Research Article

High Order Fefferman-Phong Type Inequalities in Carnot Groups and Regularity for Degenerate Elliptic Operators plus a Potential

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Let $\{X_1, X_2, \dots, X_m\}$ be the basis of space of horizontal vector fields in a Carnot group $\mathbb{G} = (\mathbb{R}^n; \circ)$ ($m < n$). We prove high order Fefferman-Phong type inequalities in $\mathbb{G}$. As applications, we derive a priori $L^p(\mathbb{G})$ estimates for the nondivergence degenerate elliptic operators $L = -\sum_{i,j=1}^m a_{ij}(x)X_iX_j + V(x)$ with VMO coefficients and a potential V belonging to an appropriate Stummel type class introduced in this paper. Some of our results are also new even for the usual Euclidean space.

1. Introduction and the Main Results

The classical L^p estimates for nondivergence elliptic operators with potentials of the form

$$\mathfrak{L}u \equiv -\sum_{i,j=1}^n a_{ij}(x)u_{x_i x_j} + Vu, \quad x \in \mathbb{R}^n, \quad (1)$$

have been extensively investigated and many results have been proved; see [1–5] and so forth. In particular, when $(a_{ij})_{n \times n} = I$, the identity matrix, and V belongs to the reverse Hölder class B_q ($n/2 \leq q < \infty$), Shen [2] established L^p ($1 < p \leq q$) boundedness for the Schrödinger operator $-\Delta + V$ and showed that the range of p is optimal. It is noted that $V \in B_q$ ($q > 1$) means that $V \in L^q_{\text{loc}}(\mathbb{R}^n)$, $V \geq 0$, and there exists a positive constant c such that the reverse Hölder inequality

$$\left(|B|^{-1} \int_B V^q(x) dx\right)^{1/q} \leq c \left(|B|^{-1} \int_B V(x) dx\right) \quad (2)$$

holds for every ball B in $\mathbb{R}^n$. More recently, when $V \in B_q$ ($n/2 \leq q < \infty$), a priori $L^p(\mathbb{R}^n)$ ($1 < p \leq q$) estimate for $\mathfrak{L}$ in (1) with VMO coefficients has been deduced by

Bramanti et al. [1] by using the representation formula for Vu in terms of $\mathfrak{L}u$, which generalized the result in [2]. The aim of this paper is to establish high order Fefferman-Phong type inequalities in Carnot groups and prove L^p regularity of degenerate elliptic operators plus a potential.

Let $X_1, X_2, \dots, X_m$ be horizontal vector fields in a Carnot group $\mathbb{G} = (\mathbb{R}^n; \circ)$, ($m < n$) (see Section 2.1). In this paper we consider the nondivergence degenerate elliptic operator of the kind

$$Lu \equiv Au + Vu \equiv -\sum_{i,j=1}^m a_{ij}(x)X_iX_ju + Vu, \quad (3)$$

where the leading coefficient $a_{ij}(x)$ satisfies $a_{ij}(x) = a_{ji}(x) \in L^\infty(\mathbb{G})$ for $i, j = 1, \dots, m$, and there exists a constant $\mu > 0$ such that, for any $x \in \mathbb{G}$ and $\xi \in \mathbb{R}^m$,

$$\mu|\xi|^2 \leq \sum_{i,j=1}^m a_{ij}(x)\xi_i\xi_j \leq \mu^{-1}|\xi|^2; \quad (4)$$

furthermore, we assume

$$a_{ij}(x) \in \text{VMO}(\mathbb{G}), \quad (5)$$

which shows that, for $i, j = 1, \dots, m$,

$$\eta_{ij} = \sup_{\rho \leq r} \sup_{x \in \mathbb{G}} \left(|B_\rho(x)|^{-1} \int_{B_\rho(x)} |a_{ij}(y) - a_{ij}^B| dy \right) \longrightarrow 0, \quad (6)$$

$$r \longrightarrow 0^+.$$

Here $a_{ij}^B = |B_\rho(x)|^{-1} \int_{B_\rho(x)} a_{ij}(y) dy$. $B_\rho(x)$ denotes a metric ball of radius ρ and center x associated with the Carnot-Carathéodory distance d (see Section 2) by

$$B_\rho(x) = B(x, \rho) = \{y \in \mathbb{G} : d(x, y) < \rho\}. \quad (7)$$

As to the potential V , inspired by Di Fazio and Zamboni [6, Definition 2.4], we introduce the following Stummel type class S_p .

