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Research Article

The Existence and Uniqueness of Coupled Best Proximity Point for Proximally Coupled Contraction in a Complete Ordered Metric Space

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We prove the existence and uniqueness of coupled best proximity point for mappings satisfying the proximally coupled contraction in a complete ordered metric space. Further, our result provides an extension of a result due to Bhaskar and Lakshmikantham.

1. Introduction

One of the most useful tools in the study of nonlinear functional equation is to describe many problems in physics, chemistry, and engineering. It can be formulated in terms of finding the fixed points of a nonlinear self-mapping. Fixed point theory investigates the techniques for determining a solution of the pattern $Tx = x$, where T is a self-mapping defined on a subset A of a metric space X. Noteworthy, a fixed point x of T on can be written by $d(x, Tx) = 0$.

A well-known principle that guarantees a unique fixed point solution is the Banach contraction principle [1] which states on a complete metric space X for a contraction self-mapping (i.e., $d(Tx, Ty) \leq \alpha d(x, y)$ for all $x, y \in X$, where α is a nonnegative number such that $\alpha < 1$). Over the years, this principle has been generalized in many ways; see also [2–6]. Recently, an interesting way is to study the extension of Banach contraction principle to the case of non-self-mappings. Certainly, a contraction non-self-mapping $T : A \to B$ does not necessarily have a fixed point, where A and B are nonempty subsets of a metric space X.

Ultimately, one proceeds to find an approximate solution $x \in A$ which is closest to Tx in the sense that distance $d(x, Tx)$ is minimum which implies that x and Tx are in close proximity to each other. Indeed, the best approximation theorems and the best proximity point theorem investigate the existence of an approximate solution to fixed point problems for the non-self- mappings. In 1969, Fan [7] guarantees at least one solution to the minimization problem $\min_{x\in A}\|x - Tx\|$, where A is a nonempty compact convex subset of a normed linear space X and $T : A \to X$ is a continuous function. Such an element $x \in A$ satisfying the above condition is called a best approximant of T in A. Note that if $x \in A$ is a best approximant, then $\|x - Tx\|$ need not be the optimum. As a matter of fact, the best proximity point theorems have been explored to find sufficient conditions for the existence of an element x such that the error $d(x, Tx)$ is minimum.

To have a concrete lower bound, let us consider two nonempty subsets A, B of a metric space X and a mapping $T : A \to B$. The natural question is whether one can find an element $x^* \in A$ such that $d(x^*, Tx^*) = d(A, B)$, where $d(A, B) := \min\{d(x, Tx) : x \in A\}$. Since $d(x, Tx) \geq d(A, B)$ for all $x \in A$, the optimal solution to the problem of minimizing the real valued function $x \to d(x, Tx)$ over the domain A of the mapping T will be the one for which the value $d(A, B)$ is attained. A point that satisfies the condition

$d(x, Tx) = d(A, B)$ is called a best proximity point of T. Furthermore, the best proximity theorems also regress to an extension of fixed point theorem; that is, a best proximity point becomes a fixed point if either $d(A, B) = 0$ or the mapping under consideration is a self-mapping.

The existence and convergence of best proximity points are an interesting topic of optimization theory. For more details on this approach, we refer the reader to De la Sen and Agarwal [8], Kumam et al. [9–11], di Bari et al. [12], Eldred and Veeramani [13], Al-Thagafi and Shahzad [14], Sadiq Basha and Veeramani [15], Kim and Lee [16], Kirk et al. [17], Sankar Raj [18], Karapınar et al. [19], and Jleli and Samet [20]. The study of best proximity point in the setting of partially ordered metric space attracted recently the attention of many authors; see [21–33].

