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Research Article

Modelling Fractal Waves on Shallow Water Surfaces via Local Fractional Korteweg-de Vries Equation

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A mathematical model of fractal waves on shallow water surfaces is developed by using the concepts of local fractional calculus. The derivations of linear and nonlinear local fractional versions of the Korteweg-de Vries equation describing fractal waves on shallow water surfaces are obtained.

1. Introduction

The mathematical model of shallow water waves, conceived by Boussinesq [1], was rediscovered by Korteweg and de Vries [2]. It is commonly known as Korteweg-de Vries equation (KdV) [2, 3] and is given by

$$\frac{\partial}{\partial t}\phi(x,t)+\frac{\partial^3}{\partial x^3}\phi(x,t)-6\phi(x,t)\frac{\partial}{\partial x}\phi(x,t)=0. \quad (1)$$

Several versions of the KdV equations found in the literature are listed below.

(i) Generalized KdV equation (GKdV) [4]

$$\frac{\partial}{\partial t}\phi(x,t)+\phi(x,t)\frac{\partial}{\partial x}\phi(x,t)-\frac{\partial^5}{\partial x^5}\phi(x,t)=0. \quad (2)$$

(ii) Generalized-generalized KdV equation (GGKdV) [5]

$$\frac{\partial}{\partial t}\phi(x,t)+\left(\frac{\partial}{\partial x}\phi(x,t)\right)^3+\frac{\partial}{\partial x}f(\phi)=0. \quad (3)$$

(iii) Deformed KdV equation (DKdV) [6]

$$\frac{\partial}{\partial t}\psi(x,t)+\frac{\partial}{\partial x}\times\left(\frac{\partial^2}{\partial x^2}\psi(x,t)-2\eta\psi^3(x,t)-\frac{3}{2}\frac{\psi(x,t)\left(\partial\psi(x,t)/\partial x\right)^2}{\eta+\psi^2(x,t)}\right)=0. \quad (4)$$

(iv) Modified-modified KdV equation (MM KdV) [6]

$$\frac{\partial}{\partial t}\phi(x,t)+\frac{\partial^3}{\partial x^3}\phi(x,t)-\frac{1}{8}\left(\frac{\partial}{\partial x}\phi(x,t)\right)^3+\frac{\partial}{\partial x}\phi(x,t)\left(Ae^{\mu}+B+Ce^{-\mu}\right)=0, \quad (5)$$

where η, A, B, and C are constants.

(i) Modified KdV equation (M KdV) [7]

$$\frac{\partial}{\partial t}\phi(x,t)+\frac{\partial^3}{\partial x^3}\phi(x,t)\pm 6\phi^2(x,t)\frac{\partial}{\partial x}\phi(x,t)=0. \quad (6)$$

(ii) Spherical KdV (SKdV) was [7]

$$\frac{\partial}{\partial t}\psi(x,t)+\frac{\partial^3}{\partial x^3}\psi(x,t)-6\psi(x,t)\frac{\partial}{\partial x}\psi(x,t)+\frac{\psi(x,t)}{t}=0. \tag{7}$$

(iii) Cylindrical KdV equation (CKdV) was [8]

$$\frac{\partial}{\partial t}\psi(x,t)+\frac{\partial^3}{\partial x^3}\psi(x,t)-6\psi(x,t)\frac{\partial}{\partial x}\psi(x,t)+\frac{\psi(x,t)}{2t}=0. \tag{8}$$

For Further versions of KdV equation, we refer the reader to [9–12] and the references cited therein.

Recently, the fractional KdV equations have been discussed by several authors. Momani et al. [13] studied the KdV equation with both space- and time-fractional derivatives, while the time-fractional derivative case has been considered by El-Wakil et al. [14]. Atangana and Secer [15] developed solutions for coupled Korteweg-de Vries equations with time-fractional derivatives [15]. Abdulaziz et al. [16] discussed the modified KdV equations with different space- and time-fractional derivatives.

It is imperative to note that the above mentioned works are based on the fractional calculus of differentiable functions. However, there are certain nondifferentiable physical quantities describing the physical parameters locally, where the concept of differentiable functions is not applicable. In such cases the local fractional calculus (LFC) concept allows to obtain solutions adequate to such nondifferentiable problems [17–25] such as local fractional Helmholtz and diffusion equations [19], local fractional Navier-Stokes equations in fractal domain [21], local fractional Poisson and Laplace equations arising in the electrostatics in fractal domain [23], fractional models in forest gap [24], inhomogeneous local fractional wave equations [25], local fractional heat conduction equation [26], and other results [26–30].

In the present work, we focus on the derivation of the linear and the nonlinear local fractional versions of the Korteweg-de Vries equation describing fractal waves on shallow water surfaces.