Definition 1 (Stummel type class). Let $V : \mathbb{G} \rightarrow \mathbb{R}$, $1 < p < \infty$, $r > 0$. One says that $V \in S_p(\mathbb{G})$, if for every $r > 0$,

$$\begin{aligned} \varphi_V(r) := \sup_{x \in \mathbb{G}} \left(\int_{d(x,y) < r} \frac{d(x,y)^2}{|B(x, d(x,y))|} \right. \\ \left. \times \left(\int_{d(z,x) < r} |V(z)| \right. \right. \\ \left. \left. \times \frac{d(z,y)^2}{|B(z, d(z,y))|} dz \right)^{1/(p-1)} dy \right)^{p-1} \end{aligned} \quad (8)$$

is finite and

$$\lim_{r \rightarrow 0} \varphi_V(r) = 0, \quad (9)$$

where $d(\cdot, \cdot)$ is the Carnot-Carathéodory distance; see Section 2.

Sometimes we will call $\varphi_V(r)$ the Stummel modulus of V .

Remark 2. We note that $L^\infty(\mathbb{G}) \subset S_p(\mathbb{G})$ ($1 < p < \infty$) and $S_p(\mathbb{G})$ is the special case of the function class in [7, page 56] with $p = 2$ and $\mathbb{G} = \mathbb{R}^n$ ($n \geq 5$). Also, note that the function $V(x) = d(0, x)^{-2}$ on $\mathbb{G}$ belongs to the classes $S_p(\mathbb{G})$ for $p > 2$, where d is the Carnot-Carathéodory distance (see Section 2).

Nondivergence degenerate elliptic operators similar to (3) including the form $-\sum_{i=1}^m X_i^2 + V$ have been studied by some authors; see [8–11] and so forth. The local L^p estimate for operator (3) with the vanishing potential $V = 0$ on the homogeneous group has been verified by Bramanti and Brandolini [12]. For the study of related operators, we refer to [13, 14] and references therein. We will prove regularity for the operator L in (3) on $\mathbb{G}$ if a_{ij} satisfy (4)-(5) and $V^p \in S_p(\mathbb{G})$; see Theorem 3 below. Our methods are different from the Euclidean case by Bramanti et al. [1], where estimates of integral operators and their commutators were used as a main tool.

Since Fefferman [15] proved the well-known imbedding inequality

$$\int_{\mathbb{R}^n} |V| |u|^2 dx \leq c \int_{\mathbb{R}^n} |\nabla u|^2 dx, \quad u \in C_0^\infty(\mathbb{R}^n), \quad (10)$$

with V belonging to the classical Morrey class $L^{r, n-2r}$, $1 < r \leq n/2$, it has been extended to many more general settings and applied to infer regularity for partial differential operators; see [6, 16–19] and so forth. One of the main jobs of this paper is to establish a high order Fefferman-Phong type inequality in Carnot groups (see Theorem 4), which is motivated by Di Fazio and Zamboni [6, Theorem 3.1]. So far as we know, there is not any result in literature on high order Fefferman-Phong inequalities. Using this inequality and proving several estimates with the potential, a priori $L^p(\mathbb{G})$ estimate for L is obtained.

We mention that the homogeneous dimension Q of $\mathbb{G}$, the horizontal gradient Xu , the second order horizontal gradient X^2u , the horizontal Sobolev spaces $HW^{2,p}(\mathbb{G})$ and $HW_V^{2,p}(\mathbb{G})$, the polynomial $P_B(x)$, and the reverse Hölder class B_q in our setting will be described in Section 2. Now we are in a position to state main results.

Theorem 3. Under the assumptions (4)-(5), if $V^p \in S_p(\mathbb{G})$, $1 < p < \infty$, then there exists a positive constant $c = c(p, \mu, \eta, V, \mathbb{G})$ such that, for any $u \in C_0^\infty(\mathbb{G})$, it follows that

$$\begin{aligned} \|u\|_{HW^{2,p}(\mathbb{G})} + \|Vu\|_{L^p(\mathbb{G})} \\ \leq c (\|Lu\|_{L^p(\mathbb{G})} + \|u\|_{L^p(\mathbb{G})}), \end{aligned} \quad (11)$$

where η in c depends only on the VMO moduli η_{ij} of the coefficients a_{ij} . Furthermore, (11) holds for $u \in HW_V^{2,p}(\mathbb{G})$.

It is noted that the L^p estimates of the operators similar to (3) with discontinuous leading coefficients and bounded lower terms were obtained by Bramanti and Brandolini [12, 20]. Here the potential V in Theorem 3 may be unbounded on $\mathbb{G}$.

The key for the proof of Theorem 3 is the following high order Fefferman-Phong type inequality.