Now we recall the definition of coupled fixed point. Let X be a nonempty set and $F : X \times X \to X$ a given mapping. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping F if $F(x, y) = x$ and $F(y, x) = y$. In 2006, Bhaskar and Lakshmikantham [34] proved some coupled fixed point theorems for mappings satisfying the mixed monotone mapping. Indeed, let $(X, \leq)$ be a partially ordered set; the mapping F is said to have the mixed monotone property if

$$\begin{aligned} &x_1, x_2 \in X, \quad x_1 \leq x_2 \Longrightarrow F(x_1, y) \leq F(x_2, y), \\ &\qquad\qquad\qquad\qquad\qquad\qquad \forall y \in X, \\ &y_1, y_2 \in X, \quad y_1 \leq y_2 \Longrightarrow F(x, y_1) \geq F(x, y_2), \\ &\qquad\qquad\qquad\qquad\qquad\qquad \forall x \in X. \end{aligned} \tag{1}$$

Their results investigate a large class of problems and show the existence and uniqueness of a solution for a periodic boundary value problem. For more details on this concept one may go through the references [35–37].

Motivated by the above theorems, we introduce the concept of proximally mixed monotone property and proximally coupled contraction. We also explore the existence and uniqueness of coupled best proximity points in the setting of partially ordered metric spaces, thereby producing optimal approximate solutions for that function with respect to both coordinates. Further, we attempt to give the generalization of the results in [34].

2. Preliminaries

Let X be a nonempty set such that (X, d) is a metric space. Unless otherwise specified, it is assumed throughout this section that A and B are nonempty subsets of the metric space (X, d); the following notions are used subsequently:

$$\begin{aligned} &d(A, B) := \inf\{d(x, y) : x \in A, y \in B\}, \\ &A_0 = \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\}, \\ &B_0 = \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}. \end{aligned} \tag{2}$$

In [17], the authors discussed sufficient conditions which guarantee the nonemptiness of A_0 and B_0. Also, in [15], the authors proved that A_0 is contained in the boundary of A. Moreover, the authors proved that A_0 is contained in the boundary of A in the setting of normed linear spaces.

Definition 1. Let $(X, d, \leq)$ be a partially ordered metric space and A, B nonempty subsets of X. A mapping $F : A \times A \to B$ is said to have proximal mixed monotone property if $F(x, y)$ is proximally nondecreasing in x and is proximally nonincreasing in y; that is, for all $x, y \in A$,

$$\begin{aligned} &\left.\begin{aligned} x_1 \leq x_2 \\ d(u_1, F(x_1, y)) = d(A, B) \\ d(u_2, F(x_2, y)) = d(A, B) \end{aligned}\right\} \Longrightarrow u_1 \leq u_2, \\ &\left.\begin{aligned} y_1 \leq y_2 \\ d(u_3, F(x, y_1)) = d(A, B) \\ d(u_4, F(x, y_2)) = d(A, B) \end{aligned}\right\} \Longrightarrow u_4 \leq u_3, \end{aligned} \tag{3}$$

where $x_1, x_2, y_1, y_2, u_1, u_2, u_3, u_4 \in A$.

One can see that, if $A = B$ in the above definition, the notion of proximal mixed monotone property reduces to that of mixed monotone property.

Lemma 2. *Let $(X, d, \leq)$ be a partially ordered metric space and A, B nonempty subsets of X. Assume that A_0 is nonempty. A mapping $F : A \times A \to B$ has proximal mixed monotone property with $F(A_0 \times A_0) \subseteq B_0$; then for any x_0, x_1, x_2, y_0, y_1 in A_0*

$$\left.\begin{aligned} x_0 \leq x_1, \qquad y_0 \geq y_1 \\ d(x_1, F(x_0, y_0)) = d(A, B) \\ d(x_2, F(x_1, y_1)) = d(A, B) \end{aligned}\right\} \Longrightarrow x_1 \leq x_2. \tag{4}$$

Proof. By hypothesis $F(A_0 \times A_0) \subseteq B_0$, $F(x_1, y_0) \in B_0$. Hence there exists $x_1^* \in A$ such that

$$d(x_1^*, F(x_1, y_0)) = d(A, B). \tag{5}$$

Using F is proximal mixed monotone (in particular F is proximally nondecreasing in x) to (4) and (5), we get

$$\left.\begin{aligned} x_0 \leq x_1 \\ d(x_1, F(x_0, y_0)) = d(A, B) \\ d(x_1^*, F(x_1, y_0)) = d(A, B) \end{aligned}\right\} \Longrightarrow x_1 \leq x_1^*. \tag{6}$$