The paper is organized as follows. In Section 2, we recall the local fractional conservation laws for the quantities in mathematical physics while the local fractional Korteweg-de Vries equation is derived from local fractional calculus in Section 3. The conclusions are outlined in Section 4.

2. Theoretical Background

2.1. Local Fractional Conservation Laws Arising in Mathematical Physics. First of all, we discuss the local fractional conservation laws of mass, energy, and momentum in fractal media.

Let us consider the quantity $\psi(r,t)$ which varies within the fractal volume $V^{(\gamma)}$. Observe that the variations in $\psi(r,t)$ with respect to the fractal time corresponds to the variation in the flux through the fractal boundary $\mathbf{S}^{(\beta)}$ or by a source inside the volume $V^{(\gamma)}$. The integral form of local fractional conservation of the quantity $\psi(r,t)$ is given by [17, 19, 21]

$$\frac{d^\alpha}{dt^\alpha}\iiint_{V^{(\gamma)}}\psi(r,t)\,dV^{(\gamma)}=-\oiint_{S^{(\beta)}}\phi(r,t)\cdot d\mathbf{S}^{(\beta)}+\iiint_{V^{(\gamma)}}H(r,t)\,dV^{(\gamma)}, \tag{9}$$

where $\phi(r,t)=\psi(r,t)v(r,t)$ is the fractal flux vector and $H(r,t)$ is the source (sink) for a nondifferentiable quantity $\psi(r,t)$.

The local fractional surface integral is defined by [17, 19–22]

$$\iint u(r_P)\,d\mathbf{S}^{(\beta)}=\lim_{N\to\infty}\sum_{P=1}^{N}u(r_P)\,\mathbf{n}_P\Delta S_P^{(\beta)}, \tag{10}$$

where $\Delta S_P^{(\beta)}$ is the local fractal surface and N denote elements of the surface with unit normal local fractional vector $\mathbf{n}_P$. When $\Delta S_P^{(\beta)}\to 0$ as $N\to\infty$, the local fractional volume integral of the function $\mathbf{u}$ takes the form [17, 19–23]

$$\iiint\mathbf{u}(r_P)\,dV^{(\gamma)}=\lim_{N\to\infty}\sum_{P=1}^{N}\mathbf{u}(r_P)\,\Delta V_P^{(\gamma)}. \tag{11}$$

The local fractional derivative of a function $f(x)$ of order α is defined by [17, 24, 25]

$$\frac{d^\alpha f(x_0)}{dx^\alpha}=\frac{\Delta^\alpha(f(x)-f(x_0))}{(x-x_0)^\alpha}, \tag{12}$$

with

$$\Delta^\alpha(f(x)-f(x_0))\cong\Gamma(1+\alpha)\left[f(x)-f(x_0)\right]. \tag{13}$$

Using (9), the local fractional differential form of the local fractional conservation balance of the quantity $\psi(r,t)$ can be expressed as

$$\frac{\partial^\alpha\psi(r,t)}{\partial t^\alpha}+\nabla^\alpha\cdot\phi(r,t)=H(r,t). \tag{14}$$

The local fractional gradient of the scale function φ emerging from (14) is [17]

$$\nabla^\alpha\varphi=\lim_{dV^{(\gamma)}\to 0}\left(\frac{1}{dV^{(\gamma)}}\oiint_{S^{(\beta)}}\varphi d\mathbf{S}^{(\beta)}\right). \tag{15}$$

In the Cantorian coordinates, the local fractional conservation equation (14) with respect to $\psi(x,y,z,t)$ can be written as

$$\frac{\partial^\alpha\psi(x,y,z,t)}{\partial t^\alpha}+\nabla^\alpha\cdot\phi(x,y,z,t)=H(x,y,z,t). \tag{16}$$

Alternatively

$$\frac{\partial^\alpha \psi(x,y,z,t)}{\partial t^\alpha} + \frac{\partial^\alpha \phi_1(x,y,z,t)}{\partial x^\alpha} + \frac{\partial^\alpha \phi_2(x,y,z,t)}{\partial y^\alpha} + \frac{\partial^\alpha \phi_3(x,y,z,t)}{\partial z^\alpha} = H(x,y,z,t), \tag{17a}$$

$$\phi(x,y,z,t) = \phi_1(x,y,z,t)\, i^\alpha + \phi_2(x,y,z,t)\, j^\alpha + \phi_3(x,y,z,t)\, k^\alpha. \tag{17b}$$

Notice that the quantity $\phi(x,y,z,t)$ can represent mass, energy, or momentum in fractal media.

If ρ denotes the fractal mass density, then the function $\phi = \rho \boldsymbol{v}$ is the mass fractal flux and $H(x,y,z,t) = 0$. In this case, the local fractional conservation of mass in fractal media reads as

$$\frac{\partial^\alpha \rho}{\partial t^\alpha} + \nabla^\alpha \cdot (\rho \boldsymbol{v}) = 0. \tag{18}$$

In passing e remark that (18) is used to describe fractal physical problems [17, 19, 20].

