Research Article

A Note on Strongly Starlike Mappings in Several Complex Variables

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Let $\mathbb{C}^n$ denote the space of $n$ complex variables $z = (z_1, \ldots, z_n)$ with the Euclidean inner product $\langle z, w \rangle = \sum_{j=1}^n z_j \overline{w}_j$ and the norm $\| z \| = \langle z, z \rangle^{1/2}$. The open unit ball $\{z \in \mathbb{C}^n : \| z \| < 1 \}$ is denoted by $\mathbb{B}^n$. In the case of one complex variable, $\mathbb{B}^1$ is denoted by $U$.

If $\Omega$ is a domain in $\mathbb{C}^n$, let $H(\Omega)$ be the set of holomorphic mappings from $\Omega$ to $\mathbb{C}^n$. If $\Omega$ is a domain in $\mathbb{C}^n$ which contains the origin and $f \in H(\Omega)$, we say that $f$ is normalized if $f(0) = 0$ and $Df(0) = I_n$, where $I_n$ is the identity matrix.

A normalized mapping $f \in H(\mathbb{B}^n)$ is said to be starlike if $f$ is biholomorphic on $\mathbb{B}^n$ and $tf(\mathbb{B}^n) \subset f(\mathbb{B}^n)$ for $t \in [0, 1]$, where the last condition says that the image $f(\mathbb{B}^n)$ is a starlike domain with respect to the origin. For a normalized locally biholomorphic mapping $f$ on $\mathbb{B}^n$, $f$ is starlike if and only if

$$
\Re \left( \langle Df(z) \rangle^{-1} f(z), z \right) > 0, \quad z \in \mathbb{B}^n \setminus \{0\} \quad (1)
$$

(see [1–4] and the references therein, cf. [5]).

Let $\alpha \in (0, 1]$. A function $f \in H(U)$, normalized by $f(0) = 0$ and $f'(0) = 1$, is said to be strongly starlike of order $\alpha$ if

$$
|\arg \frac{zf'(z)}{f(z)}| < \alpha \frac{\pi}{2}, \quad z \in U. \quad (2)
$$

If $f$ is strongly starlike of order $\alpha$, then $f$ is also starlike and thus univalent on $U$. Stankiewicz [6] proved that if $\alpha \in (0, 1)$, then a domain $\Omega \not\subset \mathbb{C}$ which contains the origin is $\alpha$-accessible if and only if $\Omega = f(U)$, where $U$ is the unit disc in $\mathbb{C}$ and $f$ is a strongly starlike function of order $1 - \alpha$ on $U$. For strongly starlike functions on $U$, see also Brannan and Kirwan [7], Ma and Minda [8], and Sugawa [9].

Kohr and Liczberski [10] introduced the following definition of strongly starlike mappings of order $\alpha$ on $\mathbb{B}^n$.

Definition 1. Let $0 < \alpha \leq 1$. A normalized locally biholomorphic mapping $f \in H(\mathbb{B}^n)$ is said to be strongly starlike of order $\alpha$ if

$$
|\arg \langle [Df(z)]^{-1} f(z), z \rangle| < \alpha \frac{\pi}{2}, \quad z \in \mathbb{B}^n \setminus \{0\}. \quad (3)
$$

Obviously, if $f$ is strongly starlike of order $\alpha$, then $f$ is also starlike, and if $\alpha = 1$ in (3), one obtains the usual notion of starlikeness on the unit ball $\mathbb{B}^n$.

Using this definition, Hamada and Honda [11], Hamada and Kohr [12], Liczberski [13], and Liu and Li [14] obtained...
various results for strongly starlike mappings of order \( \alpha \) in several complex variables.

Recently, Liczborski and Starkov [15] gave another definition of strongly starlike mappings of order \( \alpha \) on the Euclidean unit ball \( B^n \) in \( \mathbb{C}^n \), where \( \alpha \in (0, 1) \), and proved that a normalized biholomorphic mapping \( f \) on \( B^n \) is strongly starlike of order \( 1 - \alpha \) if and only if \( f \) is an \( \alpha \)-accessible domain in \( \mathbb{C}^n \) for \( \alpha \in (0, 1) \). Their definition is as follows.

**Definition 2.** Let \( 0 < \alpha \leq 1 \). A normalized locally biholomorphic mapping \( f \in H(B^n) \) is said to be strongly starlike of order \( \alpha \) (in the sense of Liczborski and Starkov) if

\[
\Re \left( \langle DF(z)\rangle^{-1} f(z), z \right) \geq \left\| \langle DF(z)\rangle^{-1} z \right\| \| f(z) \| \sin \left(1 - \alpha \frac{\pi}{2}\right),
\]

where \( \langle DF(z)\rangle^{-1} z \neq 0 \). In this case \( n = 1 \), it is obvious that both notions of strong starlikeness of order \( \alpha \) are equivalent. Thus, the following natural question arises in dimension \( n \geq 2 \).