Theorem 4. Let $B = B_{r_B}(x_0)$ be any metric ball in $\mathbb{G}$. If $V \in S_p(\mathbb{G})$ ($1 < p < \infty$), then there exists a first order polynomial $P_B(x)$ such that, for any $u \in C^\infty(B)$, one has

$$\int_B |u - P_B(x)|^p |V| dx \leq c \varphi_V(2r_B) \int_B |X^2u|^p dx, \quad (12)$$

where the positive constant c is independent of u and B . Moreover, for any $u \in C_0^\infty(B)$, one has

$$\int_B |u|^p |V| dx \leq c \varphi_V(2r_B) \int_B |X^2u|^p dx, \quad (13)$$

where $c > 0$ is independent of u and B .

The above X^2u is a set of $X_i X_j u$ for all $i, j = 1, \dots, m$. We will define X^2u precisely in Section 2.

Remark 5. The main difference between Theorem 4 and [6, Theorem 3.1] is clear; that is, the right-hand side term $\|Xu\|_{L^p(B)}$ in [6] is replaced by $\|X^2u\|_{L^p(B)}$ here. Of course, the class involving V is not the same.

We observe an important relation between the Stummel class here and the reverse Hölder class: if $V \in B_q \cap L^1(\mathbb{G})$, $q > Q/2$, then $V^p \in S_p(\mathbb{G})$, $1 < p \leq 2q/Q$. From it and Theorem 3, the following result follows.

Theorem 6. *Under the same assumptions on a_{ij} as in Theorem 3, if $V \in B_q \cap L^1(\mathbb{G})$, $q > Q/2$, then for $1 < p \leq 2q/Q$ and $u \in C_0^\infty(B)$, the estimate (11) holds.*

Remark 7. When $q = Q/2$ and $V \in B_{Q/2} \cap L^1(\mathbb{G})$, by the important property of the B_q class (see [21]), there exists $\varepsilon > 0$ such that $V \in B_{Q/2+\varepsilon}$. Therefore, estimate (11) holds for $1 < p \leq 1 + 2\varepsilon/Q$ and $u \in C_0^\infty(B)$.

The paper is organized as follows. In Section 2 we recall some basic facts about Carnot groups and function spaces. In Section 3 we first give the proof of Theorem 4. Then combining with the known result in [12, Theorem 2] and proving an estimate with the potential V , we finish the proof of Theorem 3. The proof of Theorem 6 is given in Section 4. In Section 5, we restate Theorems 3 and 4 for the Euclidean case and elliptic operators without proofs.

Dependence of Constants. Throughout this paper, the letter c denotes a positive constant which may vary from line to line.

2. Preliminaries

2.1. Background on Carnot Groups. We collect some facts about Carnot groups that will be needed in the sequel and refer the readers to [22–25] for further details.

Definition 8 (Carnot group). A Carnot group $\mathbb{G} = (\mathbb{R}^n; \circ)$ is a simply connected nilpotent Lie group such that its Lie algebra $\tilde{\mathfrak{g}}$ admits a stratification

$$\tilde{\mathfrak{g}} = V_1 \oplus V_2 \oplus \cdots \oplus V_r = \bigoplus_{j=1}^r V_j, \quad (14)$$

where $[V_1, V_j] = V_{j+1}$, $j = 1, \dots, r-1$, and $[V_1, V_r] = \{0\}$. Here r is called the step of $\mathbb{G}$.

For $k = 1, \dots, r$, let $X_{1,k}, \dots, X_{m_k,k}$ be a basis of V_k consisting of commutators of length k , where m_k is the dimension of V_k . The horizontal vector fields are ones in the first layer V_1 and for convenience, we set $m_1 = m$ and denote $X_{i,1} = X_i$, $i = 1, \dots, m$. Clearly, vector fields $X_1, \dots, X_m$ satisfy Hormander's condition [26].

Let $\{\delta_\lambda\}_{\lambda>0}$ be a family of nonisotropic dilations on $\mathbb{G}$ defined by

$$\delta_\lambda : \mathbb{G} \longrightarrow \mathbb{G}, \quad \delta_\lambda(\xi) = (\lambda\xi_1, \lambda^2\xi_2, \dots, \lambda^r\xi_r), \quad (15)$$

for any $\lambda > 0$ and $\xi = (\xi_1, \dots, \xi_r) = (x_{1,1}, \dots, x_{m_1,1}, \dots, x_{1,r}, \dots, x_{m_r,r}) \in \mathbb{G}$. The integer $Q = \sum_{k=1}^r km_k$ is said to be the homogeneous dimension of $\mathbb{G}$. In general, we assume $Q \geq 4$. We call that a vector field $X_{i,k} \in \tilde{\mathfrak{g}}$ is left invariant if for any smooth function f one has

$$X_{i,k}^x(f(y \circ x)) = (X_{i,k}f)(y \circ x), \quad \text{for any } x \in \mathbb{G}, \quad (16)$$

and $X_{i,k}$ is sth homogeneous if for any smooth function f , it follows that

$$X_{i,k}(f(\delta_\lambda(x))) = \lambda^s (X_{i,k}f)(\delta_\lambda(x)), \quad \text{for any } x \in \mathbb{G}. \quad (17)$$

