K,n+N}A_K)\Delta^{K-1}_{n+N}x_{n+N} \\
&\qquad\qquad\quad - (I-r_{K,n+N}A_K)\Delta^{K-1}_{n+N}x_{n+N}\Big\|\Big] \\
&\qquad + \|\Delta^{K-1}_{n+N}x_{n+N} - \Delta^{K-1}_n x_n\| \\
&\quad \vdots \\
&\le |r_{K,n+N}-r_{K,n}|\Big[\|A_K\Delta^{K-1}_{n+N}x_{n+N}\| + \frac{1}{r_{K,n+N}} \\
&\qquad\qquad \times \Big\|T^{(\Theta_K,\varphi_K)}_{r_{K,n+N}}(I-r_{K,n+N}A_K)\Delta^{K-1}_{n+N}x_{n+N} \\
&\qquad\qquad\quad - (I-r_{K,n+N}A_K)\Delta^{K-1}_{n+N}x_{n+N}\Big\|\Big] \\
&\qquad + \cdots + |r_{1,n+N}-r_{1,n}| \\
&\qquad \times \Big[\|A_1\Delta^0_{n+N}x_{n+N}\| + \frac{1}{r_{1,n+N}} \\
&\qquad\quad \times \Big\|T^{(\Theta_1,\varphi_1)}_{r_{1,n+N}}(I-r_{1,n+N}A_1)\Delta^0_{n+N}x_{n+N} \\
&\qquad\qquad - (I-r_{1,n+N}A_1)\Delta^0_{n+N}x_{n+N}\Big\|\Big]
\end{aligned}
$$

$$+ \left\| \Delta^0_{n+N} x_{n+N} - \Delta^0_n x_n \right\|$$
$$\leq \widetilde{M}_2 \sum_{j=1}^{K} \left| r_{j,n+N} - r_{j,n} \right| + \left\| x_{n+N} - x_n \right\|, \tag{62}$$

where $\widetilde{M}_2 > 0$ is a constant such that for each $n \geq 0$

$$\sum_{j=1}^{K} \left[\left\| A_j \Delta^{j-1}_{n+N} x_{n+N} \right\| + \frac{1}{r_{j,n+N}} \left\| T^{(\Theta_j,\varphi_j)}_{r_{j,n+N}} \left(I - r_{j,n+N} A_j \right) \Delta^{j-1}_{n+N} x_{n+N} - \left(I - r_{j,n+N} A_j \right) \Delta^{j-1}_{n+N} x_{n+N} \right\| \right] \leq \widetilde{M}_2. \tag{63}$$

Therefore, it follows from (18), (61), (62), and $\{\rho_n\}_{n=0}^{\infty} \subset (0, 2\alpha]$ that

$$\begin{aligned}
&\left\| z_{n+N} - z_n \right\| \\
&= \left\| \left(T_{n+N} v_{n+N} - \rho_{n+N} \widetilde{A}_1 T_{n+N} v_{n+N} \right) - \left(T_n v_n - \rho_n \widetilde{A}_1 T_n v_n \right) \right\| \\
&\leq \left\| \left(T_{n+N} v_{n+N} - \rho_{n+N} \widetilde{A}_1 T_{n+N} v_{n+N} \right) - \left(T_n v_n - \rho_{n+N} \widetilde{A}_1 T_n v_n \right) \right\| \\
&\quad + \left\| \left(T_n v_n - \rho_{n+N} \widetilde{A}_1 T_n v_n \right) - \left(T_n v_n - \rho_n \widetilde{A}_1 T_n v_n \right) \right\| \\
&\leq \left\| T_{n+N} v_{n+N} - T_n v_n \right\| + \left| \rho_{n+N} - \rho_n \right| \left\| \widetilde{A}_1 T_n v_n \right\| \\
&\leq \widetilde{M}_0 \sum_{k=1}^{M} \left| \lambda_{k,n+N} - \lambda_{k,n} \right| + \left\| u_{n+N} - u_n \right\| \\
&\quad + \frac{4\widetilde{M}_1}{L} \left| s_{n+N} - s_n \right| + \left| \rho_{n+N} - \rho_n \right| \left\| \widetilde{A}_1 T_n v_n \right\| \\
&\leq \widetilde{M}_0 \sum_{k=1}^{M} \left| \lambda_{k,n+N} - \lambda_{k,n} \right| + \widetilde{M}_2 \sum_{j=1}^{K} \left| r_{j,n+N} - r_{j,n} \right| \\
&\quad + \left\| x_{n+N} - x_n \right\| + \frac{4\widetilde{M}_1}{L} \left| s_{n+N} - s_n \right| + \left| \rho_{n+N} - \rho_n \right| \left\| \widetilde{A}_1 T_n v_n \right\|.
\end{aligned} \tag{64}$$

From Lemma 11 and (64), it is found that

$$\begin{aligned}
&\left\| x_{n+N} - x_n \right\| \\
&= \left\| y_{n+N-1} - \mu \alpha_{n+N-1} \widetilde{A}_2 y_{n+N-1} - \left(y_{n-1} - \mu \alpha_{n-1} \widetilde{A}_2 y_{n-1} \right) \right\| \\
&= \left\| S^{\alpha_{n+N-1}}_{[n+N]} z_{n+N-1} - S^{\alpha_{n-1}}_{[n]} z_{n-1} \right\| \\
&\leq \left\| S^{\alpha_{n+N-1}}_{[n+N]} z_{n+N-1} - S^{\alpha_{n+N-1}}_{[n+N]} z_{n-1} \right\| \\
&\quad + \left\| S^{\alpha_{n+N-1}}_{[n+N]} z_{n-1} - S^{\alpha_{n-1}}_{[n]} z_{n-1} \right\| \\
&\leq \left(1 - \alpha_{n+N-1} \tau \right) \left\| z_{n+N-1} - z_{n-1} \right\| \\
&\quad + \mu \left| \alpha_{n+N-1} - \alpha_{n-1} \right| \left\| \widetilde{A}_2 S_{[n]} z_{n-1} \right\| \\
&= \left(1 - \alpha_{n+N-1} \tau \right) \left\| z_{n+N-1} - z_{n-1} \right\| \\
&\quad + \mu \left| \alpha_{n+N-1} - \alpha_{n-1} \right| \left\| \widetilde{A}_2 y_{n-1} \right\| \\
&\leq \left(1 - \alpha_{n+N-1} \tau \right) \\
&\quad \times \left[\widetilde{M}_0 \sum_{k=1}^{M} \left| \lambda_{k,n+N-1} - \lambda_{k,n-1} \right| + \widetilde{M}_2 \sum_{j=1}^{K} \left| r_{j,n+N-1} - r_{j,n-1} \right| + \left\| x_{n+N-1} - x_{n-1} \right\| \right. \\
&\quad \left. + \frac{4\widetilde{M}_1}{L} \left| s_{n+N-1} - s_{n-1} \right| + \left| \rho_{n+N-1} - \rho_{n-1} \right| \left\| \widetilde{A}_1 T_{n-1} v_{n-1} \right\| \right] \\
&\quad + \mu \left| \alpha_{n+N-1} - \alpha_{n-1} \right| \left\| \widetilde{A}_2 y_{n-1} \right\| \\
&\leq \left(1 - \alpha_{n+N-1} \tau \right) \left\| x_{n+N-1} - x_{n-1} \right\| \\
&\quad + \widetilde{M}_0 \sum_{k=1}^{M} \left| \lambda_{k,n+N-1} - \lambda_{k,n-1} \right| + \widetilde{M}_2 \sum_{j=1}^{K} \left| r_{j,n+N-1} - r_{j,n-1} \right| \\
&\quad + \frac{4\widetilde{M}_1}{L} \left| s_{n+N-1} - s_{n-1} \right| + \left| \rho_{n+N-1} - \rho_{n-1} \right| \left\| \widetilde{A}_1 T_{n-1} v_{n-1} \right\| \\
&\quad + \mu \left| \alpha_{n+N-1} - \alpha_{n-1} \right| \left\| \widetilde{A}_2 y_{n-1} \right\| \\
&\leq \left(1 - \alpha_{n+N-1} \tau \right) \left\| x_{n+N-1} - x_{n-1} \right\| \\
&\quad + \widetilde{M}_3 \left(\sum_{k=1}^{M} \left| \lambda_{k,n+N-1} - \lambda_{k,n-1} \right| + \sum_{j=1}^{K} \left| r_{j,n+N-1} - r_{j,n-1} \right| \right. \\
&\quad \left. + \left| s_{n+N-1} - s_{n-1} \right| + \left| \rho_{n+N-1} - \rho_{n-1} \right| + \left| \alpha_{n+N-1} - \alpha_{n-1} \right| \right),
\end{aligned} \tag{65}$$

where $\sup_{n \geq 0} \{ \widetilde{M}_0 + 4\widetilde{M}_1 / L + \widetilde{M}_2 + \| \widetilde{A}_1 T_n v_n \| + \mu \| \widetilde{A}_2 y_n \| \} \leq \widetilde{M}_3$ for some $\widetilde{M}_3 > 0$. Applying Lemma 12 to (65) we obtain from conditions (i)–(vi) that

$$\lim_{n \to \infty} \left\| x_{n+N} - x_n \right\| = 0. \tag{66}$$

Step 3. We prove that $\lim_{n \to \infty} \| x_n - S_{[n+N]} \cdots S_{[n+1]} x_n \| = 0$ provided $\lim_{n \to \infty} (\| x_n - y_n \| + \| T_n v_n - v_n \|) = 0$.

