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Research Article

Some Function Spaces via Orthonormal Bases on Spaces of Homogeneous Type

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Let (X, d, μ) be a space of homogeneous type in the sense of Coifman and Weiss, where the quasi-metric d may have no regularity and the measure μ satisfies only the doubling property. Adapting the recently developed randomized dyadic structures of X and applying orthonormal bases of $L^2(X)$ constructed recently by Auscher and Hytönen, we develop the Besov and Triebel-Lizorkin spaces on such a general setting. In this paper, we establish the wavelet characterizations and provide the dualities for these spaces. The results in this paper extend earlier related results with additional assumptions on the quasi-metric d and the measure μ to the full generality of the theory of these function spaces.

1. Introduction

The aim of this paper is to introduce the Besov and Triebel-Lizorkin spaces on spaces of homogeneous type in the sense of Coifman and Weiss without any additional assumption on the quasi-metric and the doubling measure. The main tool used in this paper is the remarkable orthonormal basis constructed recently by Auscher and Hytönen [1]. It is well known that function spaces play an important role in both classical and modern analysis, ordinary and partial differential equations, and approximation theory, and so forth. Since the seventies of last century, classical theory of the Besov spaces [2] as well as the Triebel-Lizorkin spaces [3, 4] has been developed rapidly from Euclidean spaces to spaces of homogeneous type in the sense of Coifman and Weiss with some additional assumptions on the quasi-metric and the doubling measure. See, for example, [5, 6] and references therein.

Let us recall briefly spaces of homogeneous type introduced by Coifman and Weiss [7]. A function $d : X \times X \to [0, \infty)$ is called a quasi-metric on a set X if d satisfies the following: (1) $d(x, y) = d(y, x)$ for all $x, y \in X$; (2) $d(x, y) = 0$ if and only if $x = y$; and (3) the quasi-triangle inequality holds: there is a constant $A_0 \geq 1$ such that

$$d(x, y) \leq A_0 [d(x, z) + d(z, y)] \tag{1}$$

for all x, y and $z \in X$. In addition, the quasi-metric ball $B(x, r)$ with center $x \in X$ and radius $r > 0$ is defined by $B(x, r) := \{y \in X : d(x, y) < r\}$. We say that a nonzero measure μ satisfies the doubling condition if there is a constant $C_\mu > 0$ such that, for all $x \in X$ and all $r > 0$,

$$\mu(B(x, 2r)) \leq C_\mu \mu(B(x, r)) < \infty. \tag{2}$$

We point out that the doubling condition (2) implies that there exist positive constants $\mathbb{Q}$ (the *upper dimension* of μ) and C such that, for all $x \in X$ and $\lambda \geq 1$,

$$\mu(B(x, \lambda r)) \leq C\lambda^{\mathbb{Q}} \mu(B(x, r)). \tag{3}$$

A space (X, d, μ) is said to be the space of homogeneous type in the sense of Coifman and Weiss [7] if d is a quasi-metric on X and μ satisfies the doubling condition. Such spaces have many applications in the theory of singular

integrals and function spaces [8]. Unfortunately, for some applications, additional assumptions were imposed on these general spaces due to the facts that the original quasi-metric may have no (Hölder) regularity and quasi-metric balls, even Borel sets, may not be open.

A recent work on the Besov spaces and the Triebel-Lizorkin spaces on spaces (X, d, μ) of homogeneous type in the sense of Coifman and Weiss was developed by Han et al. [6]. More precisely, in [6], in order to apply Coifman's construction for the approximation to the identity, they need that the quasi-metric d satisfies the Hölder regularity and the doubling measure μ is required to satisfy the reversed doubling condition; that is, there are constants $\kappa \in (0, \mathbb{Q}]$ and $c \in (0, 1]$ such that

$$c\lambda^{\kappa}\mu(B(x,r)) \le \mu(B(x,\lambda r)) \tag{4}$$

for all $x \in X$, $0 < r < \sup_{x,y\in X} d(x,y)/2$, and $1 \le \lambda < \sup_{x,y\in X} d(x,y)/2r$. This assumption ensures that

$$\sum_{k\in\mathbb{Z}:\delta^k\ge r} \frac{1}{\mu(B(x,\delta^k))} \le \frac{C}{\mu(B(x,r))}, \tag{5}$$

which is the key to develop the theory of the Besov and Triebel-Lizorkin spaces on spaces of homogeneous type. The main tool used in [6] is the so-called frame developed by Han [9] (see also [10]) because the Fourier transform is missing in general spaces of homogeneous type and wavelets were not available at that time. However, things seem to be changed after the quite recent work of Auscher and Hytönen [1], where remarkable orthonormal bases (wavelets) were constructed by using spline functions with the original quasi-metric and the doubling measure. These orthonormal bases open the door for developing function spaces on these general settings. See the very recent work in [11] for the theory of product H^p, CMO^p, VMO and duality on such general spaces. Motivated by the work in [11], in this paper, we develop the theory of the Besov and Triebel-Lizorkin spaces on spaces of homogeneous type in the sense of Coifman and Weiss without any additional assumption on the quasi-metric and the doubling measure. Therefore, results in this paper extend earlier related results with additional assumptions on the quasi-metric d and the measure μ to the full generality of the theory of the Besov and Triebel-Lizorkin spaces.

This paper is organized as follows. In Section 2 we give some notations and preliminaries, including a useful almost orthogonality estimate (see Lemma 7). Wavelet characterizations and dualities of the inhomogeneous Besov and Triebel-Lizorkin spaces are given in Section 3. We introduce homogeneous Besov and Triebel-Lizorkin spaces and discuss relationships between homogeneous and inhomogeneous Besov and Triebel-Lizorkin spaces in Section 4.

2. Preliminaries and Notations

Throughout this paper, (X, d, μ) denotes a space of homogeneous type with quasi-triangle constant A_0 and $\mu(X) = \infty$. By $V_r(x)$ we denote the measure of $B(x, r)$, the ball centered at x with radius $r > 0$, and by $V(x, y)$ we denote the measure of $B(x, d(x, y))$, the ball centered at x with radius $d(x, y) > 0$. In addition, we use the notation $a \lesssim b$ to mean that there is a constant $C > 0$ such that $a \le Cb$ and the notation $a \sim b$ to mean that $a \lesssim b \lesssim a$. The implicit constants, C, are meant to be independent of other relevant quantities.

We first recall the orthonormal basis of $L^2(X)$ constructed by Auscher and Hytönen [1]. Let s_α^k be spline functions constructed in [1, Section 3] and let V_k be the closed linear span in $L^2(X)$ of $\{s_\alpha^k\}_{\alpha\in\mathcal{X}^k}$. By [1, Theorem 5.1] we observe that

$$V_k \subset V_{k+1}, \qquad \bigcap_{k\in\mathbb{Z}} V_k = \{0\}, \qquad \overline{\bigcup_{k\in\mathbb{Z}} V_k} = L^2(X). \tag{6}$$

Further on, by writing W_k as the orthogonal (in $L^2(X)$) complement of V_k in V_{k+1}, Auscher and Hytönen pointed out the following result.

Theorem 1 (Theorems 6.1 and 7.1, [1]). *Let $a := (1 + 2\log_2 A_0)^{-1}$. There exists an orthonormal basis $\{\phi_\alpha^k\}_{\alpha\in\mathcal{X}^k}$ of V_k satisfying the following exponential decay:*

$$\left|\phi_\alpha^k(x)\right| \le \frac{C}{\sqrt{V_{\delta^k}(x_\alpha^k)}} \exp\left\{-\nu\left(\delta^{-k} d\left(x, x_\alpha^k\right)\right)^a\right\} \tag{7}$$

and the Hölder regularity of order η:

$$\left|\phi_\alpha^k(x) - \phi_\alpha^k(y)\right| \le \frac{C}{\sqrt{V_{\delta^k}(x_\alpha^k)}} \left(\frac{d(x,y)}{\delta^k}\right)^\eta \times \exp\left\{-\nu\left(\delta^{-k} d\left(x, x_\alpha^k\right)\right)^a\right\} \tag{8}$$

whenever $d(x, y) \le \delta^k$ for some $\eta \in (0, 1]$. Here $\delta > 0$ is a fixed parameter small enough, say $\delta < (1/1000)A_0^{-10}$, and C, ν are two positive constants independent of k, α, x, and x_α^k. Clearly, $\{\phi_\alpha^k\}_{\alpha\in\mathcal{X}^k, k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(X)$.

Also, there exists an orthonormal basis $\{\psi_\alpha^k\}_{\alpha\in\mathcal{Y}^k}$ of W_k satisfying the following exponential decay:

$$\left|\psi_\alpha^k(x)\right| \le \frac{C}{\sqrt{V_{\delta^k}(y_\alpha^k)}} \exp\left\{-\nu\left(\delta^{-k} d\left(x, y_\alpha^k\right)\right)^a\right\}, \tag{9}$$

the Hölder regularity of order η:

$$\left|\psi_\alpha^k(x) - \psi_\alpha^k(y)\right| \le \frac{C}{\sqrt{V_{\delta^k}(y_\alpha^k)}} \left(\frac{d(x,y)}{\delta^k}\right)^\eta \times \exp\left\{-\nu\left(\delta^{-k} d\left(x, y_\alpha^k\right)\right)^a\right\} \tag{10}$$

whenever $d(x, y) \le \delta^k$ for some $\eta \in (0, 1]$, and the cancellation property:

$$\int_X \psi_\alpha^k(x)\, d\mu(x) = 0, \quad k \in \mathbb{Z},\ \alpha \in \mathcal{Y}^k, \tag{11}$$

where δ is given above and C, ν are two positive constants independent of k, α, x, and y_α^k. Clearly, $\{\psi_\alpha^k\}_{\alpha\in\mathcal{Y}^k, k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(X)$. In what follows, we also refer to the functions ψ_α^k as wavelets.