Analogously, using F is proximal mixed monotone (in particular F is proximally nonincreasing in y) to (4) and (5), we get

$$\left.\begin{aligned} y_1 \leq y_0 \\ d(x_2, F(x_1, y_1)) = d(A, B) \\ d(x_1^*, F(x_1, y_0)) = d(A, B) \end{aligned}\right\} \Longrightarrow x_1^* \leq x_2. \tag{7}$$

From (6) and (7), one can conclude that $x_1 \leq x_2$. Hence the proof is completed. □

Lemma 3. *Let $(X, d, \leq)$ be a partially ordered metric space and A, B nonempty subsets of X. Assume that A_0 is nonempty. A mapping $F : A \times A \to B$ has proximal mixed monotone*

property with $F(A_0 \times A_0) \subseteq B_0$; then for any x_0, x_1, y_0, y_1, y_2 in A_0

$$\left.\begin{array}{r} x_0 \le x_1, \qquad y_0 \ge y_1 \\ d(y_1, F(y_0, x_0)) = d(A, B) \\ d(y_2, F(y_1, x_1)) = d(A, B) \end{array}\right\} \Longrightarrow y_1 \ge y_2. \tag{8}$$

Proof. The proof is same as Lemma 2. □

Definition 4. Let $(X, d, \le)$ be a partially ordered metric space and A, B nonempty subsets of X. A mapping $F : A \times A \to B$ is said to be proximally coupled contraction if there exists $k \in (0, 1)$ such that whenever

$$\left.\begin{array}{r} x_1 \le x_2, \qquad y_1 \ge y_2 \\ d(u_1, F(x_1, y_1)) = d(A, B) \\ d(u_2, F(x_2, y_2)) = d(A, B) \end{array}\right\} \Longrightarrow d(u_1, u_2) \le \frac{k}{2}\left[d(x_1, x_2) + d(y_1, y_2)\right], \tag{9}$$

where $x_1, x_2, y_1, y_2, u_1, u_2 \in A$.

One can see that, if $A = B$ in the above definition, the notion of proximally coupled contraction reduces to that of coupled contraction.

3. Coupled Best Proximity Point Theorems

Let $(X, d, \le)$ be a partially ordered complete metric space endowed with the product space $X \times X$ with the following partial order:

$$\begin{gathered} \text{for } (x, y), (u, v) \in X \times X, \\ (u, v) \le (x, y) \Longleftrightarrow x \ge u, y \le v. \end{gathered} \tag{10}$$

Theorem 5. *Let $(X, \le, d)$ be a partially ordered complete metric space. Let A and B be nonempty closed subsets of the metric space (X, d) such that $A_0 \ne \emptyset$. Let $F : A \times A \to B$ satisfy the following conditions.*

(i) *F is a continuous proximally coupled contraction having the proximal mixed monotone property on A such that $F(A_0 \times A_0) \subseteq B_0$.*

(ii) *There exist elements (x_0, y_0) and (x_1, y_1) in $A_0 \times A_0$ such that*

$$\begin{aligned} d(x_1, F(x_0, y_0)) &= d(A, B) \quad \text{with } x_0 \le x_1, \\ d(y_1, F(y_0, x_0)) &= d(A, B) \quad \text{with } y_0 \ge y_1. \end{aligned} \tag{11}$$

Then there exist $(x, y) \in A \times A$ such that $d(x, F(x, y)) = d(A, B)$ and $d(y, F(y, x)) = d(A, B)$.

Proof. By hypothesis there exist elements (x_0, y_0) and (x_1, y_1) in $A_0 \times A_0$ such that

$$\begin{aligned} d(x_1, F(x_0, y_0)) &= d(A, B) \quad \text{with } x_0 \le x_1, \\ d(y_1, F(y_0, x_0)) &= d(A, B) \quad \text{with } y_0 \ge y_1. \end{aligned} \tag{12}$$

Because of the fact that $F(A_0 \times A_0) \subseteq B_0$, there exists an element (x_2, y_2) in $A_0 \times A_0$ such that

$$\begin{aligned} d(x_2, F(x_1, y_1)) &= d(A, B), \\ d(y_2, F(y_1, x_1)) &= d(A, B). \end{aligned} \tag{13}$$

Hence from Lemmas 2 and 3, we obtain $x_1 \le x_2$ and $y_1 \ge y_2$.