Indeed, from $\| x_{n+1} - y_n \| = \mu \alpha_n \| \widetilde{A}_2 y_n \| \leq \alpha_n \widetilde{M}_3$ and condition (i), we get $\lim_{n \to \infty} \| x_{n+1} - y_n \| = 0$. Now, let us show that $\| u_n - x_n \| \to 0$, $\| v_n - u_n \| \to 0$ and $\| x_n - T_n v_n \| \to 0$

as $n \to \infty$. As a matter of fact, utilizing Lemma 4, we get from (43)

$$\begin{aligned}\|y_n - x^*\|^2 &= \|S_{[n+1]}(T_n v_n - \rho_n \widetilde{A}_1 T_n v_n) - x^*\|^2 \\ &\le \|T_n v_n - x^* - \rho_n \widetilde{A}_1 T_n v_n\|^2 \\ &\le \|T_n v_n - x^*\|^2 - 2\rho_n \langle \widetilde{A}_1 T_n v_n, z_n - x^* \rangle \\ &\le \|v_n - x^*\|^2 + 2\rho_n \|\widetilde{A}_1 T_n v_n\| \|z_n - x^*\|.\end{aligned} \tag{67}$$

Observe that

$$\begin{aligned}&\|\Delta_n^j x_n - x^*\|^2 \\ &= \|T_{r_{j,n}}^{(\Theta_j,\varphi_j)}(I - r_{j,n}A_j)\Delta_n^{j-1}x_n - T_{r_{j,n}}^{(\Theta_j,\varphi_j)}(I - r_{j,n}A_j)x^*\|^2 \\ &\le \|(I - r_{j,n}A_j)\Delta_n^{j-1}x_n - (I - r_{j,n}A_j)x^*\|^2 \\ &\le \|\Delta_n^{j-1}x_n - x^*\|^2 + r_{j,n}(r_{j,n} - 2\zeta_j)\|A_j\Delta_n^{j-1}x_n - A_j x^*\|^2 \\ &\le \|x_n - x^*\|^2 + r_{j,n}(r_{j,n} - 2\zeta_j)\|A_j\Delta_n^{j-1}x_n - A_j x^*\|^2, \\ &\|\Lambda_n^k u_n - x^*\|^2 \\ &= \|J_{R_k,\lambda_{k,n}}(I - \lambda_{k,n}B_k)\Lambda_n^{k-1}u_n - J_{R_k,\lambda_{k,n}}(I - \lambda_{k,n}B_k)x^*\|^2 \\ &\le \|(I - \lambda_{k,n}B_k)\Lambda_n^{k-1}u_n - (I - \lambda_{k,n}B_k)x^*\|^2 \\ &\le \|\Lambda_n^{k-1}u_n - x^*\|^2 + \lambda_{k,n}(\lambda_{k,n} - 2\eta_k)\|B_k\Lambda_n^{k-1}u_n - B_k x^*\|^2 \\ &\le \|u_n - x^*\|^2 + \lambda_{k,n}(\lambda_{k,n} - 2\eta_k)\|B_k\Lambda_n^{k-1}u_n - B_k x^*\|^2 \\ &\le \|x_n - x^*\|^2 + \lambda_{k,n}(\lambda_{k,n} - 2\eta_k)\|B_k\Lambda_n^{k-1}u_n - B_k x^*\|^2,\end{aligned} \tag{68}$$

for $j \in \{1, 2, \ldots, K\}$ and $k \in \{1, 2, \ldots, M\}$. Combining (67)-(68), we get

$$\begin{aligned}&\|y_n - x^*\|^2 \\ &\le \|v_n - x^*\|^2 + 2\rho_n \|\widetilde{A}_1 T_n v_n\| \|z_n - x^*\| \\ &\le \|\Lambda_n^k u_n - x^*\|^2 + 2\rho_n \|\widetilde{A}_1 T_n v_n\| \|z_n - x^*\| \\ &\le \|u_n - x^*\|^2 + \lambda_{k,n}(\lambda_{k,n} - 2\eta_k)\|B_k\Lambda_n^{k-1}u_n - B_k x^*\|^2 \\ &\quad + 2\rho_n \|\widetilde{A}_1 T_n v_n\| \|z_n - x^*\| \\ &\le \|\Delta_n^j x_n - x^*\|^2 + \lambda_{k,n}(\lambda_{k,n} - 2\eta_k)\|B_k\Lambda_n^{k-1}u_n - B_k x^*\|^2 \\ &\quad + 2\rho_n \|\widetilde{A}_1 T_n v_n\| \|z_n - x^*\| \\ &\le \|x_n - x^*\|^2 + r_{j,n}(r_{j,n} - 2\zeta_j)\|A_j\Delta_n^{j-1}x_n - A_j x^*\|^2 \\ &\quad + \lambda_{k,n}(\lambda_{k,n} - 2\eta_k)\|B_k\Lambda_n^{k-1}u_n - B_k x^*\|^2 \\ &\quad + 2\rho_n \|\widetilde{A}_1 T_n v_n\| \|z_n - x^*\|,\end{aligned} \tag{69}$$

which immediately yields

$$\begin{aligned}&r_{j,n}(2\zeta_j - r_{j,n})\|A_j\Delta_n^{j-1}x_n - A_j x^*\|^2 \\ &\quad + \lambda_{k,n}(2\eta_k - \lambda_{k,n})\|B_k\Lambda_n^{k-1}u_n - B_k x^*\|^2 \\ &\le \|x_n - x^*\|^2 - \|y_n - x^*\|^2 + 2\rho_n \|\widetilde{A}_1 T_n v_n\| \|z_n - x^*\| \\ &\le \|x_n - y_n\|(\|x_n - x^*\| + \|y_n - x^*\|) \\ &\quad + 2\rho_n \|\widetilde{A}_1 T_n v_n\| \|z_n - x^*\|.\end{aligned} \tag{70}$$

Since $\{\lambda_{k,n}\}_{n=0}^{\infty} \subset [a_k, b_k] \subset (0, 2\eta_k)$ and $\{r_{j,n}\}_{n=0}^{\infty} \subset [c_j, d_j] \subset (0, 2\zeta_j)$ for $j = 1, 2, \ldots, K$ and $k = 1, 2, \ldots, M$ and $\{x_n\}, \{y_n\}, \{\widetilde{A}_1 T_n v_n\}$ and $\{z_n\}$ are bounded sequences, we deduce from $\rho_n \to 0$ and $\|x_n - y_n\| \to 0$ that

$$\begin{aligned}\lim_{n\to\infty} \|A_j\Delta_n^{j-1}x_n - A_j x^*\| &= 0, \\ \lim_{n\to\infty} \|B_k\Lambda_n^{k-1}u_n - B_k x^*\| &= 0,\end{aligned} \tag{71}$$

for all $j \in \{1, 2, \ldots, K\}$ and $k \in \{1, 2, \ldots, M\}$.

Furthermore, by Proposition 3(ii) and Lemma 5(a), we have

$$\begin{aligned}&\|\Delta_n^j x_n - x^*\|^2 \\ &= \|T_{r_{j,n}}^{(\Theta_j,\varphi_j)}(I - r_{j,n}A_j)\Delta_n^{j-1}x_n - T_{r_{j,n}}^{(\Theta_j,\varphi_j)}(I - r_{j,n}A_j)x^*\|^2 \\ &\le \langle (I - r_{j,n}A_j)\Delta_n^{j-1}x_n - (I - r_{j,n}A_j)x^*, \Delta_n^j x_n - x^* \rangle \\ &= \frac{1}{2} \\ &\quad \times \Big(\|(I - r_{j,n}A_j)\Delta_n^{j-1}x_n - (I - r_{j,n}A_j)x^*\|^2 \\ &\qquad + \|\Delta_n^j x_n - x^*\|^2 \\ &\qquad - \|(I - r_{j,n}A_j)\Delta_n^{j-1}x_n - (I - r_{j,n}A_j)x^* - (\Delta_n^j x_n - x^*)\|^2\Big) \\ &\le \frac{1}{2}\Big(\|\Delta_n^{j-1}x_n - x^*\|^2 + \|\Delta_n^j x_n - x^*\|^2 \\ &\qquad - \|\Delta_n^{j-1}x_n - \Delta_n^j x_n - r_{j,n}(A_j\Delta_n^{j-1}x_n - A_j x^*)\|^2\Big),\end{aligned} \tag{72}$$

which implies that

$$\begin{aligned}
&\left\|\Delta_n^j x_n - x^*\right\|^2 \\
&\le \left\|\Delta_n^{j-1} x_n - x^*\right\|^2 \\
&\quad - \left\|\Delta_n^{j-1} x_n - \Delta_n^j x_n - r_{j,n}(A_j \Delta_n^{j-1} x_n - A_j x^*)\right\|^2 \\
&= \left\|\Delta_n^{j-1} x_n - x^*\right\|^2 - \left\|\Delta_n^{j-1} x_n - \Delta_n^j x_n\right\|^2 \\
&\quad - r_{j,n}^2 \left\|A_j \Delta_n^{j-1} x_n - A_j x^*\right\|^2 \\
&\quad + 2 r_{j,n} \left\langle \Delta_n^{j-1} x_n - \Delta_n^j x_n, A_j \Delta_n^{j-1} x_n - A_j x^* \right\rangle \\
&\le \left\|\Delta_n^{j-1} x_n - x^*\right\|^2 - \left\|\Delta_n^{j-1} x_n - \Delta_n^j x_n\right\|^2 \\
&\quad + 2 r_{j,n} \left\|\Delta_n^{j-1} x_n - \Delta_n^j x_n\right\| \left\|A_j \Delta_n^{j-1} x_n - A_j x^*\right\| \\
&\le \left\|x_n - x^*\right\|^2 - \left\|\Delta_n^{j-1} x_n - \Delta_n^j x_n\right\|^2 \\
&\quad + 2 r_{j,n} \left\|\Delta_n^{j-1} x_n - \Delta_n^j x_n\right\| \left\|A_j \Delta_n^{j-1} x_n - A_j x^*\right\|.
\end{aligned} \tag{73}$$