In order to introduce the (inhomogeneous) Besov and Triebel-Lizorkin spaces, we need the following concepts of test functions and distributions. Refer to [12] for details and also see [11].

Definition 2. For fixed $x_0 \in X$, $r > 0$, $\beta \in (0, \eta]$, where η is given in Theorem 1, and $\gamma > 0$. A function f is said to be a test function of type (x_0, r, β, γ) centered at x_0 with width $r > 0$ if f satisfies the following decay and Hölder regularity properties:

(i) $|f(x)| \le C(1/(V_r(x_0) + V(x, x_0)))(r/(r + d(x, x_0)))^\gamma$ for all $x \in X$,

(ii) $|f(x) - f(y)| \le C(d(x, y)/(r + d(x, x_0)))^\beta (1/(V_r(x_0) + V(x, x_0)))(r/(r + d(x, x_0)))^\gamma$ for all $x, y \in X$ such that $d(x, y) \le (1/2A_0)(r + d(x, x_0))$.

If f is a test function of type (x_0, r, β, γ), we write $f \in G(x_0, r, \beta, \gamma)$. The norm of f on $G(x_0, r, \beta, \gamma)$ is defined by

$$\|f\|_{G(x_0,r,\beta,\gamma)} := \inf\{C > 0 : \text{(i) and (ii) hold}\}. \tag{12}$$

We denote $G(\beta, \gamma) := G(x_0, 1, \beta, \gamma)$. It is easy to check that $G(x_1, r, \beta, \gamma) = G(\beta, \gamma)$ with equivalent norms for any fixed $x_1 \in X$ and $r > 0$. Furthermore, it is also easy to see that $G(\beta, \gamma)$ is a Banach space with respect to the norm on $G(\beta, \gamma)$. For given $\epsilon \in (0, \eta]$, let $\mathring{G}(\beta, \gamma)$ be the completion of the space $G(\epsilon, \epsilon)$ in $G(\beta, \gamma)$ with $0 < \beta, \gamma \le \epsilon$. Obviously, $\mathring{G}(\epsilon, \epsilon) = G(\epsilon, \epsilon)$. In addition, we say that $f \in \mathring{G}(\beta, \gamma)$ if $f \in G(\beta, \gamma)$ and there is a sequence $\{f_n\} \subset G(\epsilon, \epsilon)$ such that $\|f_n - f\|_{G(\beta,\gamma)} \to 0$ as $n \to \infty$. For given $f \in \mathring{G}(\beta, \gamma)$, we define $\|f\|_{\mathring{G}(\beta,\gamma)} := \|f\|_{G(\beta,\gamma)}$. Obviously, $\mathring{G}(\beta, \gamma)$ is a Banach space with respect to the norm $\|\cdot\|_{\mathring{G}(\beta,\gamma)}$. As usual, we denote by $(\mathring{G}(\beta, \gamma))'$ the dual space of $\mathring{G}(\beta, \gamma)$ consisting of all continuous linear functionals on $\mathring{G}(\beta, \gamma)$.

Han et al. [11] showed that the wavelets constructed by Auscher and Hytönen [1] are test functions defined above.

Lemma 3 (see [11], Theorem 3.3). *Let ψ_α^k be given in Theorem 1 with the Hölder regularity of η. Then $\psi_\alpha^k / \sqrt{V_{\delta^k}(x_\alpha^k)} \in G(x_\alpha^k, \delta^k, \eta, \epsilon)$ for each $\epsilon > 0$.*

By analogous arguments, we can obtain the next result.

Proposition 4. *Let ϕ_α^k be given in Theorem 1 with Hölder regularity of η. Then $\phi_\alpha^k / \sqrt{V_{\delta^k}(x_\alpha^k)} \in G(x_\alpha^k, \delta^k, \eta, \epsilon)$ for each $\epsilon > 0$.*

Let P_k and Q_k be orthogonal projections onto V_k and W_k, respectively. We next give some estimates on the kernels of operators P_k and Q_k.

Lemma 5. *The kernel $P_k(x, y)$ is symmetric in x and y, and, for any $\epsilon > 0$, $P_k(x, y)$ satisfies that*

(i) *for all $x, y \in X$,*

$$|P_k(x, y)| \le C \frac{1}{V_{\delta^k}(x) + V(x, y)} \left(\frac{\delta^k}{\delta^k + d(x, y)} \right)^\epsilon, \tag{13}$$

(ii) *for $d(y, y') \le (1/2A_0)(\delta^k + d(x, y))$,*

$$\begin{aligned} &|P_k(x, y) - P_k(x, y')| \\ &\quad \le C \left(\frac{d(y, y')}{\delta^k + d(x, y)} \right)^\eta \frac{1}{V_{\delta^k}(x) + V(x, y)} \\ &\quad \times \left(\frac{\delta^k}{\delta^k + d(x, y)} \right)^\epsilon, \end{aligned} \tag{14}$$

(iii) *for $d(x, x') \le (1/2A_0)(\delta^k + d(x, y))$ and $d(y, y') \le (1/2A_0)(\delta^k + d(x, y'))$,*

$$\begin{aligned} &|[P_k(x, y) - P_k(x', y)] - [P_k(x, y') - P_k(x', y')]| \\ &\quad \le C \left(\frac{d(x, x')}{\delta^k + d(x, y)} \right)^\eta \left(\frac{d(y, y')}{\delta^k + d(x, y)} \right)^\eta \\ &\quad \times \frac{1}{V_{\delta^k}(x) + V(x, y)} \left(\frac{\delta^k}{\delta^k + d(x, y)} \right)^\epsilon. \end{aligned} \tag{15}$$

Proof. Note that, for any $x, x_0 \in X$ and $r > 0$ with $r \le d(x, x_0)$, from the doubling property on the measure μ it follows that

$$V(x, x_0) \le C \left(\frac{d(x, x_0)}{r} \right)^{\mathbb{Q}} V_r(x_0), \tag{16}$$

$$\frac{1}{V_r(x_0)} \le C \frac{1}{V(x, x_0) + V_r(x_0)} \left(\frac{d(x, x_0)}{r} \right)^{\mathbb{Q}}. \tag{17}$$

Also, note that the kernel $P_k(x, y)$ is symmetric in x and y due to [1, Lemma 10.1]. Thus, it suffices to prove (13) for the cases that $\delta^k > d(x, y)$ and $\delta^k \le d(x, y)$. Again, by [1, Lemma 10.1] we have

$$|P_k(x, y)| \le C \frac{1}{V_{\delta^k}(x)} \left(\frac{\delta^k}{\delta^k + d(x, y)} \right)^\Gamma \tag{18}$$

for all $\Gamma > 0$, where the constant C is independent of k. If $\delta^k > d(x, y)$, then $V(x, y) < V_{\delta^k}(x)$ and hence

$$\begin{aligned} &\frac{1}{V_{\delta^k}(x)} \left(\frac{\delta^k}{\delta^k + d(x, y)} \right)^\Gamma \\ &\quad \le C \frac{1}{V(x, y) + V_{\delta^k}(x)} \left(\frac{\delta^k}{\delta^k + d(x, y)} \right)^\Gamma. \end{aligned} \tag{19}$$

On the other hand, if $\delta^k < d(x,y)$, then (17) implies that

$$\frac{1}{V_{\delta^k}(x)}\left(\frac{\delta^k}{\delta^k + d(x,y)}\right)^{\Gamma} \le C\frac{1}{V(x,y)+V_{\delta^k}(x)}\left(\frac{\delta^k}{\delta^k + d(x,y)}\right)^{\Gamma-\mathbb{Q}}. \tag{20}$$

By letting $\epsilon := \Gamma - \mathbb{Q}$, we obtain (13), immediately.