Continuing this process, we can construct the sequences (x_n) and (y_n) in A_0 such that

$$d(x_{n+1}, F(x_n, y_n)) = d(A, B), \quad \forall n \in \mathbb{N} \tag{14}$$

with $x_0 \le x_1 \le x_2 \le \cdots x_n \le x_{n+1} \cdots$ and

$$d(y_{n+1}, F(y_n, x_n)) = d(A, B), \quad \forall n \in \mathbb{N} \tag{15}$$

with $y_0 \ge y_1 \ge y_2 \ge \cdots y_n \ge y_{n+1} \cdots$.

Then $d(x_n, F(x_{n-1}, y_{n-1})) = d(A, B)$, $d(x_{n+1}, F(x_n, y_n)) = d(A, B)$ and also we have $x_{n-1} \le x_n$, $y_{n-1} \ge y_n$, $\forall n \in \mathbb{N}$. Now using that F is proximally coupled contraction on A we get

$$d(x_n, x_{n+1}) \le \frac{k}{2}\left[d(x_{n-1}, x_n) + d(y_{n-1}, y_n)\right], \quad \forall n \in \mathbb{N}. \tag{16}$$

Similarly

$$d(y_n, y_{n+1}) \le \frac{k}{2}\left[d(y_{n-1}, y_n) + d(x_{n-1}, x_n)\right], \quad \forall n \in \mathbb{N}. \tag{17}$$

Adding (16) and (17), we get

$$\begin{aligned} &d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \\ &\quad \le k\left[d(x_{n-1}, x_n) + d(y_{n-1}, y_n)\right] \\ &\quad \le k\left[k\left[d(x_{n-2}, x_{n-1}) + d(y_{n-2}, y_{n-1})\right]\right] \\ &\quad \le \cdots \\ &\quad \le k^n\left[d(x_0, x_1) + d(y_0, y_1)\right]. \end{aligned} \tag{18}$$

Finally, we get

$$d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \le k^n\left[d(x_0, x_1) + d(y_0, y_1)\right]. \tag{19}$$

Now we prove that (x_n) and (y_n) are Cauchy sequences.

For $n > m$, regarding triangle inequality and (19), one can observe that

$$\begin{aligned} &d(x_n, x_m) + d(y_n, y_m) \\ &\quad \le d(x_n, x_{n-1}) + d(y_n, y_{n-1}) + \cdots + d(x_m, x_{m+1}) \\ &\qquad + d(y_m, y_{m+1}) \\ &\quad \le \left(k^{n-1} + \cdots + k^m\right)\left[d(x_0, x_1) + d(y_0, y_1)\right] \\ &\quad \le \frac{k^m}{1-k}\left[d(x_0, x_1) + d(y_0, y_1)\right]. \end{aligned} \tag{20}$$

Let $\epsilon > 0$ be given. Choose a natural number M such that $(k^m/(1-k))[d(x_0, x_1) + d(y_0, y_1)] < \epsilon$ for all $m > M$. Thus, $d(x_n, x_m) + d(y_n, y_m) < \epsilon$ for $n > m$. Therefore, the sequences (x_n) and (y_n) are Cauchy.

Since A is closed subset of a complete metric space X, these sequences have limits. Thus, there exists $x, y \in A$ such that $x_n \to x$ and $y_n \to y$. Therefore $(x_n, y_n) \to (x, y)$ in $A \times A$. Since F is continuous, we have $F(x_n, y_n) \to F(x, y)$ and $F(y_n, x_n) \to F(y, x)$.