By Lemma 5(a) and Lemma 14, we obtain

$$\begin{aligned}
&\left\|\Lambda_n^k u_n - x^*\right\|^2 \\
&= \left\|J_{R_k,\lambda_{k,n}}(I - \lambda_{k,n} B_k)\Lambda_n^{k-1} u_n - J_{R_k,\lambda_{k,n}}(I - \lambda_{k,n} B_k) x^*\right\|^2 \\
&\le \left\langle (I - \lambda_{k,n} B_k)\, \Lambda_n^{k-1} u_n - (I - \lambda_{k,n} B_k)\, x^*, \Lambda_n^k u_n - x^* \right\rangle \\
&= \frac{1}{2} \Big(\left\|(I - \lambda_{k,n} B_k)\, \Lambda_n^{k-1} u_n - (I - \lambda_{k,n} B_k)\, x^*\right\|^2 \\
&\qquad + \left\|\Lambda_n^k u_n - x^*\right\|^2 \\
&\qquad - \Big\|(I - \lambda_{k,n} B_k)\, \Lambda_n^{k-1} u_n \\
&\qquad\quad - (I - \lambda_{k,n} B_k)\, x^* - \left(\Lambda_n^k u_n - x^*\right)\Big\|^2 \Big) \\
&\le \frac{1}{2} \Big(\left\|\Lambda_n^{k-1} u_n - x^*\right\|^2 + \left\|\Lambda_n^k u_n - x^*\right\|^2 \\
&\qquad - \left\|\Lambda_n^{k-1} u_n - \Lambda_n^k u_n - \lambda_{k,n}(B_k \Lambda_n^{k-1} u_n - B_k x^*)\right\|^2 \Big) \\
&\le \frac{1}{2} \Big(\left\|u_n - x^*\right\|^2 + \left\|\Lambda_n^k u_n - x^*\right\|^2 \\
&\qquad - \left\|\Lambda_n^{k-1} u_n - \Lambda_n^k u_n - \lambda_{k,n}(B_k \Lambda_n^{k-1} u_n - B_k x^*)\right\|^2 \Big) \\
&\le \frac{1}{2} \Big(\left\|x_n - x^*\right\|^2 + \left\|\Lambda_n^k u_n - x^*\right\|^2 \\
&\qquad - \left\|\Lambda_n^{k-1} u_n - \Lambda_n^k u_n - \lambda_{k,n}(B_k \Lambda_n^{k-1} u_n - B_k x^*)\right\|^2 \Big),
\end{aligned} \tag{74}$$

which immediately leads to

$$\begin{aligned}
&\left\|\Lambda_n^k u_n - x^*\right\|^2 \\
&\le \left\|x_n - x^*\right\|^2 \\
&\quad - \left\|\Lambda_n^{k-1} u_n - \Lambda_n^k u_n - \lambda_{k,n}(B_k \Lambda_n^{k-1} u_n - B_k x^*)\right\|^2 \\
&= \left\|x_n - x^*\right\|^2 - \left\|\Lambda_n^{k-1} u_n - \Lambda_n^k u_n\right\|^2 \\
&\quad - \lambda_{k,n}^2 \left\|B_k \Lambda_n^{k-1} u_n - B_k x^*\right\|^2 \\
&\quad + 2\lambda_{k,n} \left\langle \Lambda_n^{k-1} u_n - \Lambda_n^k u_n, B_k \Lambda_n^{k-1} u_n - B_k x^* \right\rangle \\
&\le \left\|x_n - x^*\right\|^2 - \left\|\Lambda_n^{k-1} u_n - \Lambda_n^k u_n\right\|^2 \\
&\quad + 2\lambda_{k,n} \left\|\Lambda_n^{k-1} u_n - \Lambda_n^k u_n\right\| \left\|B_k \Lambda_n^{k-1} u_n - B_k x^*\right\|.
\end{aligned} \tag{75}$$

Combining (67) and (75) we conclude that

$$\begin{aligned}
\left\|y_n - x^*\right\|^2 &\le \left\|v_n - x^*\right\|^2 + 2\rho_n \left\|\widetilde{A}_1 T_n v_n\right\| \left\|z_n - x^*\right\| \\
&\le \left\|\Lambda_n^k u_n - x^*\right\|^2 + 2\rho_n \left\|\widetilde{A}_1 T_n v_n\right\| \left\|z_n - x^*\right\| \\
&\le \left\|x_n - x^*\right\|^2 - \left\|\Lambda_n^{k-1} u_n - \Lambda_n^k u_n\right\|^2 \\
&\quad + 2\lambda_{k,n} \left\|\Lambda_n^{k-1} u_n - \Lambda_n^k u_n\right\| \left\|B_k \Lambda_n^{k-1} u_n - B_k x^*\right\| \\
&\quad + 2\rho_n \left\|\widetilde{A}_1 T_n v_n\right\| \left\|z_n - x^*\right\|,
\end{aligned} \tag{76}$$

which yields

$$\begin{aligned}
&\left\|\Lambda_n^{k-1} u_n - \Lambda_n^k u_n\right\|^2 \\
&\le \left\|x_n - x^*\right\|^2 - \left\|y_n - x^*\right\|^2 \\
&\quad + 2\lambda_{k,n} \left\|\Lambda_n^{k-1} u_n - \Lambda_n^k u_n\right\| \left\|B_k \Lambda_n^{k-1} u_n - B_k x^*\right\| \\
&\quad + 2\rho_n \left\|\widetilde{A}_1 T_n v_n\right\| \left\|z_n - x^*\right\| \\
&\le \left\|x_n - y_n\right\| \left(\left\|x_n - x^*\right\| + \left\|y_n - x^*\right\|\right) \\
&\quad + 2\lambda_{k,n} \left\|\Lambda_n^{k-1} u_n - \Lambda_n^k u_n\right\| \left\|B_k \Lambda_n^{k-1} u_n - B_k x^*\right\| \\
&\quad + 2\rho_n \left\|\widetilde{A}_1 T_n v_n\right\| \left\|z_n - x^*\right\|.
\end{aligned} \tag{77}$$

Since $\{\lambda_{k,n}\}_{n=0}^{\infty} \subset [a_k, b_k] \subset (0, 2\eta_k)$ for $k = 1, 2, \ldots, M$ and $\{u_n\}, \{x_n\}, \{y_n\}, \{\widetilde{A}_1 T_n v_n\}$ and $\{z_n\}$ are bounded sequences, we deduce from (71), $\rho_n \to 0$, and $\|x_n - y_n\| \to 0$ that

$$\lim_{n\to\infty} \left\|\Lambda_n^{k-1} u_n - \Lambda_n^k u_n\right\| = 0, \quad \forall k \in \{1, 2, \ldots, M\}. \tag{78}$$

Also, combining (51), (67), and (73), we deduce that

$$\begin{aligned}\|y_n - x^*\|^2 &\le \|v_n - x^*\|^2 + 2\rho_n \|\widetilde{A}_1 T_n v_n\| \|z_n - x^*\| \\ &\le \|u_n - x^*\|^2 + 2\rho_n \|\widetilde{A}_1 T_n v_n\| \|z_n - x^*\| \\ &\le \|\Delta_n^j x_n - x^*\|^2 + 2\rho_n \|\widetilde{A}_1 T_n v_n\| \|z_n - x^*\| \\ &\le \|x_n - x^*\|^2 - \|\Delta_n^{j-1} x_n - \Delta_n^j x_n\|^2 \\ &\quad + 2r_{j,n} \|\Delta_n^{j-1} x_n - \Delta_n^j x_n\| \|A_j \Delta_n^{j-1} x_n - A_j x^*\| \\ &\quad + 2\rho_n \|\widetilde{A}_1 T_n v_n\| \|z_n - x^*\|,\end{aligned} \tag{79}$$

which leads to

$$\begin{aligned}&\|\Delta_n^{j-1} x_n - \Delta_n^j x_n\|^2 \\ &\quad \le \|x_n - x^*\|^2 - \|y_n - x^*\|^2 \\ &\qquad + 2r_{j,n} \|\Delta_n^{j-1} x_n - \Delta_n^j x_n\| \|A_j \Delta_n^{j-1} x_n - A_j x^*\| \\ &\qquad + 2\rho_n \|\widetilde{A}_1 T_n v_n\| \|z_n - x^*\| \\ &\quad \le \|x_n - y_n\| (\|x_n - x^*\| + \|y_n - x^*\|) \\ &\qquad + 2r_{j,n} \|\Delta_n^{j-1} x_n - \Delta_n^j x_n\| \|A_j \Delta_n^{j-1} x_n - A_j x^*\| \\ &\qquad + 2\rho_n \|\widetilde{A}_1 T_n v_n\| \|z_n - x^*\|.\end{aligned} \tag{80}$$