In the context of the present analysis, the local fractional conservation of energy E in fractal media is

$$\frac{\partial^\alpha E}{\partial t^\alpha} + \nabla^\alpha \cdot (E\boldsymbol{v}) = H(x,y,z,t). \tag{19}$$

The function $\phi = E\boldsymbol{v}$ in (19) is the fractal flux vector of the energy in fractal media. Further, if the function $E = \rho C_\alpha T$ denotes the amount of heat energy per unit fractal volume in fractal media, then the transport flux is

$$\phi = \rho C T \boldsymbol{v}. \tag{20}$$

Thus, the conservation of thermal energy in fractal media can be expressed as

$$\frac{\partial^\alpha \rho C_\alpha T}{\partial t^\alpha} + \nabla^\alpha \cdot (\rho C_\alpha T \boldsymbol{v}) = \nabla^\alpha \cdot k(\nabla^\alpha T) + H(x,y,z,t). \tag{21}$$

As a consequence of (21), the local fractional Fourier law (with fractal thermal conductivity k) reads as

$$F = -k\nabla^\alpha T. \tag{22}$$

For constant C_α and k, (21) can be rewritten as [17, 22]

$$\frac{\partial^\alpha T}{\partial t^\alpha} + \boldsymbol{v} \cdot \nabla^\alpha T = a\nabla^{2\alpha} T + \frac{H(x,y,z,t)}{\rho C_\alpha}, \tag{23}$$

where the fractal thermal diffusivity a is

$$a = \frac{k}{\rho C_\alpha}. \tag{24}$$

The local fractional conservation of momentum in fractal media is

$$\frac{\partial^\alpha M}{\partial t^\alpha} + \nabla^\alpha \cdot (M\boldsymbol{v}) = H(x,y,z,t), \tag{25}$$

where the quantity M represents the momentum in fractal media while the function $\phi = M\boldsymbol{v}$ is the fractal momentum flux vector fractal media.

If the momentum per unit fractal volume is $M = \rho\boldsymbol{v}$, then the sources due to fractal stresses and fractal body forces (gravity generated) are $\nabla^\alpha \cdot \theta$ and $H(x,y,z,t) = \rho g$, respectively. With this terminology, we have

$$\frac{\partial^\alpha \rho\boldsymbol{v}}{\partial t^\alpha} + \nabla^\alpha \cdot (\rho\boldsymbol{v}\boldsymbol{v}) = \nabla^\alpha \cdot \theta + \rho\mathbf{g}. \tag{26}$$

In view of the local fractional conservation of mass (18), (26) takes the form

$$\frac{\partial^\alpha \boldsymbol{v}}{\partial t^\alpha} + (\boldsymbol{v} \cdot \nabla^\alpha)\boldsymbol{v} = \frac{1}{\rho}\nabla^\alpha \cdot \theta + \mathbf{g}. \tag{27}$$

In (26) $(\boldsymbol{v} \cdot \nabla^\alpha)\boldsymbol{v}$ is the nondifferentiable advection of momentum in fractal media.

For compressible fluids, the general form of the Navier-Stokes equation on Cantor sets is [21]

$$\rho\frac{\partial^\alpha \boldsymbol{v}}{\partial t^\alpha} = -\nabla^\alpha p + \frac{1}{3}\mu\nabla^\alpha((\nabla^\alpha \cdot \boldsymbol{v})) + \mu\nabla^{2\alpha}\boldsymbol{v} + \rho\mathbf{b} - \rho\boldsymbol{v}(\nabla^\alpha \cdot \boldsymbol{v}), \tag{28}$$

where $\boldsymbol{v}$ is the fractal fluid velocity, μ is the dynamic viscosity, p is the thermodynamic pressure, and $\mathbf{b}$ denotes the specific fractal body force.

If the term $(1/3)\mu\nabla^\alpha((\nabla^\alpha \cdot \boldsymbol{v})) + \mu\nabla^{2\alpha}\boldsymbol{v}$ is zero, then (28) reduces to

$$\rho\frac{\partial^\alpha \boldsymbol{v}}{\partial t^\alpha} = -\nabla^\alpha p + \rho\mathbf{b} - \rho\boldsymbol{v}(\nabla^\alpha \cdot \boldsymbol{v}), \tag{29}$$

which is known as Cauchy's equation of motion of flows on Cantor sets [21].