As in [23], the homogeneous norm of $\xi \in \mathbb{G}$ is defined by

$$d_{\mathbb{G}}(\xi) = \left[\sum_{k=1}^r \left(\sum_{i=1}^{m_k} |\xi_{i,k}|^2 \right)^{r/k} \right]^{1/2r}. \quad (18)$$

It is natural to define the pseudo distance by the homogeneous norm

$$d_{\mathbb{G}}(\xi, \eta) = d_{\mathbb{G}}(\eta^{-1} \circ \xi), \quad \text{for any } \xi, \eta \in \mathbb{G}, \quad (19)$$

where η^{-1} is the inverse of η . A polynomial on $\mathbb{G}$ [24] is a function which can be expressed in exponential coordinates (see, e.g., [22, Section 2.1.9] and [27]) as

$$P(x) = \sum_I a_I x^I, \quad a_I \in \mathbb{R}, \quad (20)$$

where $I = (i_{j,k})_{j=1, \dots, m_k}^{k=1, \dots, r}$ are multi-indices and

$$x^I = \prod_{k=1, \dots, r; j=1, \dots, m_k} x_{j,k}^{i_{j,k}}. \quad (21)$$

The homogeneous degree of monomial x^I is the sum $d(I) = \sum_{k=1}^r \sum_{j=1}^{m_k} ki_{j,k}$ and the homogeneous degree of $P(x)$ is $\max\{d(I) \mid a_I \neq 0\}$.

From [28], the left invariant vector fields $X_1, \dots, X_m$ can induce the corresponding Carnot-Carathéodory distance d : for any $\delta > 0$, let $A(\delta)$ be the set of absolutely continuous curves $\gamma : [0, 1] \rightarrow \mathbb{G}$ such that for a.e. $t \in [0, 1]$,

$$\gamma'(t) = \sum_{i=1}^m a_i(t) X_i(\gamma(t)) \quad (22)$$

$$\text{with } \sum_{i=1}^m |a_i(t)| \leq \delta.$$

By [29], it is known that for δ large enough the set $A(\delta)$ is nonempty. We define the Carnot-Carathéodory distance by

$$d(\xi, \eta) = \inf \{ \delta > 0 \mid \exists \delta \in A(\delta) \text{ with } \gamma(0) = \xi, \gamma(1) = \eta \}. \quad (23)$$

It is well known that the distance d is equivalent to the pseudo distance $d_{\mathbb{G}}$ (see [28]). In this paper, we will mainly use the Carnot-Carathéodory distance d to study regularity of (3).

Associated with the distance, we define the metric ball of center ξ and radius r in $\mathbb{G}$ by

$$B_r(\xi) = \{\eta \in \mathbb{G} \mid d(\xi, \eta) < r\}. \quad (24)$$

The Lebesgue measure in $\mathbb{R}^n$ is the Haar measure on $\mathbb{G}$ ([25, page 619]). Due to (15), one has

$$|B_r(\xi)| = C_Q r^Q, \quad (25)$$

where $|B_r(\xi)|$ is the measure of $B_r(\xi)$ and C_Q is a positive constant.

2.2. Function Spaces. Denote $X = (X_1, \dots, X_m)$, $Xu = (X_1u, \dots, X_mu)$, $X^2u = \{X_iX_ju\}_{i,j=1}^m$, $|Xu| = \sum_{i=1}^m |X_iu|$, and $|X^2u| = \sum_{i,j=1}^m |X_iX_ju|$.

Definition 9 (Horizontal Sobolev space). For any $p \geq 1$ and a domain $\Omega \subset \mathbb{G}$, one defines the Horizontal Sobolev spaces by

$$HW^{2,p}(\Omega) = \{u \in L^p(\Omega) \mid \|u\|_{HW^{2,p}(\Omega)} < +\infty\} \quad (26)$$

with the norm

$$\|u\|_{HW^{2,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|Xu\|_{L^p(\Omega)} + \|X^2u\|_{L^p(\Omega)}, \quad (27)$$

where $\|Xu\|_{L^p(\Omega)} = \sum_{i=1}^m \|X_iu\|_{L^p(\Omega)}$, $\|X^2u\|_{L^p(\Omega)} = \sum_{i,j=1}^m \|X_iX_ju\|_{L^p(\Omega)}$.