Since $\{r_{j,n}\}_{n=0}^{\infty} \subset [c_j, d_j] \subset (0, 2\zeta_j)$ for $j = 1, 2, \ldots, K$ and $\{x_n\}$, $\{y_n\}$, $\{\widetilde{A}_1 T_n v_n\}$ and $\{z_n\}$ are bounded sequences, we conclude from (71), $\rho_n \to 0$, and $\|x_n - y_n\| \to 0$ that

$$\lim_{n\to\infty} \|\Delta_n^{j-1} x_n - \Delta_n^j x_n\| = 0, \quad \forall j \in \{1, 2, \ldots, K\}. \tag{81}$$

Hence from (78) and (81) we get

$$\begin{aligned}\|x_n - u_n\| &= \|\Delta_n^0 x_n - \Delta_n^K x_n\| \\ &\le \|\Delta_n^0 x_n - \Delta_n^1 x_n\| + \|\Delta_n^1 x_n - \Delta_n^2 x_n\| + \cdots \\ &\quad + \|\Delta_n^{K-1} x_n - \Delta_n^K x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,\end{aligned} \tag{82}$$

$$\begin{aligned}\|u_n - v_n\| &= \|\Lambda_n^0 u_n - \Lambda_n^M u_n\| \\ &\le \|\Lambda_n^0 u_n - \Lambda_n^1 u_n\| + \|\Lambda_n^1 u_n - \Lambda_n^2 u_n\| + \cdots \\ &\quad + \|\Lambda_n^{M-1} u_n - \Lambda_n^M u_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,\end{aligned} \tag{83}$$

respectively. Thus, from (82) and (83), we obtain

$$\|x_n - v_n\| \le \|x_n - u_n\| + \|u_n - v_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{84}$$

together with $\|v_n - T_n v_n\| \to 0$, which implies that

$$\lim_{n\to\infty} \|x_n - T_n v_n\| = 0. \tag{85}$$

On the other hand, we observe that the following relation holds:

$$\begin{aligned}x_{n+N} - x_n &= x_{n+N} - S_{[n+N]} \left(I - \rho_{n+N-1} \widetilde{A}_1\right) T_{n+N-1} v_{n+N-1} \\ &\quad + S_{[n+N]} \left(I - \rho_{n+N-1} \widetilde{A}_1\right) T_{n+N-1} v_{n+N-1} \\ &\quad - S_{[n+N]} S_{[n+N-1]} \left(I - \rho_{n+N-2} \widetilde{A}_1\right) T_{n+N-2} v_{n+N-2} \\ &\quad + \cdots + S_{[n+N]} \cdots S_{[n+2]} \left(I - \rho_{n+1} \widetilde{A}_1\right) T_{n+1} v_{n+1} \\ &\quad - S_{[n+N]} \cdots S_{[n+1]} \left(I - \rho_n \widetilde{A}_1\right) T_n v_n \\ &\quad + S_{[n+N]} \cdots S_{[n+1]} \left(I - \rho_n \widetilde{A}_1\right) T_n v_n - x_n.\end{aligned} \tag{86}$$

Since $\|x_{n+1} - y_n\| \to 0$ and $\rho_n \to 0$ as $n \to \infty$, from the nonexpansivity of each S_i $(i = 1, 2, \ldots, N)$ and boundedness of $\{\widetilde{A}_1 T_n v_n\}$ it follows from (85) that as $n \to \infty$ we have

$$\begin{aligned}&\left\|x_{n+N} - S_{[n+N]} \left(I - \rho_{n+N-1} \widetilde{A}_1\right) T_{n+N-1} v_{n+N-1}\right\| \\ &\quad = \|x_{n+N} - y_{n+N-1}\| \longrightarrow 0, \\ &\left\|S_{[n+N]} \left(I - \rho_{n+N-1} \widetilde{A}_1\right) T_{n+N-1} v_{n+N-1}\right. \\ &\quad \left. - S_{[n+N]} S_{[n+N-1]} \left(I - \rho_{n+N-2} \widetilde{A}_1\right) T_{n+N-2} v_{n+N-2}\right\| \\ &\le \left\| \left(I - \rho_{n+N-1} \widetilde{A}_1\right) T_{n+N-1} v_{n+N-1}\right. \\ &\quad \left. - S_{[n+N-1]} \left(I - \rho_{n+N-2} \widetilde{A}_1\right) T_{n+N-2} v_{n+N-2}\right\| \\ &\le \left\|T_{n+N-1} v_{n+N-1}\right. \\ &\quad \left. - S_{[n+N-1]} \left(I - \rho_{n+N-2} \widetilde{A}_1\right) T_{n+N-2} v_{n+N-2}\right\| \\ &\quad + \rho_{n+N-1} \|\widetilde{A}_1 T_{n+N-1} v_{n+N-1}\| \\ &\le \|T_{n+N-1} v_{n+N-1} - x_{n+N-1}\| \\ &\quad + \left\|x_{n+N-1} - S_{[n+N-1]} \left(I - \rho_{n+N-2} \widetilde{A}_1\right) T_{n+N-2} v_{n+N-2}\right\| \\ &\quad + \rho_{n+N-1} \|\widetilde{A}_1 T_{n+N-1} v_{n+N-1}\| \\ &= \|T_{n+N-1} v_{n+N-1} - x_{n+N-1}\| + \|x_{n+N-1} - y_{n+N-2}\| \\ &\quad + \rho_{n+N-1} \|\widetilde{A}_1 T_{n+N-1} v_{n+N-1}\| \longrightarrow 0, \\ &\qquad \vdots \\ &\left\|S_{[n+N]} \cdots S_{[n+2]} \left(I - \rho_{n+1} \widetilde{A}_1\right) T_{n+1} v_{n+1}\right. \\ &\quad \left. - S_{[n+N]} \cdots S_{[n+1]} \left(I - \rho_n \widetilde{A}_1\right) T_n v_n\right\| \\ &\le \left\|\left(I - \rho_{n+1} \widetilde{A}_1\right) T_{n+1} v_{n+1} - S_{[n+1]} \left(I - \rho_n \widetilde{A}_1\right) T_n v_n\right\| \\ &\le \left\|T_{n+1} v_{n+1} - S_{[n+1]} \left(I - \rho_n \widetilde{A}_1\right) T_n v_n\right\|\end{aligned}$$

$$
\begin{aligned}
&+\rho_{n+1}\left\|\widetilde{A}_1 T_{n+1} v_{n+1}\right\| \\
&\leq \left\|T_{n+1} v_{n+1}-x_{n+1}\right\|+\left\|x_{n+1}-S_{[n+1]}\left(I-\rho_n \widetilde{A}_1\right) T_n v_n\right\| \\
&\quad+\rho_{n+1}\left\|\widetilde{A}_1 T_{n+1} v_{n+1}\right\| \\
&=\left\|T_{n+1} v_{n+1}-x_{n+1}\right\|+\left\|x_{n+1}-y_n\right\| \\
&\quad+\rho_{n+1}\left\|\widetilde{A}_1 T_{n+1} v_{n+1}\right\| \longrightarrow 0.
\end{aligned} \tag{87}
$$

Therefore, from (66) and (86), we obtain

$$
\lim_{n \rightarrow \infty}\left\|S_{[n+N]} \cdots S_{[n+1]}\left(I-\rho_n \widetilde{A}_1\right) T_n v_n-x_n\right\|=0. \tag{88}
$$

So, it follows that

$$
\begin{aligned}
&\left\|S_{[n+N]} \cdots S_{[n+1]}\left(I-\rho_n \widetilde{A}_1\right) x_n-x_n\right\| \\
&\leq\left\|S_{[n+N]} \cdots S_{[n+1]}\left(I-\rho_n \widetilde{A}_1\right) x_n\right. \\
&\quad\left.-S_{[n+N]} \cdots S_{[n+1]}\left(I-\rho_n \widetilde{A}_1\right) T_n v_n\right\| \\
&\quad+\left\|S_{[n+N]} \cdots S_{[n+1]}\left(I-\rho_n \widetilde{A}_1\right) T_n v_n-x_n\right\| \longrightarrow 0.
\end{aligned} \tag{89}
$$

Observe that

$$
\begin{aligned}
&\left\|S_{[n+N]} \cdots S_{[n+1]} x_n-x_n\right\| \\
&\leq\left\|S_{[n+N]} \cdots S_{[n+1]} x_n-S_{[n+N]} \cdots S_{[n+1]}\left(x_n-\rho_n \widetilde{A}_1 x_n\right)\right\| \\
&\quad+\left\|S_{[n+N]} \cdots S_{[n+1]}\left(x_n-\rho_n \widetilde{A}_1 x_n\right)-x_n\right\| \\
&\leq \rho_n\left\|\widetilde{A}_1 x_n\right\|+\left\|S_{[n+N]} \cdots S_{[n+1]}\left(x_n-\rho_n \widetilde{A}_1 x_n\right)-x_n\right\| \\
&\longrightarrow 0 \quad(n \longrightarrow \infty).
\end{aligned} \tag{90}
$$

That is,

$$
\lim_{n \rightarrow \infty}\left\|S_{[n+N]} \cdots S_{[n+1]} x_n-x_n\right\|=0. \tag{91}
$$

Step 4. We prove that $\limsup_{n \rightarrow \infty}\langle\widetilde{A}_1 x^*, x^*-x_n\rangle \leq 0$ provided $\lim_{n \rightarrow \infty}(\|x_n-y_n\|+\|T_n v_n-v_n\|)=0$.