For

$$\frac{1}{\rho}\nabla^\alpha \cdot \theta = -\frac{1}{\rho}\nabla^\alpha p, \qquad \mathbf{b} = -\mathbf{g}, \tag{30}$$

the Navier-Stokes equation on Cantor sets for a compressible fluid becomes

$$\frac{\partial^\alpha \boldsymbol{v}}{\partial t^\alpha} + \boldsymbol{v}(\nabla^\alpha \cdot \boldsymbol{v}) = -\frac{1}{\rho}\nabla^\alpha p - \mathbf{g}. \tag{31}$$

2.2. Fractal Water Waves

2.2.1. Linear Theory for Fractal Water Waves. Let us consider the following local fractional conservation equations of fluid motion in fractal media (Cauchy's equation of motion of flows on Cantor sets):

$$\frac{\partial^\alpha \rho}{\partial t^\alpha} + \nabla^\alpha \cdot (\rho\boldsymbol{v}) = 0, \tag{32}$$

$$\frac{\partial^\alpha \boldsymbol{v}}{\partial t^\alpha} + \boldsymbol{v}(\nabla^\alpha \cdot \boldsymbol{v}) = -\frac{1}{\rho}\nabla^\alpha p - \mathbf{g}. \tag{33}$$

If the fractal fluid is incompressible and locally fractional irrotational, then we have

$$\nabla^{\alpha}\rho = 0, \tag{34}$$

$$\frac{\partial^{\alpha}\rho}{\partial t^{\alpha}} = 0, \tag{35}$$

$$\nabla^{\alpha}\times \boldsymbol{v} = 0. \tag{36}$$

From (32) and (37), we have

$$\boldsymbol{v} = \nabla^{\alpha}\varphi, \tag{37}$$

$$\nabla^{2\alpha}\varphi = 0. \tag{38}$$

The local fractional Laplace operator is

$$\nabla^{\alpha}\cdot\nabla^{\alpha} = \nabla^{2\alpha}. \tag{39}$$

We notice that (38) is the local fractional Laplace equation (see [21, 23]).

If the following relationship is valid [21]

$$\boldsymbol{v}\cdot\nabla^{\alpha}\boldsymbol{v} = \nabla^{\alpha}\left(\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{2}\right) - \left(\nabla^{\alpha}\times\boldsymbol{v}\right)\times\boldsymbol{v}. \tag{40}$$

Then, we have

$$\boldsymbol{v}\cdot\nabla^{\alpha}\boldsymbol{v} = \nabla^{\alpha}\left(\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{2}\right). \tag{41}$$

Hence, from (33) and (41), we get

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}\left(\nabla^{\alpha}\varphi\right) + \boldsymbol{v}\left(\nabla^{\alpha}\cdot\boldsymbol{v}\right) = -\frac{1}{\rho}\nabla^{\alpha}p - \mathbf{g}, \tag{42}$$

which leads to

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}\left(\nabla^{\alpha}\varphi\right) + \nabla^{\alpha}\left(\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{2}\right) = -\frac{1}{\rho}\nabla^{\alpha}p - \mathbf{g}. \tag{43}$$

Equation (43) can be rewritten in terms of local fractional gradient as

$$\nabla^{\alpha}\left(\frac{\partial^{\alpha}\varphi}{\partial t^{\alpha}} + \frac{\boldsymbol{v}\cdot\boldsymbol{v}}{2} + \frac{p}{\rho} + \mathbf{g}y\right) = 0, \tag{44}$$

or

$$\nabla^{\alpha}\left(\frac{\partial^{\alpha}\varphi}{\partial t^{\alpha}} + \frac{\nabla^{\alpha}\varphi\cdot\nabla^{\alpha}\varphi}{2} + \frac{p}{\rho} + \mathbf{g}\frac{y^{\alpha}}{\Gamma(1+\alpha)}\right) = 0. \tag{45}$$

From (45), we have

$$\frac{\partial^{\alpha}\varphi}{\partial t^{\alpha}} + \frac{\nabla^{\alpha}\varphi\cdot\nabla^{\alpha}\varphi}{2} + \frac{p-p_0}{\rho} + \mathbf{g}y = n(t), \tag{46}$$

where p_0 is the initial pressure.

Let us suggest that the velocity of the fractal flow normal to the fractal interface can be described as

$$f(x,z,t) = \eta(x,z,t) - y \tag{47}$$

and is equal to the velocity of the fractal interface normal to itself. With these suggestions, we obtain

$$\frac{\partial^{\alpha}\eta}{\partial t^{\alpha}} + \nabla^{\alpha}\varphi\cdot\nabla^{\alpha}\eta = \boldsymbol{v}_y, \tag{48}$$

which is the fractal kinematic equation on the fractal boundary with

$$\boldsymbol{v}_y = \frac{\partial^{\alpha}\varphi}{\partial y^{\alpha}}. \tag{49}$$