Analogously to [1], the space $HW_V^{2,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the norm

$$\|u\|_{HW_V^{2,p}(\Omega)} = \|u\|_{HW^{2,p}(\Omega)} + \|Vu\|_{L^p(\Omega)}. \quad (28)$$

Definition 10 (Reverse Hölder class). (1) A nonnegative locally L^q integrable function $V(x)$ on $\mathbb{G}$ is said to belong to the reverse Hölder class B_q ($1 < q < \infty$), if there exists a positive constant c such that

$$\left(|B|^{-1} \int_B V^q(x) dx\right)^{1/q} \leq c \left(|B|^{-1} \int_B V(x) dx\right), \quad (29)$$

for any metric ball B in $\mathbb{G}$.

(2) Let $V(x) > 0$ a.e. and $V(x) \in L_{\text{loc}}^\infty(\mathbb{G})$; one says $V(x) \in B_\infty$ if there exists a positive constant c such that

$$\sup_B V(x) \leq c \left(|B|^{-1} \int_B V(x) dx\right), \quad (30)$$

for any metric ball B in $\mathbb{G}$.

It is easy to see that $B_\infty \subset B_q \subset B_p, 1 < p < q < \infty$.

3. Proofs of Theorems 3 and 4

We first prove Theorem 4 and then prove Theorem 3.

3.1. Proof of Theorem 4. The following lemma is due to Lu and Wheeden [30, 31]. It will play a key role in our proof.

Lemma 11. Let $B = B_{r_B}(x_0)$ be a metric ball in $\mathbb{G}$. If $u \in C^\infty(B)$, then there exists a first order polynomial $P_B(x)$ such that, for a.e. $x \in B$,

$$|u(x) - P_B(x)| \leq c \int_B |X^2u(y)| \frac{d(x,y)^2}{|B(x,d(x,y))|} dy, \quad (31)$$

where the positive constant c is independent of u , x , and B . Moreover, if $u \in C_0^\infty(B)$, then for a.e. $x \in B$,

$$|u(x)| \leq c \int_B |X^2u(y)| \frac{d(x,y)^2}{|B(x,d(x,y))|} dy, \quad (32)$$

where $c > 0$ is independent of u , x , and B .

Proof of Theorem 4. By (31), Fubini's Theorem, and Hölder's inequality, we have

$$\begin{aligned} & \int_B |u(x) - P_B(x)|^p |V(x)| dx \\ & \leq c \int_B |V(x)| |u(x) - P_B(x)|^{p-1} \\ & \quad \times \left(\int_B |X^2u(y)| \frac{d(x,y)^2}{|B(x,d(x,y))|} dy \right) dx \\ & = c \int_B |X^2u(y)| \\ & \quad \times \left(\int_B |V(x)| |u(x) - P_B(x)|^{p-1} \right. \\ & \quad \times \left. \frac{d(x,y)^2}{|B(x,d(x,y))|} dx \right) dy \\ & \leq c \left(\int_B |X^2u(y)|^p dy \right)^{1/p} \\ & \quad \cdot \left(\int_B \left(\int_B |V(x)| |u(x) - P_B(x)|^{p-1} \right. \right. \\ & \quad \times \left. \left. \frac{d(x,y)^2}{|B(x,d(x,y))|} dx \right)^{p/(p-1)} dy \right)^{(p-1)/p} \\ & \equiv c \left(\int_B |X^2u(y)|^p dy \right)^{1/p} \cdot I. \end{aligned} \quad (33)$$

Now a computation yields

$$\begin{aligned}
& I^{p/(p-1)} \\
& \leq \int_B \left(\int_B |V(z)| \frac{d(z, y)^2}{|B(z, d(z, y))|} dz \right)^{1/(p-1)} \\
& \quad \cdot \left(\int_B |u(x) - P_B(x)|^p |V(x)| \right. \\
& \quad \quad \times \left. \frac{d(x, y)^2}{|B(x, d(x, y))|} dx \right) dy \\
& = \int_B |u(x) - P_B(x)|^p |V(x)| \int_B \frac{d(x, y)^2}{|B(x, d(x, y))|} \\
& \quad \cdot \left(\int_B |V(z)| \frac{d(z, y)^2}{|B(z, d(z, y))|} dz \right)^{1/(p-1)} dy dx \\
& \leq \int_{B(x_0, r_B)} |u(x) - P_B(x)|^p |V(x)| \\
& \quad \times \int_{B(x, 2r_B)} \frac{d(x, y)^2}{|B(x, d(x, y))|} \\
& \quad \cdot \left(\int_{B(x, 2r_B)} |V(z)| \right. \\
& \quad \quad \times \left. \frac{d(z, y)^2}{|B(z, d(z, y))|} dz \right)^{1/(p-1)} dy dx \\
& \leq (\varphi_V(2r_B))^{1/(p-1)} \int_B |u(x) - P_B(x)|^p |V(x)| dx.
\end{aligned} \tag{34}$$

Therefore,

$$\begin{aligned}
& \int_B |u(x) - P_B(x)|^p |V(x)| dx \\
& \leq c (\varphi_V(2r_B))^{1/p} \left(\int_B |u(x) - P_B(x)|^p |V(x)| dx \right)^{(p-1)/p} \\
& \quad \times \left(\int_B |X^2 u(y)|^p dy \right)^{1/p}.
\end{aligned} \tag{35}$$

It implies (12).