Indeed, choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$
\limsup_{n \rightarrow \infty}\left\langle\widetilde{A}_1 x^*, x^*-x_n\right\rangle=\lim_{i \rightarrow \infty}\left\langle\widetilde{A}_1 x^*, x^*-x_{n_i}\right\rangle. \tag{92}
$$

The boundedness of $\{x_{n_i}\}$ implies the existence of a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ and a point $\widehat{x} \in H$ such that $x_{n_{i_j}} \rightharpoonup \widehat{x}$. We may assume without loss of generality that $x_{n_i} \rightharpoonup \widehat{x}$; that is,

$$
\begin{aligned}
\limsup_{n \rightarrow \infty}\left\langle\widetilde{A}_1 x^*, x^*-x_n\right\rangle &=\lim_{i \rightarrow \infty}\left\langle\widetilde{A}_1 x^*, x^*-x_{n_i}\right\rangle \\
&=\left\langle\widetilde{A}_1 x^*, x^*-\widehat{x}\right\rangle.
\end{aligned} \tag{93}
$$

First, we can readily see that $\widehat{x} \in \cap_{i=1}^N \operatorname{Fix}(S_i)$. Since the pool of mappings $\{S_i : i \leq i \leq N\}$ is finite, we may further assume (passing to a further subsequence if necessary) that, for some integer $l \in\{1,2, \ldots, N\}$,

$$
S_{[n_i]} \equiv S_l, \quad \forall i \geq 1. \tag{94}
$$

Then, it follows from (91) that

$$
x_{n_i}-S_{[i+N]} \cdots S_{[i+1]} x_n-x_{n_i} \longrightarrow 0. \tag{95}
$$

Hence, by Lemma 9, we conclude that

$$
\widehat{x} \in \operatorname{Fix}\left(S_{[i+N]} \cdots S_{[i+1]}\right). \tag{96}
$$

Together with the assumption

$$
\begin{aligned}
\bigcap_{i=1}^N \operatorname{Fix}\left(S_i\right) &=\operatorname{Fix}\left(S_1 S_2 \cdots S_N\right) \\
&=\operatorname{Fix}\left(S_N S_1 \cdots S_{N-1}\right) \\
&=\cdots=\operatorname{Fix}\left(S_2 S_3 \cdots S_N S_1\right),
\end{aligned} \tag{97}
$$

this implies that $\widehat{x} \in \cap_{i=1}^N \operatorname{Fix}(S_i)$. Now, since

$$
x^* \in \operatorname{VI}\left(\bigcap_{i=i}^N \operatorname{Fix}\left(S_i\right), \widetilde{A}_1\right), \tag{98}
$$

we obtain

$$
\begin{aligned}
\limsup_{n \rightarrow \infty}\left\langle\widetilde{A}_1 x^*, x^*-x_n\right\rangle &=\lim_{i \rightarrow \infty}\left\langle\widetilde{A}_1 x^*, x^*-x_{n_i}\right\rangle \\
&=\left\langle\widetilde{A}_1 x^*, x^*-\widehat{x}\right\rangle \leq 0.
\end{aligned} \tag{99}
$$

Step 5. We prove that $\lim_{n \rightarrow \infty}\|x_n-x^*\|=0$ provided $\|x_n-y_n\|+\|T_n v_n-v_n\|=o(\rho_n)$.

Indeed, first of all, let us show that $\limsup_{n \rightarrow \infty}\langle\widetilde{A}_1 x^*, x^*-x_n\rangle \leq 0$. We choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$
\limsup_{n \rightarrow \infty}\left\langle\widetilde{A}_2 x^*, x^*-x_n\right\rangle=\lim_{k \rightarrow \infty}\left\langle\widetilde{A}_2 x^*, x^*-x_{n_k}\right\rangle. \tag{100}
$$

The boundedness of $\{x_{n_k}\}$ implies that there is a subsequence of $\{x_{n_k}\}$ which converges weakly to a point $\overline{x} \in H$. Without loss of generality, we may assume that $x_{n_k} \rightharpoonup \overline{x}$; that is,

$$
\begin{aligned}
\limsup_{n \rightarrow \infty}\left\langle\widetilde{A}_2 x^*, x^*-x_n\right\rangle &=\lim_{k \rightarrow \infty}\left\langle\widetilde{A}_2 x^*, x^*-x_{n_k}\right\rangle \\
&=\left\langle\widetilde{A}_2 x^*, x^*-\overline{x}\right\rangle.
\end{aligned} \tag{101}
$$

Repeating the same argument as in the proof of $\widehat{x} \in \cap_{i=1}^N \operatorname{Fix}(S_i)$, we have $\overline{x} \in \cap_{i=1}^N \operatorname{Fix}(S_i)$. Let $p \in \cap_{i=1}^N \operatorname{Fix}(S_i)$ be fixed arbitrarily. Note that $\cap_{i=i}^N \operatorname{Fix}(S_i) \subset \cap_{j=1}^K \operatorname{GMEP}(\Theta_j, \varphi_j, A_j) \cap \cap_{k=1}^M I(B_k, R_k) \cap \Gamma$. Then, it follows from the nonexpansivity of each

S_i $(i = 1, 2, \ldots, N)$ and monotonicity of $\widetilde{A}_1$ that, for all $n \geq 0$,

$$\begin{aligned}
\|y_n - p\|^2 &= \left\|S_{[n+1]}(I - \rho_n \widetilde{A}_1)T_n v_n - S_{[n+1]}p\right\|^2 \\
&\leq \left\|(T_n v_n - p) - \rho_n \widetilde{A}_1 T_n v_n\right\|^2 \\
&= \|T_n v_n - p\|^2 + 2\rho_n \left\langle \widetilde{A}_1 T_n v_n, p - T_n v_n \right\rangle \\
&\quad + \rho_n^2 \left\|\widetilde{A}_1 T_n v_n\right\|^2 \\
&= \|T_n v_n - p\|^2 + 2\rho_n \left\langle \widetilde{A}_1 T_n v_n - \widetilde{A}_1 p, p - T_n v_n \right\rangle \\
&\quad + 2\rho_n \left\langle \widetilde{A}_1 p, p - T_n v_n \right\rangle + \rho_n^2 \left\|\widetilde{A}_1 T_n v_n\right\|^2 \\
&\leq \|v_n - p\|^2 + 2\rho_n \left\langle \widetilde{A}_1 p, p - T_n v_n \right\rangle \\
&\quad + \rho_n^2 \left\|\widetilde{A}_1 T_n v_n\right\|^2 \\
&\leq \|x_n - p\|^2 + 2\rho_n \left\langle \widetilde{A}_1 p, p - T_n v_n \right\rangle + \rho_n^2 \widetilde{M}_3^2,
\end{aligned} \tag{102}$$

which implies that

$$\begin{aligned}
&\lim_{n\to\infty} \left\langle \widetilde{A}_1 p, p - T_n v_n \right\rangle \\
&\quad \leq \lim_{n\to\infty} \frac{1}{2\rho_n} \left[\|x_n - p\|^2 - \|y_n - p\|^2 + \rho_n^2 \widetilde{M}_3^2\right] \\
&\quad \leq \lim_{n\to\infty} \frac{\|x_n - y_n\|}{2\rho_n} \left(\|x_n - p\| + \|y_n - p\|\right) + \lim_{n\to\infty} \frac{\rho_n}{2} \widetilde{M}_3^2.
\end{aligned} \tag{103}$$