The proof of (14) is based on the estimates

$$\begin{aligned}\left|P_k(x,y) - P_k(x,y')\right| &\le C\left(\frac{d(y,y')}{\delta^k}\right)^{\eta} \\ &\times\left(\frac{\exp\left\{-\nu\left(\delta^{-k}d(x,y)\right)^s\right\}}{\sqrt{V_{\delta^k}(x)V_{\delta^k}(y)}} + \frac{\exp\left\{-\nu\left(\delta^{-k}d(x,y')\right)^s\right\}}{\sqrt{V_{\delta^k}(x)V_{\delta^k}(y')}}\right)\end{aligned} \tag{21}$$

given in [1, Lemma 10.1]. Indeed, since $d(y,y') \le (1/2A_0)(\delta^k + d(x,y))$ implies that $\delta^k + d(x,y') \ge (1/2A_0)(\delta^k + d(x,y))$ and that $V_{\delta^k}(x) + V(x,y') \sim V_{\delta^k}(x) + V(x,y)$, it follows from (17) and (21) that

$$\begin{aligned}II &\le C\left(\frac{d(y,y')}{\delta^k}\right)^{\eta}\frac{1}{V_{\delta^k}(x)+V(x,y)} \times\left(\frac{\delta^k}{\delta^k+d(x,y)}\right)^{\Gamma} \\ &\le C\left(\frac{d(y,y')}{\delta^k+d(x,y)}\right)^{\eta}\frac{1}{V_{\delta^k}(x)+V(x,y)} \times\left(\frac{\delta^k}{\delta^k+d(x,y)}\right)^{\Gamma-\eta}.\end{aligned} \tag{22}$$

Applying $\epsilon := \Gamma - \eta$ yields (14), immediately. The proof of (15) is similar to that of (13) and (14), and we omit it here. □

Lemma 6 (see [11], Lemma 3.6). *The kernel $Q_k(x,y)$ is symmetric in x and y and, for any $\epsilon > 0$, satisfies that*

(i) *for all $x, y \in X$,*

$$\left|Q_k(x,y)\right| \le C\frac{1}{V_{\delta^k}(x)+V(x,y)}\left(\frac{\delta^k}{\delta^k+d(x,y)}\right)^{\epsilon}, \tag{23}$$

(ii) *for $d(y,y') \le (1/2A_0)(\delta^k + d(x,y))$,*

$$\begin{aligned}&\left|Q_k(x,y) - Q_k(x,y')\right| \\ &\le C\left(\frac{d(y,y')}{\delta^k+d(x,y)}\right)^{\eta} \times\frac{1}{V_{\delta^k}(x)+V(x,y)}\left(\frac{\delta^k}{\delta^k+d(x,y)}\right)^{\epsilon},\end{aligned} \tag{24}$$

(iii) *for $d(x,x') \le (1/2A_0)(\delta^k + d(x,y))$ and $d(y,y') \le (1/2A_0)(\delta^k + d(x,y))$*

$$\begin{aligned}&\left|\left[Q_k(x,y) - Q_k(x',y)\right] - \left[Q_k(x,y') - Q_k(x',y')\right]\right| \\ &\le C\left(\frac{d(x,x')}{\delta^k+d(x,y)}\right)^{\eta}\left(\frac{d(y,y')}{\delta^k+d(x,y)}\right)^{\eta} \times\frac{1}{V_{\delta^k}(x)+V(x,y)}\left(\frac{\delta^k}{\delta^k+d(x,y)}\right)^{\epsilon}.\end{aligned} \tag{25}$$

Let $\{\varphi_\beta^l\}_{l\in\mathbb{Z},\beta\in\mathscr{Y}^l}$ be another orthonormal basis of $L^2(X)$ given by the algorithm in Theorem 1, and let E_l be the projection onto W_l with respect to the basis $\{\varphi_\beta^l\}_{l\in\mathbb{Z},\beta\in\mathscr{Y}^l}$. Finally, we give the almost orthogonality estimate of the kernel of operator Q_kE_l to end this section. The proof is similar to that in [6, Lemma 3.1], and we omit it here.

Lemma 7. *Let $\epsilon > 0$, and let $\eta > 0$ be the order of the Hölder regularity given in (8). There exists a constant $C = C_\epsilon > 0$, dependent only on ϵ, such that, for all $\sigma \in (0,\eta]$,*

$$\left|Q_kE_l(x,y)\right| \le C\delta^{|k-l|\sigma}\frac{1}{V_{\delta^{(k\wedge l)}}(x)+V(x,y)} \times\left(\frac{\delta^{(k\wedge l)}}{\delta^{(k\wedge l)}+d(x,y)}\right)^{\epsilon}, \tag{26}$$

(ii) for all $\tau \in (0,1)$,

$$\begin{aligned}&\left|Q_kE_l(x,y) - Q_kE_l(x,y')\right| \\ &\le C\delta^{|k-l|\sigma\tau}\left(\frac{d(y,y')}{\delta^{(k\wedge l)}+d(x,y)}\right)^{\eta(1-\tau)} \times\frac{1}{V_{\delta^{(k\wedge l)}}(x)+V(x,y)}\left(\frac{\delta^{(k\wedge l)}}{\delta^{(k\wedge l)}+d(x,y)}\right)^{\epsilon}\end{aligned} \tag{27}$$

whenever $d(y,y') < (1/(2A_0)^2)d(x,y)$. The same estimate holds with x and y interchanged.

3. Inhomogeneous Besov and Triebel-Lizorkin Spaces on Spaces of Homogeneous Type

In this section, we introduce the inhomogeneous Besov and Triebel-Lizorkin spaces on spaces of homogeneous type and give wavelet characterizations of corresponding spaces.

We would like to point out that the wavelet expansion (Theorem 9) is the key to develop the theory of the Besov and Triebel-Lizorkin spaces. As defined previously, by P_k and Q_k we denote the orthogonal projections onto V_k and W_k, respectively. These two projections provide us with regular Littlewood-Paley decompositions for spaces of homogeneous type.

Theorem 8 (Theorem 10.2, [1]). *Let $f \in L^2(X)$. Then*

$$f = \sum_{k\in\mathbb{Z}} Q_k(f) = \sum_{k\in\mathbb{Z}} Q_k^2(f), \tag{28}$$

which gives homogeneous Littlewood-Paley decomposition, while

$$f = P_l(f) + \sum_{k\geq l} Q_k(f) = P_l^2(f) + \sum_{k\geq l} Q_k^2(f) \tag{29}$$

provides inhomogeneous Littlewood-Paley decomposition.

Furthermore, following the proof of Theorem 3.4 in [11], we can show that such wavelet expansions also hold in the space of distributions.

Theorem 9. *Let $0 < \beta, \gamma < \eta$. Then wavelet expansion (29) holds in $\mathring{G}(\beta', \gamma')$ with $\beta' \in (0, \beta)$ and $\gamma' \in (0, \gamma)$ and hence holds in $(\mathring{G}(\beta', \gamma'))'$ with $\beta' \in (\beta, \eta)$ and $\gamma' \in (\gamma, \eta)$ as well.*

Proof. Since $G(\beta, \gamma) \subset L^2(X)$ and (29) holds in the sense of $L^2(X)$, (29) holds for such test function f in the (a.e.) pointwise sense (for some subsequence with respect to the convergence).

By analogous arguments given in [11, Theorem 3.4], we can show the convergence of $\sum_{k\geq 1} Q_k(f)$ and $P_1(f)$ in $G(\beta', \gamma')$ by using the size and smooth conditions of kernels $Q_k(x, y)$ and $P_1(x, y)$, respectively. Therefore,

$$f = P_1(f) + \sum_{k\geq 1} Q_k(f) \tag{30}$$

in the sense of $G(\beta', \gamma')$. The second equality of (29) holds for the same reason. □

We now introduce the inhomogeneous Besov and Triebel-Lizorkin spaces on spaces of homogeneous type.

Definition 10. Let $1 < p, q \leq \infty$ and $|s| < \eta$. The inhomogeneous Besov space $B_p^{s,q}(X)$ is defined by

$$B_p^{s,q}(X) := \left\{ f \in \left(\mathring{G}(\beta, \gamma)\right)' : \|f\|_{B_p^{s,q}} < \infty \right\}, \tag{31}$$

where

$$\|f\|_{B_p^{s,q}} := \|P_1(f)\|_p + \left\{ \sum_{k\geq 1} \left(\delta^{-ks} \|Q_k(f)\|_p\right)^q \right\}^{1/q}. \tag{32}$$

For $1 < p < \infty$ and $1 < q \leq \infty$, the inhomogeneous Triebel-Lizorkin space $F_p^{s,q}(X)$ is defined by

$$F_p^{s,q}(X) := \left\{ f \in \left(\mathring{G}(\beta, \gamma)\right)' : \|f\|_{F_p^{s,q}} < \infty \right\}, \tag{33}$$

where

$$\|f\|_{F_p^{s,q}} := \|P_1(f)\|_p + \left\| \left\{ \sum_{k\geq 1} \left(\delta^{-ks} |Q_k(f)|\right)^q \right\}^{1/q} \right\|_p. \tag{34}$$

Remark 11. Observe that

$$\|f\|_{B_p^{s,q}} \sim \left\{ \|P_1(f)\|_p^q + \sum_{k\geq 1} \left(\delta^{-ks} \|Q_k(f)\|_p\right)^q \right\}^{1/q}, \quad f \in B_p^{s,q}, \tag{35}$$

as well as

$$\|f\|_{F_p^{s,q}} \sim \left\| \left\{ P_1(f)^q + \sum_{k\geq 1} \left(\delta^{-ks} |Q_k(f)|\right)^q \right\}^{1/q} \right\|_p, \quad f \in F_p^{s,q}. \tag{36}$$

The next result shows that the Besov and Triebel-Lizorkin norms given in Definition 10 are both independent of choice of the orthonormal basis $\{\psi_\alpha^k\}_{\alpha\in\mathscr{Y}^k, k\in\mathbb{Z}}$ of $L^2(X)$ and hence, they are independent of the choice of operators P_1 and Q_k with $k \geq 1$. Thus, the Besov and Triebel-Lizorkin spaces given above are both well defined. To see this, let $\{\phi_\alpha^k\}_{\alpha\in\mathscr{Y}^k, k\in\mathbb{Z}}$ be another orthonormal basis of $L^2(X)$ given in Theorem 1, and let E_1 and D_k be projections onto V_1 and W_k, respectively, with respect to the basis $\{\phi_\alpha^k\}_{\alpha\in\mathscr{Y}^k, k\in\mathbb{Z}}$.