By using (32) and repeating the argument above for (12), we immediately obtain (13). $\square$

3.2. Proof of Theorem 3. Let us recall L^p estimates for the operator $A = \sum_{i,j=1}^m a_{ij} X_i X_j$ by Bramanti and Brandolini [12, Theorem 2].

Lemma 12. Under the assumptions (4) and (5), for every $p \in (1, \infty)$, there exist positive constants $c = c(p, \mu, \mathbb{G})$ and

$r = r(p, \mu, \eta, \mathbb{G})$, where η denotes the VMO moduli of coefficients a_{ij} , such that, for any $u \in C_0^\infty(\mathbb{G})$ and $\text{spt } u \subset B_r$ (B_r any metric ball of radius r),

$$\|X^2 u\|_{L^p(B_r)} \leq c \|Au\|_{L^p(B_r)}. \tag{36}$$

Based on it and Theorem 4, we have the following L^p estimates for L in (3).

Lemma 13. Under the assumptions (4) and (5), for every $p \in (1, \infty)$ and $V^p \in S_p(\mathbb{G})$, there exist positive constants $r_0 = r_0(p, \mu, \eta, \mathbb{G})$ and $c = c(p, r_0, \mu, V, \mathbb{G})$ such that, for any $u \in C_0^\infty(\mathbb{G})$ and $\text{spt } u \subset B_{r_0}$,

$$\|X^2 u\|_{L^p(B_{r_0})} + \|Vu\|_{L^p(B_{r_0})} \leq c \|Lu\|_{L^p(B_{r_0})}. \tag{37}$$

Proof. By Theorem 4,

$$\|Vu\|_{L^p(B_{r_0})}^p \leq c \varphi_{V^p}(2r_0) \int_{B_{r_0}} |X^2 u(x)|^p dx. \tag{38}$$

Applying Lemma 12, it follows that

$$\begin{aligned}
& \|X^2 u\|_{L^p(B_{r_0})} + \|Vu\|_{L^p(B_{r_0})} \\
& \leq c (\|Au\|_{L^p(B_{r_0})} + \|Vu\|_{L^p(B_{r_0})}) \\
& \leq c (\|Lu\|_{L^p(B_{r_0})} + \|Vu\|_{L^p(B_{r_0})}) \\
& \leq c (\|Lu\|_{L^p(B_{r_0})} + \varphi_{V^p}(2r_0)^{1/p} \|X^2 u\|_{L^p(B_{r_0})}).
\end{aligned} \tag{39}$$

Choosing $r_0 > 0$ such that $c \varphi_{V^p}(2r_0)^{1/p} \leq 1/2$, we derive (37). $\square$

Proof of Theorem 3. We consult the way in [1, pages 342-343] and apply our previous results. By the basic theorem on the partition of unity (e.g., see [32, page 66]), there exists a partition of unity of nonnegative functions $\{\varphi_i\}_{i=1}^\infty$ in $\mathbb{G}$ such that $\varphi_i \in C_0^\infty(B_{r_0}(z_i))$ with r_0 in Lemma 13 and a family of metric balls $\{B_{r_0}(z_i)\}_{i=1}^\infty$ satisfying the finite overlapping property. We have from Lemma 13 that

$$\begin{aligned}
& \|X^2 u\|_{L^p(\mathbb{G})} + \|Vu\|_{L^p(\mathbb{G})} \\
& = \left\| \sum_i X^2(\varphi_i u) \right\|_{L^p(\mathbb{G})} + \left\| \sum_i V\varphi_i u \right\|_{L^p(\mathbb{G})} \\
& \leq c \left(\sum_i \|X^2(\varphi_i u)\|_{L^p(B_{r_0}(z_i))} + \sum_i \|V\varphi_i u\|_{L^p(B_{r_0}(z_i))} \right) \\
& \leq c \sum_i \|L(\varphi_i u)\|_{L^p(B_{r_0}(z_i))} \\
& \leq \sum_i (\|Lu\|_{L^p(B_{r_0}(z_i))} + \|Du\|_{L^p(B_{r_0}(z_i))} + \|u\|_{L^p(B_{r_0}(z_i))}) \\
& \leq c (\|Lu\|_{L^p(\mathbb{G})} + \|Xu\|_{L^p(\mathbb{G})} + \|u\|_{L^p(\mathbb{G})}).
\end{aligned} \tag{40}$$

Combining with the interpolation inequality (see [12, Proposition 2])

$$\|Xu\|_{L^p(\mathbb{G})} \leq \varepsilon \|X^2u\|_{L^p(\mathbb{G})} + \frac{2}{\varepsilon} \|u\|_{L^p(\mathbb{G})}, \quad (41)$$

for any $\varepsilon > 0$,

we obtain (11). $\square$

4. Proof of Theorem 6

Several preliminary conclusions are necessary.