So, from $\|x_n - y_n\| = o(\rho_n)$ and the boundedness of $\{x_n\}$ and $\{y_n\}$, we get

$$\limsup_{n\to\infty} \left\langle \widetilde{A}_1 p, p - T_n v_n \right\rangle \leq 0, \tag{104}$$

together with (85), which implies that

$$\begin{aligned}
&\left\langle \widetilde{A}_1 p, p - \overline{x} \right\rangle \\
&\quad = \lim_{k\to\infty} \left\langle \widetilde{A}_1 p, p - x_{n_k} \right\rangle \\
&\quad \leq \limsup_{n\to\infty} \left\langle \widetilde{A}_1 p, p - x_n \right\rangle \\
&\quad \leq \limsup_{n\to\infty} \left(\left\langle \widetilde{A}_1 p, p - T_n v_n \right\rangle + \left\langle \widetilde{A}_1 p, T_n v_n - x_n \right\rangle\right) \\
&\quad \leq \limsup_{n\to\infty} \left\langle \widetilde{A}_1 p, p - T_n v_n \right\rangle \\
&\quad \leq 0.
\end{aligned} \tag{105}$$

Thus, we have

$$\left\langle \widetilde{A}_1 p, p - \overline{x} \right\rangle \leq 0, \quad \forall p \in \bigcap_{i=1}^{N} \operatorname{Fix}(S_i). \tag{106}$$

Since $\widetilde{A}_1$ is monotone and $1/\alpha$-Lipschitz continuous, in terms of Minty's lemma [12], we deduce that $\overline{x} \in \operatorname{VI}(\cap_{i=1}^{N} \operatorname{Fix}(S_i), \widetilde{A}_1)$. Therefore, from $\{x^*\} = \operatorname{VI}(\operatorname{VI}(\Omega, \widetilde{A}_1), \widetilde{A}_2)$, we have

$$\begin{aligned}
\limsup_{n\to\infty} \left\langle \widetilde{A}_2 x^*, x^* - x_n \right\rangle &= \lim_{k\to\infty} \left\langle \widetilde{A}_2 x^*, x^* - x_{n_k} \right\rangle \\
&= \left\langle \widetilde{A}_2 x^*, x^* - \overline{x} \right\rangle \leq 0.
\end{aligned} \tag{107}$$

Finally, let us show that $\|x_n - x^*\| \to 0$ as $n \to \infty$. By utilizing Lemma 11, we deduce from (52) and $S_{[n+1]}^{\alpha_n} x^* = x^* - \alpha_n \mu \widetilde{A}_2 x^*$ that for all $n \geq 0$

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&= \left\|S_{[n+1]}^{\alpha_n} z_n - x^*\right\|^2 \\
&= \left\|S_{[n+1]}^{\alpha_n} z_n - S_{[n+1]}^{\alpha_n} x^* + S_{[n+1]}^{\alpha_n} x^* - x^*\right\|^2 \\
&\leq \left\|S_{[n+1]}^{\alpha_n} z_n - S_{[n+1]}^{\alpha_n} x^*\right\|^2 + 2\left\langle S_{[n+1]}^{\alpha_n} x^* - x^*, x_{n+1} - x^* \right\rangle \\
&\leq (1 - \alpha_n \tau) \|z_n - x^*\|^2 - 2\alpha_n \mu \left\langle \widetilde{A}_2 x^*, x_{n+1} - x^* \right\rangle \\
&= (1 - \alpha_n \tau) \left\|T_n v_n - x^* - \rho_n \widetilde{A}_1 T_n v_n\right\|^2 \\
&\quad - 2\alpha_n \mu \left\langle \widetilde{A}_2 x^*, x_{n+1} - x^* \right\rangle \\
&= (1 - \alpha_n \tau) \Big[\|T_n v_n - x^*\|^2 \\
&\qquad + 2\rho_n \left\langle \widetilde{A}_1 T_n v_n, x^* - T_n v_n \right\rangle + \rho_n^2 \left\|\widetilde{A}_1 T_n v_n\right\|^2\Big] \\
&\quad - 2\alpha_n \mu \left\langle \widetilde{A}_2 x^*, x_{n+1} - x^* \right\rangle \\
&= (1 - \alpha_n \tau) \Big[\|T_n v_n - x^*\|^2 \\
&\qquad + 2\rho_n \left\langle \widetilde{A}_1 T_n v_n - \widetilde{A}_1 x^*, x^* - T_n v_n \right\rangle \\
&\qquad + 2\rho_n \left\langle \widetilde{A}_1 x^*, x^* - T_n v_n \right\rangle + \rho_n^2 \left\|\widetilde{A}_1 T_n v_n\right\|^2\Big] \\
&\quad - 2\alpha_n \mu \left\langle \widetilde{A}_2 x^*, x_{n+1} - x^* \right\rangle \\
&\leq (1 - \alpha_n \tau) \Big[\|v_n - x^*\|^2 + 2\rho_n \left\langle \widetilde{A}_1 x^*, x^* - T_n v_n \right\rangle \\
&\qquad + \rho_n^2 \left\|\widetilde{A}_1 T_n v_n\right\|^2\Big] \\
&\quad - 2\alpha_n \mu \left\langle \widetilde{A}_2 x^*, x_{n+1} - x^* \right\rangle \\
&\leq (1 - \alpha_n \tau) \left[\|x_n - x^*\|^2 + 2\rho_n \left\langle \widetilde{A}_1 x^*, x^* - T_n v_n \right\rangle + \rho_n^2 \widetilde{M}_3^2\right] \\
&\quad - 2\alpha_n \mu \left\langle \widetilde{A}_2 x^*, x_{n+1} - x^* \right\rangle \\
&\leq (1 - \alpha_n \tau) \|x_n - x^*\|^2 \\
&\quad + 2\rho_n (1 - \alpha_n \tau) \left\langle \widetilde{A}_1 x^*, x^* - T_n v_n \right\rangle + \rho_n^2 \widetilde{M}_3^2 \\
&\quad - 2\alpha_n \mu \left\langle \widetilde{A}_2 x^*, x_{n+1} - x^* \right\rangle
\end{aligned}$$

$$= (1-\alpha_n\tau)\|x_n - x^*\|^2 + \alpha_n\tau\cdot\frac{1}{\tau}\left[2\frac{\rho_n}{\alpha_n}(1-\alpha_n\tau)\langle\widetilde{A}_1x^*, x^* - T_nv_n\rangle + \frac{\rho_n^2}{\alpha_n}\widetilde{M}_3^2 + 2\mu\langle\widetilde{A}_2x^*, x^* - x_{n+1}\rangle\right]. \tag{108}$$

Since $\sum_{n=0}^{\infty}\alpha_n = \infty, \rho_n \le \alpha_n$ for all $n \ge 0$ and $\alpha_n \to 0$ as $n \to \infty$, we obtain, from (107) and (104) with $p = x^*$, that $\sum_{n=0}^{\infty}\alpha_n\tau = \infty, 2(\rho_n/\alpha_n)(1-\alpha_n\tau) \le 2$, and

$$\limsup_{n\to\infty}\frac{1}{\tau}\left[2\frac{\rho_n}{\alpha_n}(1-\alpha_n\tau)\langle\widetilde{A}_1x^*, x^* - T_nv_n\rangle + \frac{\rho_n^2}{\alpha_n}\widetilde{M}_3^2 + 2\mu\langle\widetilde{A}_2x^*, x^* - x_{n+1}\rangle\right] \le 0. \tag{109}$$

Applying Lemma 12 to (108), we infer that

$$\lim_{n\to\infty}\|x_n - x^*\| = 0. \tag{110}$$

This completes the proof. □

In Theorem 18, putting $f(x) \equiv 0, \forall x \in C$, we obtain that $\Gamma = C$ and $T_n = I$ which is the identity mapping of C. Hence Theorem 18 reduces to the following.

Corollary 19. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $M, N, K \ge 1$ be three integers. Let Θ_j be a bifunction from $C\times C$ to $\mathbf{R}$ satisfying (A1)–(A4), $\varphi_j : C \to \mathbf{R}$ a lower semicontinuous and convex functional with the restriction (B1) or (B2), and $A_j : H \to H$ ζ_j-inverse strongly monotone for $j = 1, 2, \ldots, K$. Let $R_k : C \to 2^H$ be a maximal monotone mapping and let $B_k : C \to H$ be η_k-inverse strongly monotone for $k = 1, 2, \ldots, M$. Let $\{S_i\}_{i=1}^N$ be a finite family of nonexpansive mappings on H. Let $\widetilde{A}_1 : H \to H$ be α-inverse strongly monotone and let $\widetilde{A}_2 : H \to H$ be β-strongly monotone and κ-Lipschitz continuous. Assume that $\mathrm{VI}(\cap_{i=1}^N \mathrm{Fix}(S_i), \widetilde{A}_1) \ne \emptyset$ with $(\cap_{i=1}^N \mathrm{Fix}(S_i)) \subset (\cap_{j=1}^K \mathrm{GMEP}(\Theta_j, \varphi_j, A_j)) \cap (\cap_{k=1}^M I(B_k, R_k))$. Let $\mu \in (0, 2\beta/\kappa^2)$, $\{\alpha_n\}_{n=0}^\infty \subset (0,1]$, $\{\rho_n\}_{n=0}^\infty \subset (0, 2\alpha]$, $\{\lambda_{k,n}\}_{n=0}^\infty \subset [a_k, b_k] \subset (0, 2\eta_k)$, and $\{r_{j,n}\}_{n=0}^\infty \subset [c_j, d_j] \subset (0, 2\zeta_j)$ where $j \in \{1, 2, \ldots, K\}$ and $k \in \{1, 2, \ldots, M\}$. For arbitrarily given $x_0 \in H$, let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned}
u_n &= T_{r_{K,n}}^{(\Theta_K,\varphi_K)}(I - r_{K,n}A_K)\,T_{r_{K-1,n}}^{(\Theta_{K-1},\varphi_{K-1})}(I - r_{K-1,n}A_{K-1}) \cdots T_{r_{1,n}}^{(\Theta_1,\varphi_1)}(I - r_{1,n}A_1)\,x_n,\\
v_n &= J_{R_M,\lambda_{M,n}}(I - \lambda_{M,n}B_M)\,J_{R_{M-1},\lambda_{M-1,n}}(I - \lambda_{M-1,n}B_{M-1}) \cdots J_{R_1,\lambda_{1,n}}(I - \lambda_{1,n}B_1)\,u_n,\\
y_n &= S_{[n+1]}\left(I - \rho_n\widetilde{A}_1\right)v_n,\\
x_{n+1} &= y_n - \mu\alpha_n\widetilde{A}_2y_n, \quad \forall n \ge 0.
\end{aligned} \tag{111}$$