Proposition 12. *Let $|s| < \eta$, and let $f \in (\mathring{G}(\beta, \gamma))'$. For all $1 < p, q \leq \infty$, then*

$$\|P_1(f)\|_p + \left\{ \sum_{k\geq 1} \left(\delta^{-ks} \|Q_k(f)\|_p\right)^q \right\}^{1/q} \sim \|E_1(f)\|_p + \left\{ \sum_{l\geq 1} \left(\delta^{-ls} \|D_l(f)\|_p\right)^q \right\}^{1/q}, \tag{37}$$

and for all $1 < p < \infty$ and $1 < q \leq \infty$

$$\|P_1(f)\|_p + \left\| \left\{ \sum_{k\geq 1} \left(\delta^{-ks} |Q_k(f)|\right)^q \right\}^{1/q} \right\|_p \sim \|E_1(f)\|_p + \left\| \left\{ \sum_{l\geq 1} \left(\delta^{-ls} |D_l(f)|\right)^q \right\}^{1/q} \right\|_p. \tag{38}$$

Proof. Clearly, the equivalence of $\|P_1(f)\|_p$ and $\|E_1(f)\|_p$ is a direct consequence of the size condition (13) and the wavelet expansion (29). We next show that

$$\left\{\sum_{k\geq 1}\left(\delta^{-ks}\|Q_k(f)\|_p\right)^q\right\}^{1/q} \lesssim \left\{\sum_{l\geq 1}\left(\delta^{-ls}\|D_l(f)\|_p\right)^q\right\}^{1/q}. \tag{39}$$

Indeed, by the wavelet expansion (29) and almost orthogonal estimate (26), we have

$$\begin{aligned}
&|Q_k(f)(x)| \\
&\leq \sum_{l\geq 1}\int_X |Q_kD_l(x,y)|\,|D_l(f)(y)|\,d\mu(y) \\
&\leq C\sum_{l\geq 1}\delta^{|k-l|\eta}\int_X \frac{1}{V_{\delta^{(k\wedge l)}}+V(x,y)}\left(\frac{\delta^{(k\wedge l)}}{\delta^{(k\wedge l)}+d(x,y)}\right)^{\epsilon} \\
&\qquad\qquad \times|D_l(f)(y)|\,d\mu(y) \\
&\leq C\sum_{l\geq 1}\delta^{|k-l|\eta}M(D_l(f))(x),
\end{aligned} \tag{40}$$

where M denotes the Hardy-Littlewood maximal operator. Thus,

$$\left\{\sum_{k\geq 1}\left(\delta^{-ks}\|Q_k(f)(x)\|_p\right)^q\right\}^{1/q} \leq C\left\{\sum_{l\geq 1}\left(\delta^{-ls}\|D_l(f)\|_p\right)^q\right\}^{1/q} \tag{41}$$

due to the fact that $|s|<\eta$. This proves (39), and hence, we obtain (37) symmetrically.

We now prove (38). Clearly, by symmetry, it suffices to show that

$$\left\|\left\{\sum_{k\geq 1}\left(\delta^{-ks}|Q_k(f)|\right)^q\right\}^{1/q}\right\|_p \leq \left\|\left\{\sum_{l\geq 1}\left(\delta^{-ls}|D_l(f)|\right)^q\right\}^{1/q}\right\|_p. \tag{42}$$

To this end, from (40) it follows that

$$\begin{aligned}
&\sum_{k\geq 1}\left(\delta^{-ks}|Q_k(f)(x)|\right)^q \\
&\leq C\sum_{k\geq 1}\left(\sum_{l\geq 1}\delta^{(l-k)s}\delta^{|k-l|\eta}\delta^{-ls}M(D_l(f))(x)\right)^q \\
&\leq C\sum_{k\geq 1}\sum_{l\geq 1}\delta^{(l-k)s}\delta^{|k-l|\eta}\left(\delta^{-ls}M(D_l(f))(x)\right)^q \\
&\leq C\sum_{l\geq 1}\left(\delta^{-ls}M(D_l(f))(x)\right)^q
\end{aligned} \tag{43}$$

due to the fact that $\sum_{l\geq 1}\delta^{(l-k)s}\delta^{|k-l|\eta}<\infty$ and $\sum_{k\geq 1}\delta^{(l-k)s}\delta^{|k-l|\eta}<\infty$ for $|s|<\eta$. Thus, applying the Fefferman-Stein vector-valued maximal inequality, we obtain (42), immediately. The proof is complete. □

Let $\{\psi_\alpha^k\}_{\alpha\in\mathcal{Y}^k}$ be an orthonormal basis of W_k and let $\{\phi_\alpha^1\}_{\alpha\in\mathcal{X}^1}$ be an orthonormal basis of V_1 as shown in Theorem 1. Applying the wavelet expansion, given in Theorem 9,

$$f=\sum_{\alpha\in\mathcal{X}^1}\langle\phi_\alpha^1,f\rangle\,\phi_\alpha^1+\sum_{k\geq 1}\sum_{\alpha\in\mathcal{Y}^k}\langle\psi_\alpha^k,f\rangle\,\psi_\alpha^k \tag{44}$$

we now give the wavelet characterizations of the inhomogeneous Besov and Triebel-Lizorkin spaces.

Theorem 13. *Let $|s|<\eta$ and let ϕ_α^1, ψ_α^k be that given in Theorem 1 with $k\geq 1$. For $1<p,q\leq\infty$, then*

$$\begin{aligned}
&\|f\|_{B_p^{s,q}} \\
&\sim\left\{\left[\sum_{\alpha\in\mathcal{X}^1}\left(\mu\left(Q_\alpha^1\right)^{1/p-1/2}\left|\langle\phi_\alpha^1,f\rangle\right|\right)^p\right]^{q/p}\right. \\
&\qquad\left.+\sum_{k\geq 1}\delta^{-ksq}\left[\sum_{\alpha\in\mathcal{Y}^k}\left(\mu\left(Q_\alpha^k\right)^{1/p-1/2}\left|\langle\psi_\alpha^k,f\rangle\right|\right)^p\right]^{q/p}\right\}^{1/q},
\end{aligned} \tag{45}$$

and for $1<p<\infty$, $1<p\leq\infty$, then

$$\begin{aligned}
&\|f\|_{F_p^{s,q}} \\
&\sim\left\|\left\{\left(\sum_{\alpha\in\mathcal{X}^k}\mu\left(Q_\alpha^k\right)^{-1/2}\left|\langle\phi_\alpha^1,f\rangle\right|\chi_{Q_\alpha^1}\right)^q\right.\right. \\
&\qquad\left.\left.+\sum_{k\geq 1}\delta^{-ksq}\left[\sum_{\alpha\in\mathcal{Y}^k}\mu\left(Q_\alpha^k\right)^{-1/2}\left|\langle\psi_\alpha^k,f\rangle\right|\chi_{Q_\alpha^k}\right]^q\right\}^{1/q}\right\|_p.
\end{aligned} \tag{46}$$

Proof. Observe that, by analogous arguments in the proof of (26), we have

$$\begin{aligned}
&\left|P_1\left(\frac{\phi_\alpha^1}{\sqrt{\mu\left(Q_\alpha^1\right)}}\right)(x)\right| \\
&\leq C\frac{1}{V_\delta\left(x_\alpha^1\right)+V\left(x,x_\alpha^1\right)}\left(\frac{\delta}{\delta+d\left(x,x_\alpha^1\right)}\right)^{\epsilon},
\end{aligned}$$

$$\left|Q_k\left(\frac{\psi_\alpha^k}{\sqrt{\mu\left(Q_\alpha^j\right)}}\right)(x)\right| \le \frac{C\delta^{|j-k|\eta}}{V_{\delta^{(j\wedge k)}}\left(x_\alpha^j\right)+V\left(x,x_\alpha^j\right)}\left(\frac{\delta^{(j\wedge k)}}{\delta^{(j\wedge k)}+d\left(x,x_\alpha^j\right)}\right)^\epsilon. \tag{47}$$

In order to prove (45), by Remark 11, it suffices to show that

$$\left\{\|P_1(f)\|_p^q+\sum_{k\ge1}\left(\delta^{-ks}\|Q_k(f)\|_p\right)^q\right\}^{1/q} \sim \left\{\left[\sum_{\alpha\in\mathcal{X}^1}\left(\mu\left(Q_\alpha^1\right)^{1/p-1/2}\left|\left\langle\phi_\alpha^1,f\right\rangle\right|\right)^p\right]^{q/p} + \sum_{k\ge1}\delta^{-ksq}\left[\sum_{\alpha\in\mathcal{Y}^k}\left(\mu\left(Q_\alpha^k\right)^{1/p-1/2}\left|\left\langle\psi_\alpha^k,f\right\rangle\right|\right)^p\right]^{q/p}\right\}^{1/q}. \tag{48}$$