Lemma 14. *If $V \in B_q$, $q > 1$, then there exists a constant $c > 0$ such that, for any $1 \leq p < q < \infty$ and $0 < r < R < \infty$,*

$$\begin{aligned} & \frac{1}{|B_r(x)|} \int_{B_r(x)} V(y)^p dy \\ & \leq c \left(\frac{R}{r} \right)^{pQ/q} \left(\frac{1}{|B_R(x)|} \int_{B_R(x)} V(y) dy \right)^p. \end{aligned} \quad (42)$$

Proof. By Hölder's inequality and (29), it yields

$$\begin{aligned} & \frac{1}{|B_r(x)|} \int_{B_r(x)} V(y)^p dy \\ & \leq \left(\frac{1}{|B_r(x)|} \int_{B_r(x)} V(y)^q dy \right)^{p/q} \\ & \leq c \left(\frac{R}{r} \right)^Q \left(\frac{1}{|B_R(x)|} \int_{B_R(x)} V(y)^q dy \right)^{p/q} \\ & \leq c \left(\frac{R}{r} \right)^{pQ/q} \left(\frac{1}{|B_R(x)|} \int_{B_R(x)} V(y) dy \right)^p. \end{aligned} \quad (43)$$

$\square$

Remark 15. If we take $\mathbb{G} = \mathbb{R}^n$ and $p = 1$, then Lemma 14 gives the version in [2, Lemma 1.2].

Lemma 16. *If $V \in B_q \cap L^1(\mathbb{G})$, $q > Q/2$, then $V^p \in S_p(\mathbb{G})$, $1 < p \leq 2q/Q$.*

Proof. For any $x \in \mathbb{G}$, it follows that

$$\begin{aligned} I & =: \int_{d(x,y)<r} \frac{d^2(x,y)}{|B(x,d(x,y))|} \\ & \quad \times \left(\int_{d(z,y)<r} V^p(z) \frac{d^2(z,y)}{|B(z,d(z,y))|} dz \right)^{1/(p-1)} dy \\ & \leq \int_{d(x,y)<r} \frac{d^2(x,y)}{|B(x,d(x,y))|} \\ & \quad \times \left(\int_{d(z,y)<2r} V^p(z) \frac{d^2(z,y)}{|B(z,d(z,y))|} dz \right)^{1/(p-1)} dy \\ & \equiv \int_{d(x,y)<r} \frac{d^2(x,y)}{|B(x,d(x,y))|} (I_1)^{1/(p-1)} dy. \end{aligned} \quad (44)$$

By (42), it yields

$$\begin{aligned} I_1 & \leq \sum_{k=0}^{\infty} \int_{r/2^k \leq d(z,y) < r/2^{k-1}} V^p(z) \frac{d^2(z,y)}{|B(z,d(z,y))|} dz \\ & \leq c \sum_{k=0}^{\infty} \left(\frac{r}{2^k} \right)^{2-Q} \int_{d(z,y) < r/2^{k-1}} V^p(z) dz \\ & \leq cr^{2-pQ/q} R^{pQ/q-pQ} \sum_{k=0}^{\infty} \left(\frac{1}{2^k} \right)^{2-pQ/q} \left(\int_{B(y,R)} V(z) dz \right)^p \\ & \leq cr^{2-pQ/q} R^{pQ/q-pQ} \left(\int_{\mathbb{G}} V dz \right)^p. \end{aligned} \quad (45)$$

Also, we have

$$\begin{aligned} & \int_{d(x,y)<r} \frac{d^2(x,y)}{|B(x,d(x,y))|} dy \\ & \leq \sum_{k=1}^{\infty} \int_{r/2^k \leq d(x,y) < r/2^{k-1}} \frac{d^2(x,y)}{|B(x,d(x,y))|} dy \\ & \leq c \sum_{k=1}^{\infty} \left(\frac{r}{2^k} \right)^{2-Q} \int_{d(x,y) < r/2^{k-1}} dy \\ & \leq cr^2. \end{aligned} \quad (46)$$