When the fractal boundary condition at the free surface is specified, then it follows from (46) and (48) that

$$\frac{\partial^{\alpha}\eta}{\partial t^{\alpha}} + \frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}\frac{\partial^{\alpha}\eta}{\partial x^{\alpha}} + \frac{\partial^{\alpha}\varphi}{\partial z^{\alpha}}\frac{\partial^{\alpha}\eta}{\partial z^{\alpha}} = \frac{\partial^{\alpha}\varphi}{\partial y^{\alpha}}, \tag{50}$$

$$\begin{aligned}\frac{\partial^{\alpha}\varphi}{\partial t^{\alpha}} &+ \frac{1}{2}\left(\frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}\frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}} + \frac{\partial^{\alpha}\varphi}{\partial y^{\alpha}}\frac{\partial^{\alpha}\varphi}{\partial y^{\alpha}} + \frac{\partial^{\alpha}\varphi}{\partial z^{\alpha}}\frac{\partial^{\alpha}\varphi}{\partial z^{\alpha}}\right)\\ &+ \mathbf{g}\eta = 0,\end{aligned} \tag{51}$$

where $p = p_0$, $n(t) = 0$, and $\eta(x,z,t) = y$.

If the bottom section of the flow is considered, then

$$y = -h_0(x,z). \tag{52}$$

Further, if the normal velocity of the flow is zero at the fixed solid boundary, (50) gives

$$\frac{\partial^{\alpha}\varphi}{\partial y^{\alpha}} + \frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}\frac{\partial^{\alpha}h_0}{\partial x^{\alpha}} + \frac{\partial^{\alpha}\varphi}{\partial z^{\alpha}}\frac{\partial^{\alpha}h_0}{\partial z^{\alpha}} = 0. \tag{53}$$

For a horizontal bottom, we have $y = -h_0(x,z)$ which leads to

$$\frac{\partial^{\alpha}\varphi}{\partial y^{\alpha}} = 0, \tag{54}$$

or

$$\frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}\frac{\partial^{\alpha}h_0}{\partial x^{\alpha}} + \frac{\partial^{\alpha}\varphi}{\partial z^{\alpha}}\frac{\partial^{\alpha}h_0}{\partial z^{\alpha}} = 0. \tag{55}$$

Therefore, at the free surface, we have

$$\frac{\partial^{\alpha}\eta}{\partial t^{\alpha}} = \frac{\partial^{\alpha}\varphi}{\partial y^{\alpha}}, \tag{56}$$

$$\frac{\partial^{\alpha}\varphi}{\partial t^{\alpha}} + \mathbf{g}\eta = 0, \tag{57}$$

where $\eta(x,z,t) = y$.

For $y = 0$, we find from (56) and (57) that

$$\frac{\partial^{2\alpha}\varphi}{\partial t^{2\alpha}} + \mathbf{g}\frac{\partial^{\alpha}\varphi}{\partial y^{\alpha}} = 0. \tag{58}$$

Therefore, we define the line problem for a water wave as follows:

$$\begin{aligned}&\frac{\partial^{2\alpha}\varphi}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}\varphi}{\partial y^{2\alpha}} + \frac{\partial^{2\alpha}\varphi}{\partial z^{2\alpha}} = 0, \quad -h_0(x,z) < y < 0;\\ &\frac{\partial^{2\alpha}\varphi}{\partial t^{2\alpha}} + \mathbf{g}\frac{\partial^{\alpha}\varphi}{\partial y^{\alpha}} = 0, \quad y = 0;\\ &\frac{\partial^{\alpha}\varphi}{\partial y^{\alpha}} + \frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}\frac{\partial^{\alpha}h_0}{\partial x^{\alpha}} + \frac{\partial^{\alpha}\varphi}{\partial z^{\alpha}}\frac{\partial^{\alpha}h_0}{\partial z^{\alpha}} = 0, \quad y = -h_0(x,z).\end{aligned} \tag{59}$$

From (57), we may present the fractal surface as

$$\eta(x,z,t) = -\frac{1}{\mathbf{g}}\frac{\partial^{\alpha}\varphi(x,0,z,t)}{\partial t^{\alpha}}. \tag{60}$$

2.2.2. Nonlinear Theory of Fractal Water Waves. The linear wave equation given in [21] is

$$\frac{\partial^{2\alpha}\eta}{\partial t^{2\alpha}} + \varpi\frac{\partial^{2\alpha}\eta}{\partial x^{2\alpha}} = 0, \tag{61}$$

where ϖ is a constant.