By the wavelet expansion (44), we have

$$\begin{aligned}
&\left|P_1(f)(x)\right| \\
&\le \sum_{\alpha\in\mathcal{X}^1}\mu\left(Q_\alpha^1\right)\left|\left\langle\frac{\phi_\alpha^1}{\sqrt{\mu\left(Q_\alpha^1\right)}},f\right\rangle\right| \times\left|P_1\left(\frac{\phi_\alpha^1}{\sqrt{\mu\left(Q_\alpha^1\right)}}\right)(x)\right| \\
&\le C\sum_{\alpha\in\mathcal{X}^1}\int_{Q_\alpha^1}\frac{1}{V_\delta\left(x_\alpha^1\right)+V\left(x,x_\alpha^1\right)} \times\left(\frac{\delta}{\delta+d\left(x,x_\alpha^1\right)}\right)^\epsilon \times\left|\left\langle\frac{\phi_\alpha^1}{\sqrt{\mu\left(Q_\alpha^1\right)}},f\right\rangle\right|dz \\
&\le C\sum_{\alpha\in\mathcal{X}^1}\int_{Q_\alpha^1}\frac{1}{V_\delta(z)+V(x,z)}\left(\frac{\delta}{\delta+d(x,z)}\right)^\epsilon \times\left|\left\langle\frac{\phi_\alpha^1}{\sqrt{\mu\left(Q_\alpha^1\right)}},f\right\rangle\right|dz \\
&= C\sum_{\alpha\in\mathcal{X}^1}\int_X\frac{1}{V_\delta(z)+V(x,z)}\left(\frac{\delta}{\delta+d(x,z)}\right)^\epsilon \times\left|\left\langle\frac{\phi_\alpha^1}{\sqrt{\mu\left(Q_\alpha^1\right)}},f\right\rangle\right|\chi_{Q_\alpha^1}(z)\,dz \\
&\le CM\left(\sum_{\alpha\in\mathcal{X}^1}\left|\left\langle\frac{\phi_\alpha^1}{\sqrt{\mu\left(Q_\alpha^1\right)}},f\right\rangle\right|\chi_{Q_\alpha^1}\right)(x),
\end{aligned} \tag{49}$$

so that

$$\|P_1(f)\|_p \le C\left[\sum_{\alpha\in\mathcal{X}^1}\left(\mu\left(Q_\alpha^1\right)^{1/p-1/2}\left|\left\langle\phi_\alpha^1,f\right\rangle\right|\right)^p\right]^{1/p}. \tag{50}$$

By analogous arguments, we also have

$$\begin{aligned}
&\left|Q_k(f)(x)\right| \\
&\le \sum_{\alpha\in\mathcal{Y}^k}\mu\left(Q_\alpha^k\right)\left|\left\langle\frac{\psi_\alpha^k}{\sqrt{\mu\left(Q_\alpha^k\right)}},f\right\rangle\right| \times\left|Q_k\left(\frac{\psi_\alpha^k}{\sqrt{\mu\left(Q_\alpha^k\right)}}\right)(x)\right| \\
&\le CM\left(\sum_{\alpha\in\mathcal{Y}^k}\left|\left\langle\frac{\psi_\alpha^k}{\sqrt{\mu\left(Q_\alpha^k\right)}},f\right\rangle\right|\chi_{Q_\alpha^k}\right)(x),
\end{aligned} \tag{51}$$

and hence,

$$\begin{aligned}
&\sum_{k\ge1}\left(\delta^{-ks}\|Q_k(f)\|_p\right)^q \\
&\le C\sum_{k\ge1}\delta^{-ksq}\left\|M\left(\sum_{\alpha\in\mathcal{Y}^k}\left|\left\langle\frac{\psi_\alpha^k}{\sqrt{\mu\left(Q_\alpha^k\right)}},f\right\rangle\right|\chi_{Q_\alpha^k}\right)\right\|_p^q \\
&\le C\sum_{k\ge1}\delta^{-ksq}\left\|\sum_{\alpha\in\mathcal{Y}^k}\left|\left\langle\frac{\psi_\alpha^k}{\sqrt{\mu\left(Q_\alpha^k\right)}},f\right\rangle\right|\chi_{Q_\alpha^k}\right\|_p^q \\
&\le C\sum_{k\ge1}\delta^{-ksq}\left[\sum_{\alpha\in\mathcal{Y}^k}\left(\mu\left(Q_\alpha^k\right)^{1/p-1/2}\left|\left\langle\psi_\alpha^k,f\right\rangle\right|\right)^p\chi_{Q_\alpha^k}\right]^{q/p}.
\end{aligned} \tag{52}$$

This implies that

$$\left\{\|P_1(f)\|_p^q+\sum_{k\geq 1}\left(\delta^{-ks}\|Q_k(f)\|_p\right)^q\right\}^{1/q}$$
$$\leq\left\{\left[\sum_{\alpha\in\mathcal{X}^1}\left(\mu\left(Q_\alpha^1\right)^{1/p-1/2}\left|\left\langle\phi_\alpha^1,f\right\rangle\right|\right)^p\right]^{q/p}\right.$$
$$\left.+\sum_{k\geq 1}\delta^{-ksq}\left[\sum_{\alpha\in\mathcal{Y}^k}\left(\mu\left(Q_\alpha^k\right)^{1/p-1/2}\left|\left\langle\psi_\alpha^k,f\right\rangle\right|\right)^p\right]^{q/p}\right\}^{1/q}. \tag{53}$$

On the other hand, by using the wavelet expansion and the size condition, we have

$$\sum_{\alpha\in\mathcal{X}^1}\left(\mu\left(Q_\alpha^1\right)^{1/p-1/2}\left|\left\langle\phi_\alpha^1,f\right\rangle\right|\right)^p$$
$$\leq\sum_{\alpha\in\mathcal{X}^1}\mu\left(Q_\alpha^1\right)\times\left(\int_X\left|P_1(f)(y)\right|\left|P_1\left(\frac{\phi_\alpha^1}{\sqrt{\mu\left(Q_\alpha^1\right)}}\right)(y)\right|dy\right)^p$$
$$\leq C\|P_1(f)\|_p^p \tag{54}$$

as well as

$$\sum_{\alpha\in\mathcal{Y}^i}\left(\left|\left\langle\psi_\alpha^j,f\right\rangle\right|\mu\left(Q_\alpha^j\right)^{1/p-1/2}\right)^p$$
$$\leq\sum_{\alpha\in\mathcal{Y}^i}\mu\left(Q_\alpha^j\right)\left(\sum_{k\geq 1}\int_X\left|\int_X\frac{\psi_\alpha^j(x)}{\sqrt{\mu\left(Q_\alpha^j\right)}}Q_k(x,y)d\mu(x)\right|\left|Q_k(f)(y)\right|d\mu(y)\right)^p$$
$$\leq C\int_X\left(\sum_{k\geq 1}\delta^{|j-k|\eta}\int_X\frac{1}{V_{\delta^{(j\wedge k)}}(z)+V(y,z)}\left(\frac{\delta^{(j\wedge k)}}{\delta^{(j\wedge k)}+d(y,z)}\right)^\epsilon\left|Q_k(f)(y)\right|d\mu(y)\right)^p d\mu(z) \tag{55}$$
$$\leq C\left\|\sum_{k\geq 1}\delta^{|j-k|\eta}M\left(Q_k(f)\right)\right\|_p^p$$
$$\leq C\left(\sum_{k\geq 1}\delta^{|j-k|\eta p}\|Q_k(f)\|_p\right)^p,$$

and hence, by using the Hölder inequality, we have further that

$$\left\{\left[\sum_{\alpha\in\mathcal{X}^1}\left(\mu\left(Q_\alpha^1\right)^{1/p-1/2}\left|\left\langle\phi_\alpha^1,f\right\rangle\right|\right)^p\right]^{q/p}\right.$$
$$\left.+\sum_{j\geq 1}\delta^{-jsq}\left[\sum_{\alpha\in\mathcal{Y}^j}\left(\mu\left(Q_\alpha^j\right)^{1/p-1/2}\left|\left\langle\psi_\alpha^j,f\right\rangle\right|\right)^p\right]^{q/p}\right\}^{1/q}$$
$$\leq C\left\{\|P_1(f)\|_p^q\right.$$
$$\left.+\sum_{j\geq 1}\left(\sum_{k\geq 1}\delta^{(k-j)s}\delta^{|j-k|\eta}\delta^{-ks}\|Q_k(f)\|_p\right)^q\right\}^{1/q}$$
$$\leq C\left\{\|P_1(f)\|_p^q+\sum_{k\geq 1}\left(\delta^{-ks}\|Q_k(f)\|_p\right)^q\right\}^{1/q}. \tag{56}$$

This completes the proof of (48).