Therefore combining (45) and (46) gets

$$\begin{aligned} I & \leq cr^{(1/(p-1))(2-pQ/q)} R^{(pQ/(p-1))(1/q-1)} \\ & \quad \times \left(\int_{\mathbb{G}} V(z) dz \right)^{p/(p-1)} \cdot cr^2 \\ & \leq cr^{(1/(p-1))(2p-pQ/q)} R^{(pQ/(p-1))(1/q-1)} \\ & \quad \times \left(\int_{\mathbb{G}} V(z) dz \right)^{p/(p-1)} \longrightarrow 0, \\ & \quad \text{as } r \longrightarrow 0. \end{aligned} \quad (47)$$

The result is proved. $\square$

Proof of Theorem 6. By Lemma 16 and Theorem 3, we immediately obtain Theorem 6. $\square$

Remark 17. In order to assure the convergence of the series $\sum_{k=0}^{\infty} (2^{-k})^{2-pQ/q}$ in the proof of Lemma 16, we require the assumption $p \leq 2q/Q$, which leads to the range of p in Theorem 6 smaller than [1, Theorem 1].

5. Results to the Euclidean Case and Elliptic Operators

Here for convenience of readers, we restate Theorems 3 and 4 corresponding to the Euclidean case but omit their proofs because the proofs are analogous to Theorems 3 and 4. It will be assumed for the leading coefficients a_{ij} in (1) that

(H_1) $a_{ij}(x) = a_{ji}(x) \in L^\infty(\mathbb{R}^n)$ for all $i, j = 1, \dots, n$ and there exists a positive constant μ such that, for any $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$,

$$\mu^{-1} |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \leq \mu |\xi|^2; \quad (48)$$

(H_2) $a_{ij}(x) \in VMO(\mathbb{R}^n)$; that is, for $i, j = 1, \dots, n$,

$$\eta_{ij} = \sup_{\rho \leq r} \sup_{x \in \mathbb{R}^n} \left(|B_\rho(x)|^{-1} \int_{B_\rho(x)} |a_{ij}(y) - a_{ij}^B| dy \right) \rightarrow 0, \quad (49)$$

as $r \rightarrow 0^+$,

where $a_{ij}^B = |B_\rho(x)|^{-1} \int_{B_\rho(x)} a_{ij}(y) dy$.

A function $V \in S_p(\mathbb{R}^n)$ for $1 < p < \infty$ means that, for each $r > 0$,

$$\varphi_V(r) := \sup_{x \in \mathbb{R}^n} \left(\int_{|x-y|<r} \frac{1}{|x-y|^{n-2}} \times \left(\int_{|z-y|<r} \frac{|V(z)|}{|z-y|^{n-2}} dz \right)^{1/(p-1)} dy \right)^{p-1} \quad (50)$$

is finite and

$$\lim_{r \rightarrow 0} \varphi_V(r) = 0. \quad (51)$$

Theorem 18. Under assumptions (H_1) and (H_2) , if $V^p \in S_p(\mathbb{R}^n)$, $1 < p < \infty$, then there exists a positive constant $c = c(n, p, \mu, \eta, V)$ such that, for any $u \in C_0^\infty(\mathbb{R}^n)$, one has

$$\begin{aligned} & \|u\|_{W^{2,p}(\mathbb{R}^n)} + \|Vu\|_{L^p(\mathbb{R}^n)} \\ & \leq c (\|\mathfrak{L}u\|_{L^p(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)}), \end{aligned} \quad (52)$$

where η depends only on the VMO moduli of the coefficients a_{ij} .

Remark 19. If $|V| \equiv 0$, $x \in \mathbb{R}^n$ or $|V| \leq \text{const.}$, $x \in \mathbb{R}^n$, the L^p theory of (1) with discontinuous leading coefficients was intensively studied and the result was proved in [33–36] and so forth. Bramanti et al. [1] obtained a prior $L^p(\mathbb{R}^n)$ estimate for (1) with $a_{ij} \in VMO$ and $V \in B_q$. Here $B_q \not\subset S_p(\mathbb{R}^n)$ and $S_p(\mathbb{R}^n) \not\subset B_q$.

Theorem 20. Let $B = B_{r_B}(x_0)$ be any ball in $\mathbb{R}^n$. If $V \in S_p(\mathbb{R}^n)$, then there exists a first order polynomial $P_B(x)$ in $\mathbb{R}^n$ such that, for any $u \in C^\infty(B)$, one has

$$\int_B |u - P_B(x)|^p |V| dx \leq c \varphi_V(2r_B) \int_B |X^2 u|^p dx, \quad (53)$$

where the positive constant c is independent of u and B . Moreover, for any $u \in C_0^\infty(B)$, one has

$$\int_B |u|^p |V| dx \leq c \varphi_V(2r_B) \int_B |X^2 u|^p dx, \quad (54)$$

where $c > 0$ is independent of u and B .

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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