From (43), we get

$$\frac{1}{\rho}\nabla^{\alpha}p - \mathbf{g} = 0 \tag{62}$$

or

$$\frac{1}{\rho}\frac{\partial^{\alpha}\eta}{\partial y^{\alpha}}p - \mathbf{g} = 0. \tag{63}$$

Then

$$p - p_0 = \mathbf{g}\rho(\eta - y). \tag{64}$$

From (33) and (64), we have

$$\frac{\partial^{\alpha}\boldsymbol{v}}{\partial t^{\alpha}} + \boldsymbol{v}(\nabla^{\alpha}\cdot\boldsymbol{v}) = -\mathbf{g}\nabla^{\alpha}\eta. \tag{65}$$

Consequently

$$\begin{aligned}
&\frac{\partial^{2\alpha}\varphi}{\partial t^{\alpha}\partial x^{\alpha}} + \frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}\frac{\partial^{2\alpha}\varphi}{\partial x^{2\alpha}} + \frac{\partial^{\alpha}\varphi}{\partial y^{\alpha}}\frac{\partial^{2\alpha}\varphi}{\partial y^{\alpha}\partial x^{\alpha}} \\
&\quad + \frac{\partial^{\alpha}\varphi}{\partial z^{\alpha}}\frac{\partial^{2\alpha}\varphi}{\partial z^{\alpha}\partial x^{\alpha}} = -\mathbf{g}\frac{\partial^{\alpha}\eta}{\partial x^{\alpha}}, \\
&\frac{\partial^{2\alpha}\varphi}{\partial t^{\alpha}\partial z^{\alpha}} + \frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}\frac{\partial^{2\alpha}\varphi}{\partial x^{\alpha}\partial z^{\alpha}} + \frac{\partial^{\alpha}\varphi}{\partial y^{\alpha}}\frac{\partial^{2\alpha}\varphi}{\partial y^{\alpha}\partial z^{\alpha}} \\
&\quad + \frac{\partial^{\alpha}\varphi}{\partial z^{\alpha}}\frac{\partial^{2\alpha}\varphi}{\partial z^{2\alpha}} = -\mathbf{g}\frac{\partial^{\alpha}\eta}{\partial z^{\alpha}}.
\end{aligned} \tag{66}$$

If the conditions $(\partial^{2\alpha}\varphi/\partial y^{\alpha}\partial z^{\alpha}) = 0$ and $(\partial^{2\alpha}\varphi/\partial y^{\alpha}\partial x^{\alpha}) = 0$ are satisfied, then from (66), we get

$$\begin{aligned}
&\frac{\partial^{2\alpha}\varphi}{\partial t^{\alpha}\partial x^{\alpha}} + \frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}\frac{\partial^{2\alpha}\varphi}{\partial x^{2\alpha}} + \frac{\partial^{\alpha}\varphi}{\partial z^{\alpha}}\frac{\partial^{2\alpha}\varphi}{\partial z^{\alpha}\partial x^{\alpha}} = -\mathbf{g}\frac{\partial^{\alpha}\eta}{\partial x^{\alpha}}, \\
&\frac{\partial^{2\alpha}\varphi}{\partial t^{\alpha}\partial z^{\alpha}} + \frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}\frac{\partial^{2\alpha}\varphi}{\partial x^{\alpha}\partial z^{\alpha}} + \frac{\partial^{\alpha}\varphi}{\partial z^{\alpha}}\frac{\partial^{2\alpha}\varphi}{\partial z^{2\alpha}} = -\mathbf{g}\frac{\partial^{\alpha}\eta}{\partial z^{\alpha}}.
\end{aligned} \tag{67}$$

Further, from (32) and (35), we obtain

$$\frac{\partial^{\alpha}h}{\partial t^{\alpha}} + \nabla^{\alpha}\cdot(h\boldsymbol{v}) = 0, \tag{68}$$

where

$$h = h_0 + \eta. \tag{69}$$

Using (51), (68), and (69), we obtain the local fractional conservation equations for one-dimensional waves on the bottom given by

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}\left(\frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}\right) + \frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}\frac{\partial^{2\alpha}\varphi}{\partial x^{2\alpha}} + \mathbf{g}\frac{\partial^{\alpha}h}{\partial x^{\alpha}} = 0, \tag{70}$$

$$\frac{\partial^{\alpha}h}{\partial t^{\alpha}} + \frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(h\frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}\right) = 0, \tag{71}$$

which lead to

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}\left(h\frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}\right) + \frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\left(\frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}\right)^{2}h + \frac{1}{2}\mathbf{g}h^{2}\right) = 0, \tag{72}$$

$$\begin{aligned}
&\frac{\partial^{\alpha}}{\partial t^{\alpha}}\left(\frac{1}{2}\left(\frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}\right)^{2}h + \frac{1}{2}\mathbf{g}h^{2}\right) \\
&\quad + \frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{1}{2}\left(\frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}\right)^{3}h + \frac{1}{2}\mathbf{g}h^{2}\frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}\right) = 0.
\end{aligned} \tag{73}$$