Analogously, we can also obtain (46) by using the Fefferman-Stein vector-valued maximal inequality. □

As an application of wavelet characterizations, we now turn to discuss the dual spaces of the inhomogeneous Besov and Triebel-Lizorkin spaces. For convenience, we rewrite (44) as

$$f=\sum_{k\geq 0}\sum_{\alpha\in\mathcal{Z}^k}\left\langle\varphi_\alpha^k,f\right\rangle\varphi_\alpha^k, \tag{57}$$

where $\mathcal{Z}^0=\mathcal{X}^1$, $\varphi_\alpha^0=\phi_\alpha^1$, $\mathcal{Z}^k=\mathcal{Y}^k$, and $\varphi_\alpha^k=\psi_\alpha^k$ for $k\geq 1$. The following concepts of spaces of sequences were first introduced by Frazier and Jawerth [13].

Definition 14. Let $\lambda := \{\lambda_\alpha^k : k \geq 0, \alpha \in \mathcal{Z}^k\}$, and let $|s| < \eta$ and $1 < p, q \leq \infty$. The space $b_p^{s,q}$ of sequences is defined by $b_p^{s,q} := \{\lambda : \|\lambda\|_{b_p^{s,q}} < \infty\}$ with

$$\|\lambda\|_{b_p^{s,q}} := \left\{ \sum_{k\geq 0} \delta^{-ksq} \left[\sum_{\alpha\in\mathcal{Z}^k} \left(\mu\left(Q_\alpha^k\right)^{1/p-1/2} \left|\lambda_\alpha^k\right| \right)^p \right]^{q/p} \right\}^{1/q}, \tag{58}$$

and, for $1 < p < \infty$, the space $f_p^{s,q}$ is defined by $f_p^{s,q} := \{\lambda : \|\lambda\|_{f_p^{s,q}} < \infty\}$ with

$$\|\lambda\|_{f_p^{s,q}} := \left\| \left\{ \sum_{k\geq 0} \delta^{-ksq} \left[\sum_{\alpha\in\mathcal{Z}^k} \left(\mu\left(Q_\alpha^k\right)^{-1/2} \left|\lambda_\alpha^k\right| \chi_{Q_\alpha^k} \right) \right]^q \right\}^{1/q} \right\|_p. \tag{59}$$

Remark 15. It is easy to see that $f \in B_p^{s,q} \Leftrightarrow \{\langle \varphi_\alpha^k, f\rangle\} \in b_p^{s,q}$ and $f \in F_p^{s,q} \Leftrightarrow \{\langle \varphi_\alpha^k, f\rangle\} \in f_p^{s,q}$ due to (57), (45), and (46).

The next result shows that L^2 is dense in $B_p^{s,q}$ as well as $F_p^{s,q}$.

Proposition 16. *Let $0 < \beta, \gamma < \eta$, and $|s| < \eta$. Then L^2 is dense in both $B_p^{s,q}$ for $1 < p, q \leq \infty$ and $F_p^{s,q}$ for $1 < p < \infty$ and $1 < q \leq \infty$.*

Proof. Let $|s| < \eta' < \eta$. Since $G(\eta', \eta') \subset L^2$, to show the conclusion, it suffices to show that $G(\eta', \eta')$ is dense in $B_p^{s,q}$ and $F_p^{s,q}$. Let $f \in B_p^{s,q}$. For any fixed M, write

$$f_M := P_1^2(f) + \sum_{k=1}^{M} Q_k^2(f). \tag{60}$$

By the estimates (26) and (27), we can show that $f_M \in G(\eta', \eta')$ and $\|f_M\|_{B_p^{s,q}} \leq \|f\|_{B_p^{s,q}}$. Further on, we can also show that

$$\lim_{M\to\infty} \left\| \sum_{k>M} Q_k^2(f) \right\|_{B_p^{s,q}} = 0. \tag{61}$$

This is a direct consequence of the wavelet expansion (29) and estimate (26). Indeed,

$$\left|D_l Q_k(f)\right| \leq C\delta^{|k-l|\eta'} M\left(Q_k(f)\right)(x), \tag{62}$$

where D_l is defined as the same as E_k for $l \geq 0$. By wavelet expansion (29), we then have

$$\left\| \sum_{k\geq M} Q_k^2(f) \right\|_{B_p^{s,q}} \leq C \left\{ \sum_{k>M} \left(\delta^{-ks} \|Q_k(f)\|_p \right)^q \right\}^{1/q}. \tag{63}$$

This implies that $f_M \to f$ in $B_p^{s,q}$ as $M \to \infty$, and hence, L^2 is dense in $B_p^{s,q}$. By analogous arguments, we can also show that L^2 is dense in $F_p^{s,q}$. □

We are ready to give the dualities of the Besov and Triebel-Lizorkin spaces.

Theorem 17. *Let $1 < p, q < \infty$ and $|s| < \eta$, and let $1/p + 1/p' = 1/q + 1/q' = 1$. Then $(B_p^{s,q})^* = B_{p'}^{-s,q'}$. More precisely, given $g \in B_{p'}^{-s,q'}$, then $\pounds_g(f) = \langle f, g\rangle$ defines a linear functional on $B_p^{s,q} \cap G(\eta', \eta')$ with $|s| < \eta' < \eta$ such that*

$$\left|\pounds_g(f)\right| \leq C\|f\|_{B_p^{s,q}} \|g\|_{B_{p'}^{-s,q'}}, \tag{64}$$

and this linear functional can be extended to $B_p^{s,q}$ with norm at most $C\|g\|_{B_{p'}^{-s,q'}}$.

Conversely, if $\pounds$ is a linear functional on $B_p^{s,q}$, then there exists a unique $g \in B_{p'}^{-s,q'}$ such that

$$\pounds_g(f) = \langle f, g\rangle \tag{65}$$

defines a linear functional on $B_p^{s,q} \cap G(\eta', \eta')$ with $|s| < \eta' < \eta$, and $\pounds$ is the extension of $\pounds_g$ with $\|g\|_{B_{p'}^{-s,q'}} \leq C\|\pounds\|$.

Remark 18. Theorem 17 also holds with $B_p^{s,q}$ and $B_{p'}^{-s,q'}$ replaced by $F_p^{s,q}$ and $F_{p'}^{-s,q'}$, respectively.

Proof of Theorem 17.

(i) Suppose that $g \in B_{p'}^{-s,q'}$ and $f \in B_p^{s,q} \cap G(\eta', \eta')$. By Remark 15, we observe that

$$g = \sum_{k\geq 0} \sum_{\alpha\in\mathcal{Z}^k} \left\langle \varphi_\alpha^k, g\right\rangle \varphi_\alpha^k \tag{66}$$

is an element of $B_{p'}^{-s,q'}$, and hence, $\{\langle \varphi_\alpha^k, g\rangle\} \in b_{p'}^{-s,q'}$. Also,

$$f = \sum_{k\geq 0} \sum_{\alpha\in\mathcal{Z}^k} \varphi_\alpha^k \left\langle \varphi_\alpha^k, f\right\rangle \tag{67}$$

is an element of $B_p^{s,q} \cap G(\eta', \eta')$, and hence, $\{\langle \varphi_\alpha^k, f\rangle\} \in b_p^{s,q}$. Write now

$$\pounds_g(f) := \sum_{k\geq 0} \sum_{\alpha\in\mathcal{Z}^k} \left\langle \varphi_\alpha^k, g\right\rangle \left\langle \varphi_\alpha^k, f\right\rangle. \tag{68}$$

By the Hölder inequality and Minkowski inequality, we have

$$\left|\pounds_g(f)\right| \leq \left\|\left\langle \varphi_\alpha^k, g\right\rangle\right\|_{b_{p'}^{-s,q'}} \left\|\left\langle \varphi_\alpha^k, f\right\rangle\right\|_{b_p^{s,q}} \leq \|g\|_{B_{p'}^{-s,q'}} \|f\|_{B_p^{s,q}}, \tag{69}$$

and hence, $\pounds_g$ is a linear functional defined on $B_p^{s,q} \cap G(\eta', \eta')$. Clearly, $\pounds_g$ has bounded extension to $B_p^{s,q}$ due to the density of $B_p^{s,q} \cap G(\eta', \eta')$ in $B_p^{s,q}$. This proves that $B_{p'}^{-s,q'} \subset (B_p^{s,q})^*$.