**Question 1.** Let \( \alpha \in (0, 1) \). Is there any relation between the above two definitions of strong starlikeness of order \( \alpha \)?

Let \( f \) be a normalized biholomorphic mapping on the Euclidean unit ball \( B^n \) in \( \mathbb{C}^n \) and let \( \alpha \in (0, 1) \). In this paper, we will show that if \( f \) is strongly starlike of order \( \alpha \) in the sense of Definition 2, then it is also strongly starlike of order \( \alpha \) in the sense of Definition 1. As a corollary, the results obtained in [11–14] for strongly starlike mappings of order \( \alpha \) in the sense of Definition 1 also hold for strongly starlike mappings of order \( \alpha \) in the sense of Definition 2. We also give an example which shows that the converse of the above result does not hold in dimension \( n \geq 2 \).

### 2. Main Results

Let \( \mathcal{L}(a, b) \) denote the angle between \( a, b \in \mathbb{C}^n \) regarding \( a, b \) as real vectors in \( \mathbb{R}^{2n} \).

**Lemma 3.** Let \( a, b \in \mathbb{C}^n \setminus \{0\} \) be such that \( \mathcal{L}(a, b) \neq 0 \). If \( |\mathcal{L}(a, b)| \leq \pi \) and \( \mathcal{L}(a, b) < \pi/2 \), then

\[
|\mathcal{L}(a, b)| \leq \mathcal{L}(a, b).
\]

**Proof.** Let \( \theta = \mathcal{L}(a, b) \), \( \varphi = \mathcal{L}(a, b) \). Then we have \( \langle a, b \rangle = re^{i\theta} \) for some \( r \geq 0 \) and

\[
\Re \left( \langle a, b \rangle \right) = \|a\| \|b\| \cos \varphi = r \cos \theta.
\]

Since \( \cos \varphi > 0 \) and \( r = |\langle a, b \rangle| \leq \|a\| \|b\| \), we have

\[
\cos \varphi \leq \cos \theta.
\]

Therefore, we have \( |\theta| \leq \varphi \), as desired.

**Theorem 4.** Let \( f \) be a normalized biholomorphic mapping on the Euclidean unit ball \( B^n \) in \( \mathbb{C}^n \) and let \( \alpha \in (0, 1) \). If \( f \) is strongly starlike of order \( \alpha \) in the sense of Definition 2, then it is also strongly starlike of order \( \alpha \) in the sense of Definition 1.

**Proof.** Assume that \( f \) is strongly starlike of order \( \alpha \) in the sense of Definition 2. Then by (4), we have

\[
\Re \left( \langle DF(z)\rangle^{-1} f(z), z \right) \leq \alpha \frac{\pi}{2}, \quad z \in B^n \setminus \{0\}.
\]

Using Lemma 3, we have

\[
|\arg \left( \langle DF(z)\rangle^{-1} f(z), z \right)| = |\arg \left( f(z), \langle DF(z)\rangle^{-1} z \right)|
\]

\[
\leq \mathcal{L}(\langle DF(z)\rangle^{-1} z, f(z))
\]

\[
\leq \alpha \frac{\pi}{2}, \quad z \in B^n \setminus \{0\}.
\]

For fixed \( z \in B^n \setminus \{0\} \), let \( w = z/\|z\| \) and

\[
p(\zeta) = \begin{cases} 
\frac{1}{\zeta} \Re \left( \langle DF(\zeta w)\rangle^{-1} f(\zeta w), w \right), & \text{for } \zeta \in U \setminus \{0\}, \\
1, & \text{for } \zeta = 0.
\end{cases}
\]

Then \( p \) is a holomorphic function on \( U \) with \( |\arg p(\zeta)| \leq \pi \alpha/2 \) for \( \zeta \in U \). Since \( \arg p \) is a harmonic function on \( U \) and \( \arg p(0) = 0 \), by applying the maximum and minimum principles for harmonic functions, we obtain \( |\arg p(\zeta)| \leq \pi \alpha/2 \) for \( \zeta \in U \). Thus, we have

\[
|\arg \left( \langle DF(z)\rangle^{-1} f(z), z \right)| < \alpha \frac{\pi}{2}, \quad z \in B^n \setminus \{0\}.
\]

Hence \( f \) is strongly starlike of order \( \alpha \) in the sense of Definition 1, as desired.

The following example shows that the converse of the above theorem does not hold in dimension \( n \geq 2 \).