Furthermore, from (38), (50), (51), and (59), we get

$$\frac{\partial^{2\alpha}\varphi}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}\varphi}{\partial y^{2\alpha}} = 0, \quad -\infty < x < +\infty,\ 0 \le y \le \eta(x,t), \tag{74}$$

$$\frac{\partial^{\alpha}\eta}{\partial t^{\alpha}} + \frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}\frac{\partial^{\alpha}\eta}{\partial x^{\alpha}} = \frac{\partial^{\alpha}\varphi}{\partial y^{\alpha}}, \quad y = 1 + \varsigma\eta, \tag{75}$$

$$\frac{\partial^{\alpha}\varphi}{\partial t^{\alpha}} + \frac{1}{2}\left(\frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}\frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}} + \frac{\partial^{\alpha}\varphi}{\partial y^{\alpha}}\frac{\partial^{\alpha}\varphi}{\partial y^{\alpha}}\right) + \mathbf{g}\eta = 0, \tag{76}$$

with

$$\frac{\partial^{\alpha}\varphi}{\partial y^{\alpha}} = 0, \quad y = 0. \tag{77}$$

3. Local Fractional Korteweg-de Vries Equation

Using (74) and (76), it is possible to expand the fractal velocity potential into a nondifferentiable series with respect to y in the following form:

$$\varphi(x,y,t) = \sum_{i=0}^{\infty}\frac{y^{n\alpha}}{\Gamma(1+n\alpha)}\varphi_n(x,t). \tag{78}$$

Then, it follows from (74) and (78) that

$$\frac{\partial^{2\alpha}\varphi}{\partial x^{2\alpha}} = \sum_{i=0}^{\infty}\frac{y^{n\alpha}}{\Gamma(1+n\alpha)}\frac{\partial^{2\alpha}\varphi_n(x,t)}{\partial x^{2\alpha}}, \tag{79}$$

$$\begin{aligned}
\frac{\partial^{2\alpha}\varphi}{\partial y^{2\alpha}} &= \sum_{i=2}^{\infty}\frac{y^{(n-2)\alpha}}{\Gamma(1+(n-2)\alpha)}\varphi_n(x,t) \\
&= \sum_{i=0}^{\infty}\Gamma(1+(n+2)\alpha)\,y^{n\alpha}\varphi_{n+2}(x,t).
\end{aligned} \tag{80}$$

Hence,

$$\frac{\partial^{2\alpha}\varphi_n(x,t)}{\partial x^{2\alpha}}+\frac{\Gamma(1+(n+2)\alpha)}{\Gamma(1+n\alpha)}\varphi_{n+2}(x,t)=0,\quad \forall n\in N. \tag{81}$$

Thus, from (77), we get

$$\begin{aligned}\frac{\partial^{\alpha}\varphi}{\partial y^{\alpha}}&=\sum_{i=1}^{\infty}\frac{y^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)}\varphi_n(x,t)\\&=\varphi_1(x,t)=0.\end{aligned} \tag{82}$$

Equations (70) and (73) lead to

$$\begin{aligned}
n=0&:\frac{\partial^{2\alpha}\varphi_0(x,t)}{\partial x^{2\alpha}}+\frac{\Gamma(1+(n+2)\alpha)}{\Gamma(1+n\alpha)}\varphi_2(x,t)=0,\\
n=1&:\frac{\partial^{2\alpha}\varphi_1(x,t)}{\partial x^{2\alpha}}+\frac{\Gamma(1+(n+2)\alpha)}{\Gamma(1+n\alpha)}\varphi_3(x,t)=0,\\
n=2&:\frac{\partial^{2\alpha}\varphi_2(x,t)}{\partial x^{2\alpha}}+\frac{\Gamma(1+(n+2)\alpha)}{\Gamma(1+n\alpha)}\varphi_4(x,t)=0,\\
n=3&:\frac{\partial^{2\alpha}\varphi_3(x,t)}{\partial x^{2\alpha}}+\frac{\Gamma(1+(n+2)\alpha)}{\Gamma(1+n\alpha)}\varphi_5(x,t)=0,\\
n=4&:\frac{\partial^{2\alpha}\varphi_4(x,t)}{\partial x^{2\alpha}}+\frac{\Gamma(1+(n+2)\alpha)}{\Gamma(1+n\alpha)}\varphi_6(x,t)=0,\\
&\vdots\\
n=k&:\frac{\partial^{2\alpha}\varphi_k(x,t)}{\partial x^{2\alpha}}+\frac{\Gamma(1+(k+2)\alpha)}{\Gamma(1+k\alpha)}\varphi_{k+2}(x,t)=0.
\end{aligned} \tag{83}$$