Hence the continuity of the metric function d implies that $d(x_{n+1}, F(x_n, y_n)) \to d(x, F(x, y))$ and $d(y_{n+1}, F(y_n, x_n)) \to d(y, F(y, x))$. But from (14) and (15) we get that the sequences $d(x_{n+1}, F(x_n, y_n))$ and $d(y_{n+1}, F(y_n, x_n))$ are constant sequences with the value $d(A, B)$. Therefore, $d(x, F(x, y)) = d(A, B)$ and $d(y, F(y, x)) = d(A, B)$. This completes the proof of the theorem. □

Corollary 6. *Let $(X, \leq, d)$ be a partially ordered complete metric space. Let A be nonempty closed subset of the metric space (X, d). Let $F : A \times A \to A$ satisfy the following conditions.*

(i) *F is continuous having the proximal mixed monotone property and proximally coupled contraction on A.*

(ii) *There exist (x_0, y_0) and (x_1, y_1) in $A \times A$ such that $x_1 = F(x_0, y_0)$ with $x_0 \leq x_1$ and $y_1 = F(y_0, x_0)$ with $y_0 \geq y_1$.*

Then there exist $(x, y) \in A \times A$ such that $d(x, F(x, y)) = 0$ and $d(y, F(y, x)) = 0$.

In what follows we prove that Theorem 5 is still valid for F not necessarily continuous, assuming the following hypothesis in A:

$$\begin{aligned} &(x_n) \text{ is a nondecreasing sequence in } A \\ &\quad \text{such that } x_n \longrightarrow x; \text{ then } x_n \leq x. \\ &(y_n) \text{ is a nonincreasing sequence in } A \\ &\quad \text{such that } y_n \longrightarrow y; \text{ then } y \leq y_n. \end{aligned} \tag{21}$$

Theorem 7. *Assume the conditions* (21) *and A_0 is closed in X instead of continuity of F in Theorem 5; then the conclusion of Theorem 5 holds.*

Proof. Following the proof of Theorem 5, there exist sequences (x_n) and (y_n) in A satisfying the following conditions:

$$d(x_{n+1}, F(x_n, y_n)) = d(A, B) \quad \text{with } x_n \leq x_{n+1}, \ \forall n \in \mathbb{N}, \tag{22}$$

$$d(y_{n+1}, F(y_n, x_n)) = d(A, B) \quad \text{with } y_n \geq y_{n+1}, \ \forall n \in \mathbb{N}. \tag{23}$$

Moreover, (x_n) converges to x and y_n converges to y in A. From (21), we get $x_n \leq x$ and $y_n \geq y$. Note that the sequences (x_n) and (y_n) are in A_0 and A_0 is closed. Therefore, $(x, y) \in A_0 \times A_0$. Since $F(A_0 \times A_0) \subseteq B_0$, there exists $F(x, y)$ and $F(y, x)$ are in B_0. Therefore, there exists $(x^*, y^*) \in A_0 \times A_0$ such that

$$d(x^*, F(x, y)) = d(A, B), \tag{24}$$

$$d(y^*, F(y, x)) = d(A, B). \tag{25}$$

Since $x_n \leq x$ and $y_n \geq y$. By using F is proximally coupled contraction for (22) and (24) also for (25) and (23), we get

$$\begin{aligned} d(x_{n+1}, x^*) &\leq \frac{k}{2}[d(x_n, x) + d(y_n, y)], \quad \forall n, \\ d(y^*, y_{n+1}) &\leq \frac{k}{2}[d(y, y_n) + d(x, x_n)], \quad \forall n. \end{aligned} \tag{26}$$

Since $x_n \to x$ and $y_n \to y$, by taking limit on the above two inequality, we get $x = x^*$ and $y = y^*$. Consequently the result follows. □

Corollary 8. *Assume the conditions* (21) *instead of continuity of F in Corollary 6; then the conclusion of Corollary 6 holds.*

Now, we present an example where it can be appreciated that hypotheses in Theorems 5 and 7 do not guarantee uniqueness of the coupled best proximity point.

Example 9. Let $X = \{(0,1), (1,0), (-1,0), (0,-1)\} \subset \mathbb{R}^2$ and consider the usual order $(x, y) \preceq (z, t) \Leftrightarrow x \leq z$ and $y \leq t$.