**Example 5.** For \( \alpha \in (0, 1) \), let

\[
f(z) = f_\alpha(z) = (z_1 + b z_2^2, z_2), \quad z = (z_1, z_2) \in B^2,
\]

where

\[
b = \frac{3\sqrt{3}}{2} \sin \left(\alpha \frac{\pi}{2}\right).
\]

Then

\[
DF(z) = \begin{bmatrix} 1 & 2bz_2 \\ 0 & 1 \end{bmatrix}, \quad [DF(z)]^{-1} = \begin{bmatrix} 1 & -2bz_2 \\ 0 & 1 \end{bmatrix}.
\]

Therefore,

\[
\langle [DF(z)]^{-1} f(z), z \rangle = (z_1 + b z_2^2 - 2bz_2^2) \overline{z_1}
\]

\[
+ |z_2|^2 = |z_1|^2 + |z_2|^2 - b \overline{z_1} z_2^2.
\]
Since \(|z_1z_2| \leq \frac{2}{3 \sqrt{3}}\), for \(z \in \partial \mathbb{B}^2\), we obtain that 
\(|b_{z_1}z_2| \leq \sin(\pi/3)|z|^2\) for \(z \in \mathbb{B}^2\). This implies that 
\(\langle [Df(z)]^{-1} f(z), z \rangle\) lies in the disc of center \(|z|^2\) and radius 
\(\sin(\pi/3)|z|^2\) for each \(z \in \mathbb{B}^2 \setminus \{0\}\) and thus 
\[
\arg\left(\langle [Df(z)]^{-1} f(z), z \rangle\right) < \frac{\alpha \pi}{2}, \quad z \in \mathbb{B}^2 \setminus \{0\}.
\]
Therefore, \(f = f_\alpha\) is strongly starlike of order \(\alpha\) in the sense of Definition 1.

On the other hand,
\[
\langle [Df(z)]^{-1} f(z), z \rangle = 1 - m,
\]
for \(m \in [1/3, 1]\). Then, we obtain
\[
\|([Df(z)]^{-1})^* z^0\|^2 \|f(z^0)\|^2 \sin^2 \left(1 - \left(1 - \frac{\alpha \pi}{2}\right)\right)
= (1 - m) \left\{\frac{1}{3} \left(1 + 3m^2 + \frac{2}{3}\right) \left(1 + 3m^2 + \frac{4}{3}\right) \sin \left(1 - \left(1 - \frac{\alpha \pi}{2}\right)\right)\right\}
\times (1 + m) - (1 - m).
\]
Since
\[
\left[\frac{1}{3} \left(1 + 3m^2 + \frac{2}{3}\right) \left(1 + 3m^2 + \frac{4}{3}\right) \sin \left(1 - \left(1 - \frac{\alpha \pi}{2}\right)\right)\right]
\]
is increasing on \([1/3, 1]\) and positive for \(m = 1/3\), we have
\[
\Re \langle [Df(z^0)]^{-1} f(z^0), z^0 \rangle < \left\langle ([Df(z^0)]^{-1})^* z^0\right\rangle \times \|f(z^0)\| \sin \left(1 - \left(1 - \frac{\alpha \pi}{2}\right)\right)
\]
for \(m \in [1/3, 1]\).

On the other hand, for \(z^0 = (i/\sqrt{3}, \sqrt{3}/\sqrt{3})\), we have
\[
\langle [Df(z^0)]^{-1} f(z^0), z^0 \rangle = 1 + mi,
\]
\[
\|([Df(z^0)]^{-1})^* z^0\|^2 = \frac{1}{3} + \frac{2}{3} \left[1 - 3m^2\right] = 6m^2 + 1,
\]
\[
\|f(z^0)\|^2 = \frac{1}{3} i + 3m^2 + \frac{2}{3} = 3m^2 + 1.
\]
Then, we obtain
\[
\left\langle ([Df(z^0)]^{-1})^* z^0\right\rangle \|f(z^0)\|^2 \sin^2 \left(1 - \left(1 - \frac{\alpha \pi}{2}\right)\right)
- \Re \left\langle [Df(z^0)]^{-1} f(z^0), z^0 \right\rangle^2
= (6m^2 + 1) \left(3m^2 + 1\right) (1 - m^2) - 1
= m^2 (1 - 18m^2 + 9m^4 + 8) .
\]
Since \(-18m^4 + 9m^4 + 8\) is positive for \(m \in [0, 1/3]\), we have
\[
\Re \langle [Df(z^0)]^{-1} f(z^0), z^0 \rangle < \left\langle ([Df(z^0)]^{-1})^* z^0\right\rangle \times \|f(z^0)\| \sin \left(1 - \left(1 - \frac{\alpha \pi}{2}\right)\right)
\]
for \(m \in (0, 1/3]\).

Thus, \(f = f_\alpha\) is not strongly starlike of order \(\alpha\) in the sense of Definition 2 for \(\alpha \in (0, 1)\).

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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**References**