(ii) Suppose that $\pounds \in (B_p^{s,q})^*$. We need to prove that there exists a $g \in B_{p'}^{-s,q'}$ such that

$$\pounds_g(f) = \langle f, g\rangle, \quad f \in B_p^{s,q} \cap G(\eta', \eta'). \tag{70}$$

Note that

$$|\pounds(f)| \le \|\pounds\| \|f\|_{B_p^{s,q}} \le \|\pounds\| \left\|\left\langle \varphi_\alpha^k, f\right\rangle\right\|_{b_p^{s,q}}, \tag{71}$$

and define a linear functional $\widetilde{\pounds}$ on $S = \{\langle \varphi_\alpha^k, f\rangle : f \in B_p^{s,q}\}$ by

$$\widetilde{\pounds}\left(\left\{\left\langle \varphi_\alpha^k, f\right\rangle\right\}\right) = \pounds(f). \tag{72}$$

From (71) it follows that

$$\left|\widetilde{\pounds}\left(\left\{\left\langle \varphi_\alpha^k, f\right\rangle\right\}\right)\right| = |\pounds(f)| \le \|\pounds\| \left\|\left\langle \varphi_\alpha^k, f\right\rangle\right\|_{b_p^{s,q}}. \tag{73}$$

By the Hahn-Banach theorem, $\widetilde{\pounds}$ can be extended to $b_p^{s,q}$ as a continuous linear functional. In addition, by arguments analogous to that in ([13, Remark 5.11]), we can show that $(b_p^{s,q})^* = b_{p'}^{-s,q'}$, and hence, there exists a unique sequence $\{g_k\} \in b_{p'}^{-s,q'}$ such that $\|\pounds\|_{b_{p'}^{-s,q'}} \le C\|\pounds\|$ and

$$\widetilde{\pounds}(a_k) = \sum_{k\ge 0} a_k g_k, \quad \{a_k\} \in b_p^{s,q}. \tag{74}$$

Thus, for each $f \in B_p^{s,q} \cap G(\eta', \eta')$, we have

$$\pounds(f) = \widetilde{\pounds}\left(\left\{\left\langle \varphi_\alpha^k, f\right\rangle\right\}\right) = \sum_{k\ge 0} \left\langle \varphi_\alpha^k, f\right\rangle g_k = \left\langle \sum_{k\ge 0} \varphi_\alpha^k g_k, f\right\rangle. \tag{75}$$

Let $g := \sum_{k\ge 0} \varphi_\alpha^k g_k$. By the orthogonality of φ_α^k, we have

$$\begin{aligned}
\|g\|_{b_{p'}^{-s,q'}} &= \left\{\sum_{k\ge 0} \delta^{-ksq'} \times \left[\sum_\alpha \left(\mu\left(Q_\alpha^k\right)^{1/p-1/2} \times \left|\left\langle \varphi_\alpha^k, \sum_{l\ge 0} g_l \varphi_\alpha^l\right\rangle\right|\right)^{p'}\right]^{q'/p'}\right\}^{1/q'} \\
&\le C\left\{\sum_{k\ge 0} \delta^{-ksq'} \times \left[\sum_\alpha \left(\mu\left(Q_\alpha^k\right)^{1/p-1/2} \times |g_k|\right)^{p'}\right]^{q'/p'}\right\}^{1/q'} \le C\|\pounds\|,
\end{aligned} \tag{76}$$

and $|\pounds(f)| = |\pounds_g(f)| = |\langle g, f\rangle| \le \|\pounds\|\|f\|_{B_p^{s,q}}$. Note that $B_p^{s,q} \cap G(\eta', \eta')$ is dense in $B_p^{s,q}$. The linear functional $\pounds$ can be extended to $B_p^{s,q}$. Thus, $\pounds_g(f) = \langle g, f\rangle$ for all $f \in B_p^{s,q}$. □

The next result gives elementary embedding properties of Besov and Triebel-Lizorkin spaces which can be proved by an approach analogous to that of Proposition 2 in [14, Section 2.3.2] and we omit the details here.

Proposition 19. (i) *Let $1 < q_0 \le q_1 < \infty, 1 < p < \infty$, and $|s| < \eta$. Then*

$$B_p^{s,q_0} \subset B_p^{s,q_1}, \quad F_p^{s,q_0} \subset F_p^{s,q_1}. \tag{77}$$

(ii) *Let $1 < q_0 \le q_1 < \infty$ and $|s+\theta| < \eta$. Then*

$$B_p^{s+\theta,q_0} \subset B_p^{s,q_1}, \quad F_p^{s+\theta,q_0} \subset F_p^{s,q_1}. \tag{78}$$

(iii) *Let $1 < p < \infty$, $1 < q \le \infty$, and $|s| < \eta$. Then*

$$B_p^{s,\min(p,q)} \subset F_p^{s,q} \subset B_p^{s,\max(p,q)}. \tag{79}$$

4. Homogeneous Besov and Triebel-Lizorkin Spaces

In this section, we develop the theory of the homogeneous Besov and Triebel-Lizorkin spaces and discuss relationships between homogeneous and inhomogeneous versions.

In order to introduce the homogeneous Besov and Triebel-Lizorkin spaces, we recall again the spaces of test functions and distributions on spaces of homogeneous type given in [11]. Let $G(x_0, r, \beta, \gamma)$ be given in Definition 2. By $G_0(\beta, \gamma)$ we denote the collection of all test functions in $G(\beta, \gamma)$ by $\int_X f(x)d\mu(x) = 0$. For $0 < \beta, \gamma < \eta$, let $G^0(\beta, \gamma)$ be the completion of the space $G_0(\eta, \eta)$ in the norm of $G(\beta, \gamma)$. For $f \in G^0(\beta, \gamma)$, we define $\|f\|_{G^0(\beta,\gamma)} = \|f\|_{G(\beta,\gamma)}$.

The distribution space $(G^0(\beta, \gamma))'$ is defined to be the set of all linear functionals $\pounds$ from $G^0(\beta, \gamma)$ to $\mathbb{C}$ with the property that there exists $C > 0$ such that, for all $f \in G^0(\beta, \gamma)$,

$$|\pounds(f)| \le C\|f\|_{G^0(\beta,\gamma)}. \tag{80}$$

The homogeneous Besov and Triebel-Lizorkin spaces are defined as follows.

Definition 20. Let $|s| < \eta$ and $0 < \beta, \gamma < \eta$. For $1 < p, q < \infty$, the homogeneous Besov space $\dot{B}_p^{s,q}(X)$ is defined by

$$\dot{B}_p^{s,q}(X) := \left\{f \in \left(G^0(\beta, \gamma)\right)' : \|f\|_{\dot{B}_p^{s,q}} < \infty\right\}, \tag{81}$$

with

$$\|f\|_{\dot{B}_p^{s,q}} := \left\{\sum_{k\in\mathbb{Z}} \left(\delta^{-ks}\|Q_k(f)\|_p\right)^q\right\}^{1/q}, \tag{82}$$

and for $1 < p < \infty, 1 < q \le \infty$, the homogeneous Triebel-Lizorkin space $\dot{F}_p^{s,q}(X)$ is defined by

$$\dot{F}_p^{s,q}(X) = \left\{f \in \left(G^0(\beta, \gamma)\right)' : \|f\|_{\dot{F}_p^{s,q}} < \infty\right\}, \tag{83}$$

with

$$\|f\|_{\dot{F}_p^{s,q}} = \left\| \left\{ \sum_{k\in\mathbb{Z}} \left(\delta^{-ks} |Q_k(f)|\right)^q \right\}^{1/q} \right\|_p. \tag{84}$$

The authors in [11] showed that the wavelet expansion (28) holds in $G^0(\beta,\gamma)$ as well as $(G^0(\beta,\gamma))'$ with $0<\beta,\gamma<\theta$. The next result shows that Definition 20 is independent of choice of Q_k for $k\in\mathbb{Z}$.

Proposition 21. *Let $f \in (G^0(\beta,\gamma))'$ and $|s|<\eta$. For $1<p,q\le\infty$, then*

$$\left\{ \sum_{k\in\mathbb{Z}} \left(\delta^{-ks} \|Q_k(f)\|_p\right)^q \right\}^{1/q} \sim \left\{ \sum_{l\in\mathbb{Z}} \left(\delta^{-ls} \|D_l(f)\|_p\right)^q \right\}^{1/q}, \tag{85}$$

and for $1<p<\infty$ and $1<q\le\infty$, then

$$\left\| \left\{ \sum_{k\in\mathbb{Z}} \left(\delta^{-ks} |Q_k(f)|\right)^q \right\}^{1/q} \right\|_p \sim \left\| \left\{ \sum_{l\in\mathbb{Z}} \left(\delta^{-ls} |D_l(f)|\right)^q \right\}^{1/q} \right\|_p. \tag{86}$$

We next give some relationships between homogeneous and inhomogeneous spaces.

Proposition 22. *Let $0<\beta,\gamma<\eta$, and $0<s<\eta$. For $1<p,q\le\infty$, then*

$$B_p^{s,q} = \dot{B}_p^{s,q} \cap L^p, \tag{87}$$

and for $1<p<\infty$, $1<q\le\infty$, then

$$F_p^{s,q} = \dot{F}_p^{s,q} \cap L^p. \tag{88}$$

We only give the proof of Proposition 22. Others can be proved by arguments analogous to that of inhomogeneous versions.