Thus, $(X, \preceq)$ is a partially ordered set. Besides, (X, d_2) is a complete metric space considering d_2 the euclidean metric. Let $A = \{(0,1), (1,0)\}$ and $B = \{(0,-1), (-1,0)\}$ be closed subsets of X. Then, $d(A, B) = \sqrt{2}$, $A = A_0$ and $B = B_0$. Let $F : A \times A \to B$ be defined as $F((x_1, x_2), (y_1, y_2)) = (-x_2, -x_1)$. Then, it can be seen that F is continuous such that $F(A_0 \times A_0) \subseteq B_0$. The only comparable pairs of points in A are $x \preceq x$ for $x \in A$; hence proximal mixed monotone property and proximally coupled contraction on A are satisfied trivially.

It can be shown that the other hypotheses of the theorem are also satisfied. However, F has three coupled best proximity points $((0,1), (0,1))$, $((0,1), (1,0))$, and $((1,0), (1,0))$.

One can prove that the coupled best proximity point is in fact unique, provided that the product space $A \times A$ endowed with the partial order mentioned earlier has the following property:

$$\begin{aligned} &\text{Every pair of elements has either a lower bound} \\ &\text{or an upper bound.} \end{aligned} \tag{27}$$

It is known that this condition is equivalent to the following:

For every pair of $(x, y), (x^*, y^*) \in A \times A$, there exists (z_1, z_2) in $A \times A$.

$$\text{that is comparable to } (x, y), \ (x^*, y^*). \tag{28}$$

Theorem 10. *In addition to the hypothesis of Theorem 5 (resp., Theorem 7), suppose that for any two elements (x, y) and (x^*, y^*) in $A_0 \times A_0$,*

$$\begin{aligned} &\text{there exists } (z_1, z_2) \in A_0 \times A_0 \\ &\text{such that } (z_1, z_2) \text{ is comparable to } (x, y),\ (x^*, y^*); \end{aligned} \tag{29}$$

then F has a unique coupled best proximity point.

Proof. From Theorem 5 (resp., Theorem 7), the set of coupled best proximity points of F is nonempty. Suppose that there exist (x, y) and (x^*, y^*) in $A \times A$ which are coupled best proximity points. That is,

$$\begin{gathered} d(x, F(x, y)) = d(A, B), \qquad d(y, F(y, x)) = d(A, B), \\ d(x^*, F(x^*, y^*)) = d(A, B), \\ d(y^*, F(y^*, x^*)) = d(A, B). \end{gathered} \tag{30}$$

We distinguish two cases.

Case 1. Suppose that (x, y) is comparable. Let (x, y) be comparable to (x^*, y^*) with respect to the ordering in $A \times A$. Apply F as proximally coupled contraction to $d(x, F(x, y)) = d(A, B)$ and $d(x^*, F(x^*, y^*)) = d(A, B)$, there exists $k \in (0, 1)$ such that

$$d(x, x^*) \le \frac{k}{2}\left[d(x, x^*) + d(y, y^*)\right]. \tag{31}$$

Similarly, one can prove that

$$d(y, y^*) \le \frac{k}{2}\left[d(y, y^*) + d(x, x^*)\right]. \tag{32}$$

Adding (31) and (32), we get

$$d(x, x^*) + d(y, y^*) \le k\left[d(x, x^*) + d(y, y^*)\right]. \tag{33}$$

This implies that $d(x, x^*) + d(y, y^*) = 0$; hence $x = x^*$ and $y = y^*$.

Case 2. Suppose that (x, y) is not comparable. Let (x, y) be noncomparable to (x^*, y^*); then there exists $(u_1, v_1) \in A_0 \times A_0$ which is comparable to (x, y) and (x^*, y^*).

Since $F(A_0 \times A_0) \subseteq B_0$, there exists $(u_2, v_2) \in A_0 \times A_0$ such that $d(u_2, F(u_1, v_1)) = d(A, B)$ and $d(v_2, F(v_1, u_1)) = d(A, B)$. Without loss of generality assume that $(u_1, v_1) \le (x, y)$ (i.e., $x \ge u_1$ and $y \le v_1$.) Note that $(u_1, v_1) \le (x, y)$ implies that $(y, x) \le (v_1, u_1)$. From Lemmas 2 and 3, we get

$$\begin{aligned} &\left.\begin{aligned} u_1 \le x, \qquad v_1 \ge y \\ d(u_2, F(u_1, v_1)) = d(A, B) \\ d(x, F(x, y)) = d(A, B) \end{aligned}\right\} \Longrightarrow u_2 \le x, \\ &\left.\begin{aligned} u_1 \le x, \qquad v_1 \ge y \\ d(v_2, F(v_1, u_1)) = d(A, B) \\ d(y, F(y, x)) = d(A, B) \end{aligned}\right\} \Longrightarrow v_2 \ge y. \end{aligned} \tag{34}$$