Assume that

$$\begin{aligned}
\bigcap_{i=1}^N \mathrm{Fix}(S_i) &= \mathrm{Fix}(S_1S_2\cdots S_N)\\
&= \mathrm{Fix}(S_NS_1\cdots S_{N-1})\\
&= \cdots = \mathrm{Fix}(S_2S_3\cdots S_NS_1)
\end{aligned} \tag{112}$$

and that the following conditions are satisfied:

(i) $\lim_{n\to\infty}\alpha_n = 0, \sum_{n=0}^\infty \alpha_n = \infty$ *and* $\rho_n \le \alpha_n$ *for all* $n \ge 0$;

(ii) $\lim_{n\to\infty}(|\alpha_n - \alpha_{n+N}|/(\alpha_{n+N})) = 0$ *or* $\sum_{n=0}^\infty |\alpha_n - \alpha_{n+N}| < \infty$;

(iii) $\lim_{n\to\infty}(|\rho_n - \rho_{n+N}|/(\rho_{n+N})) = 0$ *or* $\sum_{n=0}^\infty |\rho_n - \rho_{n+N}| < \infty$;

(iv) $\lim_{n\to\infty}(|\lambda_{k,n} - \lambda_{k,n+N}|/(\alpha_n + N)) = 0$ *or* $\sum_{n=0}^\infty |\lambda_{k,n} - \lambda_{k,n+N}| < \infty$ *for* $k = 1, 2, \ldots, M$;

(v) $\lim_{n\to\infty}(|r_{j,n} - r_{j,n+N}|/(\alpha_n + N)) = 0$ *or* $\sum_{n=0}^\infty |r_{j,n} - r_{j,n+N}| < \infty$ *for* $j = 1, 2, \ldots, K$.

Then the following hold:

(a) $\{x_n\}_{n=0}^\infty$ *is bounded;*

(b) $\lim_{n\to\infty}\|x_n - x_{n+N}\| = 0$;

(c) $\lim_{n\to\infty}\|x_n - S_{[n+N]}\cdots S_{[n+1]}x_n\| = 0$ *provided* $\|x_n - y_n\| \to 0$ $(n \to \infty)$;

(d) $\{x_n\}_{n=0}^\infty$ *converges strongly to the unique element of* $\mathrm{VI}(\mathrm{VI}(\cap_{i=1}^N \mathrm{Fix}(S_i), \widetilde{A}_1), \widetilde{A}_2)$ *provided* $\|x_n - y_n\| = o(\rho_n)$.

In Corollary 19, putting $K = 1$ and $M = 2$, we obtain the following.

Corollary 20. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $N \ge 1$ be an integer. Let Θ be a bifunction from $C\times C$ to $\mathbf{R}$ satisfying (A1)–(A4), $\varphi : C \to \mathbf{R}$ a lower semicontinuous and convex functional with the restriction (B1) or (B2), and $A : H \to H$ ζ-inverse strongly monotone. Let $R_k : C \to 2^H$ be a maximal monotone mapping and let $B_k : C \to H$ be η_k-inverse strongly monotone for $k = 1, 2$. Let $\{S_i\}_{i=1}^N$ be a finite family of nonexpansive mappings on H. Let $\widetilde{A}_1 : H \to H$ be α-inverse strongly monotone and let $\widetilde{A}_2 : H \to H$ be β-strongly monotone and κ-Lipschitz continuous. Assume that $\mathrm{VI}(\cap_{i=1}^N \mathrm{Fix}(S_i), \widetilde{A}_1) \ne \emptyset$ with $(\cap_{i=1}^N \mathrm{Fix}(S_i)) \subset \mathrm{GMEP}(\Theta, \varphi, A) \cap I(B_2, R_2) \cap I(B_1, R_1)$. Let $\mu \in (0, 2\beta/\kappa^2)$, $\{\alpha_n\}_{n=0}^\infty \subset (0,1]$, $\{\rho_n\}_{n=0}^\infty \subset (0, 2\alpha]$, $\{\lambda_{k,n}\}_{n=0}^\infty \subset [a_k, b_k] \subset (0, 2\eta_k)$, and $\{r_n\}_{n=0}^\infty \subset [c, d] \subset (0, 2\zeta)$*

for $k = 1, 2$. For arbitrarily given $x_0 \in H$, let $\{x_n\}$ be a sequence generated by

$$\begin{aligned} &\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n\rangle \\ &\quad + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ &v_n = J_{R_2,\lambda_{2,n}}(I - \lambda_{2,n}B_2) J_{R_1,\lambda_{1,n}}(I - \lambda_{1,n}B_1) u_n, \\ &y_n = S_{[n+1]}(I - \rho_n \widetilde{A}_1) v_n, \\ &x_{n+1} = y_n - \mu\alpha_n \widetilde{A}_2 y_n, \quad \forall n \geq 0. \end{aligned} \tag{113}$$

Assume that

$$\begin{aligned} \bigcap_{i=1}^{N} \mathrm{Fix}(S_i) &= \mathrm{Fix}(S_1 S_2 \cdots S_N) \\ &= \mathrm{Fix}(S_N S_1 \cdots S_{N-1}) \\ &= \cdots = \mathrm{Fix}(S_2 S_3 \cdots S_N S_1) \end{aligned} \tag{114}$$

and that the following conditions are satisfied:

(i) $\lim_{n\to\infty}\alpha_n = 0, \sum_{n=0}^{\infty}\alpha_n = \infty$ *and* $\rho_n \leq \alpha_n$ *for all* $n \geq 0$;

(ii) $\lim_{n\to\infty}(|\alpha_n - \alpha_{n+N}|/(\alpha_{n+N})) = 0$ *or* $\sum_{n=0}^{\infty}|\alpha_n - \alpha_{n+N}| < \infty$;

(iii) $\lim_{n\to\infty}(|\rho_n - \rho_{n+N}|/(\rho_{n+N})) = 0$ *or* $\sum_{n=0}^{\infty}|\rho_n - \rho_{n+N}| < \infty$;

(iv) $\lim_{n\to\infty}(|\lambda_{k,n} - \lambda_{k,n+N}|/(\alpha_{n+N})) = 0$ *or* $\sum_{n=0}^{\infty}|\lambda_{k,n} - \lambda_{k,n+N}| < \infty$ *for* $k = 1, 2$;

(v) $\lim_{n\to\infty}(|r_n - r_{n+N}|/(\alpha_{n+N})) = 0$ *or* $\sum_{n=0}^{\infty}|r_n - r_{n+N}| < \infty$.

Then the following hold:

(a) $\{x_n\}_{n=0}^{\infty}$ *is bounded;*

(b) $\lim_{n\to\infty}\|x_n - x_{n+N}\| = 0$;

(c) $\lim_{n\to\infty}\|x_n - S_{[n+N]}\cdots S_{[n+1]}x_n\| = 0$ *provided* $\|x_n - y_n\| \to 0$ $(n \to \infty)$;

(d) $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the unique element of* $\mathrm{VI}(\mathrm{VI}(\cap_{i=1}^{N}\mathrm{Fix}(S_i), \widetilde{A}_1), \widetilde{A}_2)$ *provided* $\|x_n - y_n\| = o(\rho_n)$.

In Theorem 18, putting $K = 1$ and $M = 2$, we obtain the following.