Proof. We first show that $B_p^{s,q} \subset \dot{B}_p^{s,q} \cap L^p$. Let $f \in B_p^{s,q}$. By the wavelet expansion (29) and the Minkowski inequality, we have $f \in L^p$ and

$$\begin{aligned} \|f\|_{L^p} &\le \|P_1(f)\|_p + \sum_{k=1}^{\infty} \|Q_k(f)\|_p \\ &\le \|P_1(f)\|_p + C\left\{ \sum_{k=1}^{\infty} \left(\delta^{-ks} \|Q_k(f)\|_p\right)^q \right\}^{1/q} \\ &\le C\|f\|_{B_p^{s,q}}. \end{aligned} \tag{89}$$

Applying the wavelet expansion (29) and the size condition of Q_kP_1(26), we also have $f \in \dot{B}_p^{s,q}$ and

$$\begin{aligned} \|f\|_{\dot{B}_p^{s,q}} &\le \left\{ \sum_{k<1} \left(\delta^{-ks} \|Q_k(f)\|_p\right)^q \right\}^{1/q} \\ &\quad + \left\{ \sum_{k\ge1} \left(\delta^{-ks} \|Q_k(f)\|_p\right)^q \right\}^{1/q} \\ &\le C\|f\|_p + \|f\|_{B_p^{s,q}}. \end{aligned} \tag{90}$$

Thus, $f \in \dot{B}_p^{s,q} \cap L^p$ and $\|f\|_{L^p} + \|f\|_{\dot{B}_p^{s,q}} \lesssim \|f\|_{B_p^{s,q}}$.

Conversely, we show that $\dot{B}_p^{s,q} \cap L^p \subset B_p^{s,q}$. Indeed, let $f \in \dot{B}_p^{s,q} \cap L^p$. By the size condition of P_1, we have

$$|P_1(f)(x)| \le CM(f)(x), \tag{91}$$

and hence, $\|P_1(f)\|_p \le C\|f\|_p$. This implies that $\|f\|_{B_p^{s,q}} \le C(\|f\|_p + \|f\|_{\dot{B}_p^{s,q}})$.

By analogous arguments, we can also obtain the second desired equality for the inhomogeneous Triebel-Lizorkin spaces. □

We now introduce the homogeneous Besov and Triebel-Lizorkin spaces by wavelet coefficients.

Definition 23. Let $1<p,q\le\infty$, and $|s|<\eta$. The homogeneous Besov space $\dot{B}_{p,(2)}^{s,q}$ is defined by

$$\dot{B}_{p,(2)}^{s,q}(X) := \left\{ f \in \left(G^0(\beta,\gamma)\right)' : \|f\|_{\dot{B}_p^{s,q}} < \infty \right\}, \tag{92}$$

where

$$\|f\|_{\dot{B}_{p,(2)}^{s,q}} := \left\{ \sum_{k\in\mathbb{Z}} \delta^{-ksq} \left[\sum_{\alpha\in\mathscr{Y}^k} \left(\mu\left(Q_\alpha^k\right)^{1/p-1/2} \left|\psi_\alpha^k, f\right|\right)^p \right]^{q/p} \right\}^{1/q}. \tag{93}$$

For $1<p<\infty$, the homogeneous Triebel-Lizorkin space $\dot{F}_{p,(2)}^{s,q}$ is defined by

$$\dot{F}_{p,(2)}^{s,q}(X) = \left\{ f \in \left(G^0(\beta,\gamma)\right)' : \|f\|_{\dot{F}_p^{s,q}} < \infty \right\}, \tag{94}$$

where

$$\|f\|_{\dot{F}_{p,(2)}^{s,q}} := \left\| \left\{ \sum_{k\in\mathbb{Z}} \delta^{-ksq} \sum_{\alpha\in\mathscr{Y}^k} \left(\mu\left(Q_\alpha^k\right)^{-1/2} \left|\left\langle \psi_\alpha^k, f\right\rangle\right| \chi_{Q_\alpha^k}\right)^q \right\}^{1/q} \right\|_{L^p}. \tag{95}$$

We are ready to give the wavelet characterizations of the homogeneous Besov and Triebel-Lizorkin spaces.

Theorem 24. *Let $0 < s < \eta$ and $1 < q \le \infty$. Then*

$$\begin{aligned} \|f\|_{\dot{B}_p^{s,q}} &\sim \|f\|_{\dot{B}_{p,(2)}^{s,q}}, \quad 1 < p \le \infty, \\ \|f\|_{\dot{F}_p^{s,q}} &\sim \|f\|_{\dot{F}_{p,(2)}^{s,q}}, \quad 1 < p < \infty. \end{aligned} \tag{96}$$

Now we recall the homogenous spaces of sequences $\dot{b}_{p,}^{s,q}$ and $\dot{f}_p^{s,q}$; refer to [13] for details. Let $\lambda := \{\lambda_\alpha^k : k \in \mathbb{Z}, \alpha \in \mathscr{Y}^k\}$. For $|s| < \eta$ and $1 < p, q \le \infty$, the space of sequence $\dot{b}_p^{s,q}$ is defined by $\dot{b}_p^{s,q} := \{\lambda : \|\lambda\|_{\dot{b}_p^{s,q}} < \infty\}$ with

$$\|\lambda\|_{\dot{b}_p^{s,q}} := \left\{ \sum_{k\in\mathbb{Z}} \delta^{-ksq} \left[\sum_{\alpha\in\mathscr{Y}^k} \left(\mu\left(Q_\alpha^k\right)^{1/p-1/2} \left|\lambda_\alpha^k\right| \right)^p \right]^{q/p} \right\}^{1/q}, \tag{97}$$

and, for $1 < p < \infty$, the space $\dot{f}_p^{s,q}$ is defined by $\dot{f}_p^{s,q} := \{\lambda : \|\lambda\|_{\dot{f}_p^{s,q}} < \infty\}$ with

$$\|\lambda\|_{\dot{f}_p^{s,q}} := \left\| \left\{ \sum_{k\in\mathbb{Z}} \delta^{-ksq} \left[\sum_{\alpha\in\mathscr{Y}^k} \left(\mu\left(Q_\alpha^k\right)^{-1/2} \left|\lambda_\alpha^k\right| \chi_{Q_\alpha^k} \right) \right]^q \right\}^{1/q} \right\|_p. \tag{98}$$

Remark 25. Suppose that φ_α^k be the same as in Theorem 1 for $k \in \mathbb{Z}$, using Theorem 24 and [11, Corollary 3.5] then, we have $f \in \dot{B}_p^{s,q} \Leftrightarrow \{\langle \varphi_\alpha^k, f\rangle\} \in \dot{b}_p^{s,q}$ and $f \in \dot{F}_p^{s,q} \Leftrightarrow \{\langle \varphi_\alpha^k, f\rangle\} \in \dot{f}_p^{s,q}$.

Proposition 26. *Let $0 < s < \eta' < \eta$. Then $G(\eta', \eta')$ is dense in both $\dot{B}_p^{s,q}$ for $1 < p$ and $q \le \infty$ and $\dot{F}_p^{s,q}$ for $1 < p < \infty$ and $1 < q \le \infty$.*

Remark 27. As the proof of Proposition 16, we can also obtain that (28) converges in both the norm of $\dot{B}_p^{s,q}$ for $1 < p, q \le \infty$ and the norm of $\dot{F}_p^{s,q}$ for $1 < p < \infty$, $1 < q \le \infty$.

The homogeneous version of Theorem 17 is given as follows.

Theorem 28. *Let $1 < p, q < \infty$, and $|s| < \eta$, and let $1/p + 1/p' = 1/q + 1/q' = 1$. Then $(\dot{B}_p^{s,q})^* = \dot{B}_{p'}^{-s,q'}$. More precisely, given $g \in \dot{B}_{p'}^{-s,q'}$, then $\pounds_g(f) = \langle f, g\rangle$ defines a linear functional on $\dot{B}_p^{s,q} \cap G^0(\eta', \eta')$ with $|s| < \eta' < \eta$ such that*

$$\left|\pounds_g(f)\right| \le C\|f\|_{\dot{B}_p^{s,q}} \|g\|_{\dot{B}_{p'}^{-s,q'}}, \tag{99}$$

and this linear functional can be extended to $\dot{B}_p^{s,q}$ with norm at most $C\|g\|_{\dot{B}_{p'}^{-s,q'}}$.

Conversely, if $\pounds$ is a linear functional on $\dot{B}_p^{s,q}$, then there exists a unique $g \in \dot{B}_{p'}^{-s,q'}$ such that

$$\pounds_g(f) = \langle f, g\rangle \tag{100}$$

defines a linear functional on $\dot{B}_p^{s,q} \cap G^0(\eta', \eta')$ with $|s| < \eta' < \eta$, and $\pounds$ is the extension of $\pounds_g$ with $\|g\|_{\dot{B}_{p'}^{-s,q'}} \le C\|\pounds\|$.

Remark 29. Theorem 28 also holds with $\dot{B}_p^{s,q}$ and $\dot{B}_{p'}^{-s,q'}$ replaced by $\dot{F}_p^{s,q}$ and $\dot{F}_{p'}^{-s,q'}$, respectively.

Finally, we give the following embedding properties in homogeneous Besov and Triebel-Lizorkin spaces to end this section.

Proposition 30. (i) *Let $1 < q_0 \le q_1 \le \infty$, $1 < p < \infty$, and $|s| < \eta$. Then*

$$\dot{B}_p^{s,q_0} \subset \dot{B}_p^{s,q_1}, \qquad \dot{F}_p^{s,q_0} \subset \dot{F}_p^{s,q_1}. \tag{101}$$

(ii) *Let $1 < p < \infty$, $1 < q \le \infty$ and $|s| < \eta$. Then*

$$\dot{B}_p^{s,\min(p,q)} \subset \dot{F}_p^{s,q} \subset \dot{B}_p^{s,\max(p,q)}. \tag{102}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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