From the above two inequalities, we obtain $(u_2, v_2) \le (x, y)$. Continuing this process, we get sequences $\{u_n\}$ and $\{v_n\}$ such that $d(u_{n+1}, F(u_n, v_n)) = d(A, B)$ and $d(v_{n+1}, F(v_n, u_n)) = d(A, B)$ with $(u_n, v_n) \le (x, y)$ $\forall n \in \mathbb{N}$. By using that F is a proximally coupled contraction, we get

$$\begin{gathered} u_n \le x, \qquad v_n \ge y \\ d(u_{n+1}, F(u_n, v_n)) = d(A, B) \\ d(x, F(x, y)) = d(A, B) \\ \Downarrow \\ d(u_{n+1}, x) \le \frac{k}{2}\left[d(u_n, x) + d(v_n, y)\right], \qquad \forall n \in \mathbb{N}. \end{gathered} \tag{35}$$

Similarly, we can prove that

$$\begin{gathered} y \le v_n, \qquad x \ge u_n \\ d(y, F(y, x)) = d(A, B) \\ d(v_{n+1}, F(v_n, u_n)) = d(A, B) \\ \Downarrow \\ d(y, v_{n+1}) \le \frac{k}{2}\left[d(y, v_n) + d(x, u_n)\right], \qquad \forall n \in \mathbb{N}. \end{gathered} \tag{36}$$

Adding (35) and (36), we obtain

$$\begin{aligned} d(u_{n+1}, x) + d(y, v_{n+1}) &\le k\left[d(u_n, x) + d(v_n, y)\right]. \\ d(u_{n+1}, x) + d(y, v_{n+1}) \\ &\le k\left[d(u_n, x) + d(v_n, y)\right] \\ &\le k^2\left[d(u_{n-1}, x) + d(v_{n-1}, y)\right] \\ &\cdots \\ &\le k^{n+1}\left[d(u_0, x) + d(v_0, y)\right]. \end{aligned} \tag{37}$$

As $n \to \infty$, we get $d(u_{n+1}, x) + d(y, v_{n+1}) \to 0$, so that $u_n \to x$ and $v_n \to y$.

Analogously, one can prove that $u_n \to x^*$ and $v_n \to y^*$. Therefore, $x = x^*$ and $y = y^*$. Hence the proof is completed. □

Example 11. Let $X = \mathbb{R}$ be endowed with usual metric, and with the usual order in $\mathbb{R}$.

Suppose that $A = [1, 2]$ and $B = [-2, -1]$. Then A and B are nonempty closed subsets of X and $A_0 = 1$ and $B_0 = -1$. Also note that $d(A, B) = 2$.

Now consider the function $F : A \times A \to B$ defined as

$$F(x, y) = \frac{-x - y - 2}{4}. \tag{38}$$

Then it can be seen that F is continuous and $F(1, 1) = -1$. Hence, $F(A_0 \times A_0) \subseteq B_0$. It is easy to see that other hypotheses of the Theorem 10 are also satisfied. Further, it is easy to see that $(1, 1)$ is the unique element satisfying the conclusion of Theorem 10.

The following result, due to Fan [7], is a corollary Theorem 10 by taking $A = B$.

Corollary 12. *In addition to the hypothesis of Corollary 6 (resp., Corollary 8), suppose that for any two elements* (x, y) *and* (x^*, y^*) *in* $A \times A$,

$$\text{there exists } (z_1, z_2) \in A \times A \text{ such that } (z_1, z_2) \text{ is comparable to } (x, y), (x^*, y^*); \tag{39}$$

then F has a unique coupled fixed point.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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