Corollary 21. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $f : C \to \mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f. Let $N \geq 1$ be an integer. Let Θ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4), $\varphi : C \to \mathbf{R}$ a lower semicontinuous and convex functional with the restriction (B1) or (B2), and $A : H \to H$ ζ-inverse-strongly monotone. Let $R_k : C \to 2^H$ be a maximal monotone mapping and let $B_k : C \to H$ be η_k-inverse strongly monotone for $k = 1, 2$. Let $\{S_i\}_{i=1}^{N}$ be a finite family of nonexpansive mappings on H. Let $\widetilde{A}_1 : H \to H$ be α-inverse strongly monotone and let $\widetilde{A}_2 : H \to H$ be β-strongly monotone and κ-Lipschitz continuous. Assume that $\mathrm{VI}(\cap_{i=1}^{N}\mathrm{Fix}(S_i), \widetilde{A}_1) \neq \emptyset$ with $\cap_{i=1}^{N}\mathrm{Fix}(S_i) \subset \mathrm{GMEP}(\Theta, \varphi, A) \cap I(B_2, R_2) \cap I(B_1, R_1) \cap \Gamma$. Let $\mu \in (0, 2\beta/\kappa^2)$, $\{\alpha_n\}_{n=0}^{\infty} \subset (0, 1]$, $\{\rho_n\}_{n=0}^{\infty} \subset (0, 2\alpha]$, $\{\lambda_{k,n}\}_{n=0}^{\infty} \subset [a_k, b_k] \subset (0, 2\eta_k)$, and $\{r_n\}_{n=0}^{\infty} \subset [c, d] \subset (0, 2\zeta)$ for $k = 1, 2$. For arbitrarily given $x_0 \in H$, let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} &\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n\rangle \\ &\quad + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ &v_n = J_{R_2,\lambda_{2,n}}(I - \lambda_{2,n}B_2) J_{R_1,\lambda_{1,n}}(I - \lambda_{1,n}B_1) u_n, \\ &y_n = S_{[n+1]}(I - \rho_n \widetilde{A}_1) T_n v_n, \\ &x_{n+1} = y_n - \mu\alpha_n \widetilde{A}_2 y_n, \quad \forall n \geq 0, \end{aligned} \tag{115}$$

where $P_C(I - \lambda_n\nabla f) = s_n I + (1 - s_n)T_n$ (here T_n is nonexpansive and $s_n := s_n(\lambda_n) = (2 - \lambda_n L)/4 \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$). Assume that

$$\begin{aligned} \bigcap_{i=1}^{N} \mathrm{Fix}(S_i) &= \mathrm{Fix}(S_1 S_2 \cdots S_N) \\ &= \mathrm{Fix}(S_N S_1 \cdots S_{N-1}) \\ &= \cdots = \mathrm{Fix}(S_2 S_3 \cdots S_N S_1) \end{aligned} \tag{116}$$

and that the following conditions are satisfied:

(i) $\lim_{n\to\infty}\alpha_n = 0, \sum_{n=0}^{\infty}\alpha_n = \infty$ *and* $\rho_n \leq \alpha_n$ *for all* $n \geq 0$;

(ii) $\lim_{n\to\infty}(|\alpha_n - \alpha_{n+N}|/(\alpha_{n+N})) = 0$ *or* $\sum_{n=0}^{\infty}|\alpha_n - \alpha_{n+N}| < \infty$;

(iii) $\lim_{n\to\infty}(|s_n - s_{n+N}|/(\alpha_{n+N})) = 0$ *or* $\sum_{n=0}^{\infty}|s_n - s_{n+N}| < \infty$;

(iv) $\lim_{n\to\infty}(|\rho_n - \rho_{n+N}|/(\rho_{n+N})) = 0$ *or* $\sum_{n=0}^{\infty}|\rho_n - \rho_{n+N}| < \infty$;

(v) $\lim_{n\to\infty}(|\lambda_{k,n} - \lambda_{k,n+N}|/(\alpha_n + N)) = 0$ *or* $\sum_{n=0}^{\infty}|\lambda_{k,n} - \lambda_{k,n+N}| < \infty$ *for* $k = 1, 2$;

(vi) $\lim_{n\to\infty}(|r_n - r_{n+N}|/(\alpha_n + N)) = 0$ *or* $\sum_{n=0}^{\infty}|r_n - r_{n+N}| < \infty$.

Then the following hold:

(a) $\{x_n\}_{n=0}^{\infty}$ *is bounded;*

(b) $\lim_{n\to\infty}\|x_n - x_{n+N}\| = 0$;

(c) $\lim_{n\to\infty}\|x_n - S_{[n+N]}\cdots S_{[n+1]}x_n\| = 0$ *provided* $\lim_{n\to\infty}(\|x_n - y_n\| + \|T_n v_n - v_n\|) = 0$;

(d) $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the unique element of* $\mathrm{VI}(\mathrm{VI}(\cap_{i=1}^{N}\mathrm{Fix}(S_i), \widetilde{A}_1), \widetilde{A}_2)$ *provided* $\|x_n - y_n\| + \|T_n v_n - v_n\| = o(\rho_n)$.

In Theorem 18, putting $K = 1$ and $M = 1$, we obtain the following.

Corollary 22. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $f : C \to \mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f. Let $N \geq 1$ be an integer. Let Θ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4), $\varphi : C \to \mathbf{R}$ a lower semicontinuous and convex functional with the restriction (B1) or (B2), and $A : H \to H$ ζ-inverse-strongly monotone. Let $R : C \to 2^H$ be a maximal monotone mapping and let $B : C \to H$ be η-inverse strongly monotone. Let $\{S_i\}_{i=1}^N$ be a finite family of nonexpansive mappings on H. Let $\widetilde{A}_1 : H \to H$ be α-inverse strongly monotone and let $\widetilde{A}_2 : H \to H$ be β-strongly monotone and κ-Lipschitz continuous. Assume that $\mathrm{VI}(\cap_{i=1}^N \mathrm{Fix}(S_i), \widetilde{A}_1) \neq \emptyset$ with $(\cap_{i=1}^N \mathrm{Fix}(S_i)) \subset \mathrm{GMEP}(\Theta, \varphi, A) \cap I(B, R) \cap \Gamma$. Let $\mu \in (0, 2\beta/\kappa^2)$, $\{\alpha_n\}_{n=0}^\infty \subset (0, 1]$, $\{\rho_n\}_{n=0}^\infty \subset (0, 2\alpha]$, $\{\mu_n\}_{n=0}^\infty \subset [a, b] \subset (0, 2\eta)$, and $\{r_n\}_{n=0}^\infty \subset [c, d] \subset (0, 2\zeta)$. For arbitrarily given $x_0 \in H$, let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned}
&\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n\rangle \\
&\quad + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\
&\quad v_n = J_{R,\mu_n}(I - \mu_n B) u_n, \\
&\quad y_n = S_{[n+1]}\left(I - \rho_n \widetilde{A}_1\right) T_n v_n, \\
&\quad x_{n+1} = y_n - \mu\alpha_n \widetilde{A}_2 y_n, \quad \forall n \geq 0,
\end{aligned} \tag{117}$$

where $P_C(I - \lambda_n \nabla f) = s_n I + (1 - s_n) T_n$ (here T_n is nonexpansive and $s_n := s_n(\lambda_n) = (2 - \lambda_n L)/4 \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$). Assume that

$$\begin{aligned}
\bigcap_{i=1}^N \mathrm{Fix}(S_i) &= \mathrm{Fix}(S_1 S_2 \cdots S_N) \\
&= \mathrm{Fix}(S_N S_1 \cdots S_{N-1}) \\
&= \cdots = \mathrm{Fix}(S_2 S_3 \cdots S_N S_1)
\end{aligned} \tag{118}$$

and that the following conditions are satisfied:

(i) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^\infty \alpha_n = \infty$ *and* $\rho_n \leq \alpha_n$ *for all* $n \geq 0$;

(ii) $\lim_{n\to\infty}(|\alpha_n - \alpha_{n+N}|/(\alpha_{n+N})) = 0$ *or* $\sum_{n=0}^\infty |\alpha_n - \alpha_{n+N}| < \infty$;

(iii) $\lim_{n\to\infty}(|s_n - s_{n+N}|/(\alpha_{n+N})) = 0$ *or* $\sum_{n=0}^\infty |s_n - s_{n+N}| < \infty$;

(iv) $\lim_{n\to\infty}(|\rho_n - \rho_{n+N}|/(\rho_{n+N})) = 0$ *or* $\sum_{n=0}^\infty |\rho_n - \rho_{n+N}| < \infty$;

(v) $\lim_{n\to\infty}(|\mu_n - \mu_{n+N}|/(\alpha_n + N)) = 0$ *or* $\sum_{n=0}^\infty |\mu_n - \mu_{n+N}| < \infty$;

(vi) $\lim_{n\to\infty}(|r_n - r_{n+N}|/(\alpha_n + N)) = 0$ *or* $\sum_{n=0}^\infty |r_n - r_{n+N}| < \infty$.

Then the following hold:

(a) $\{x_n\}_{n=0}^\infty$ *is bounded;*

(b) $\lim_{n\to\infty}\|x_n - x_{n+N}\| = 0$;

(c) $\lim_{n\to\infty}\|x_n - S_{[n+N]} \cdots S_{[n+1]} x_n\| = 0$ *provided* $\lim_{n\to\infty}(\|x_n - y_n\| + \|T_n v_n - v_n\|) = 0$;

(d) $\{x_n\}_{n=0}^\infty$ *converges strongly to the unique element of* $\mathrm{VI}(\mathrm{VI}(\cap_{i=1}^N \mathrm{Fix}(S_i), \widetilde{A}_1), \widetilde{A}_2)$ *provided* $\|x_n - y_n\| + \|T_n v_n - v_n\| = o(\rho_n)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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