Hence, we get

$$\varphi(x,y,t)=\sum_{i=0}^{\infty}\frac{y^{2i\alpha}}{\Gamma(1+2i\alpha)}(-1)^i\frac{\partial^{2i\alpha}}{\partial x^{2i\alpha}}f(\varphi_0), \tag{84}$$

where

$$f(\varphi_0)=\varphi_0(x,t). \tag{85}$$

Similarly, from (84) and (85), we can get

$$\begin{aligned}
\frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}&=\frac{\partial^{\alpha}}{\partial x^{\alpha}}f(\varphi_0)-\frac{y^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{3\alpha}}{\partial x^{3\alpha}}f(\varphi_0)+O\left(x^{2\alpha}\right),\\
\frac{\partial^{\alpha}\varphi}{\partial y^{\alpha}}&=-\frac{y^{\alpha}}{\Gamma(1+\alpha)}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}f(\varphi_0)+\frac{y^{3\alpha}}{\Gamma(1+3\alpha)}\frac{\partial^{4\alpha}}{\partial x^{4\alpha}}f(\varphi_0)\\
&\quad+O\left(x^{2\alpha}\right).
\end{aligned} \tag{86}$$

In order to obtain a dimensionless form of (74)–(77), we make the following scale transformations:

$$\begin{aligned}
x&\longrightarrow x\lambda,\\
y&\longrightarrow hy,\\
\varphi&\longrightarrow\frac{\varsigma\lambda^{\alpha}c_0^{\alpha}\varphi}{h^{\alpha}},\\
t&\longrightarrow\frac{t\lambda}{c_0},\\
\eta&\longrightarrow\varsigma\eta,
\end{aligned} \tag{87}$$

so that

$$\varepsilon^{2\alpha}\frac{\partial^{2\alpha}\varphi}{\partial x^{2\alpha}}+\frac{\partial^{2\alpha}\varphi}{\partial y^{2\alpha}}=0,\quad 0\le y\le 1+a\eta, \tag{88}$$

$$\frac{\partial^{\alpha}\eta}{\partial t^{\alpha}}+a\frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}\frac{\partial^{\alpha}\eta}{\partial x^{\alpha}}-\frac{1}{\varepsilon^{2\alpha}}\frac{\partial^{\alpha}\varphi}{\partial y^{\alpha}}=0,\quad y=1+a\eta, \tag{89}$$

$$\begin{aligned}\frac{\partial^{\alpha}\varphi}{\partial t^{\alpha}}+\frac{a}{2}\frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}\frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}+\frac{1}{2}\frac{a}{\varepsilon^{2\alpha}}\frac{\partial^{\alpha}\varphi}{\partial y^{\alpha}}\frac{\partial^{\alpha}\varphi}{\partial y^{\alpha}}+\eta=0,\\ y=1+a\eta,\end{aligned} \tag{90}$$

$$\frac{\partial^{\alpha}\varphi}{\partial y^{\alpha}}=0,\quad y=0. \tag{91}$$

In this context, the equation for the free water surface is

$$y=1+a\eta, \tag{92}$$

Here $c_0=\sqrt{gh}$ is the linear wave velocity in shallow water. The two small parameters are $a=\varsigma/h^{\alpha}$ and $\varepsilon=h/\lambda$ with depth of the water h, while ς and λ are the typical height and length of the solitary wave, respectively

Equations (84), (88), and (91) allow developing a nondifferentiable series with respect to ε in the form

$$\varphi(x,\eta,t)=\sum_{i=0}^{\infty}\frac{y^{2i\alpha}}{\Gamma(1+2i\alpha)}(-1)^i\frac{\partial^{2i\alpha}}{\partial x^{2i\alpha}}f(\varphi_0)\varepsilon^{2i\alpha}. \tag{93}$$

In view of (84), (89), and (90), we have

$$\begin{aligned}\varphi&=f(\varphi_0)-\frac{y^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}f(\varphi_0)\varepsilon^{2\alpha}\\&\quad+\frac{y^{4\alpha}}{\Gamma(1+4\alpha)}\frac{\partial^{4\alpha}}{\partial x^{4\alpha}}f(\varphi_0)\varepsilon^{4\alpha}+O\left(\varepsilon^{4\alpha}\right),\end{aligned} \tag{94}$$

so that

$$\begin{aligned}\frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}&=\frac{\partial^{\alpha}f(\varphi_0)}{\partial x^{\alpha}}-\frac{(1+a\eta)^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{3\alpha}}{\partial x^{3\alpha}}f(\varphi_0)\varepsilon^{2\alpha}\\&\quad+\frac{(1+a\eta)^{4\alpha}}{\Gamma(1+4\alpha)}\frac{\partial^{5\alpha}}{\partial x^{5\alpha}}f(\varphi_0)\varepsilon^{4\alpha}+O\left(\varepsilon^{4\alpha}\right),\end{aligned} \tag{95}$$

$$\frac{\partial^\alpha \varphi}{\partial y^\alpha} = -\frac{(1+a\eta)^\alpha}{\Gamma(1+\alpha)}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}} f(\varphi_0)\varepsilon^{2\alpha} + \frac{(1+a\eta)^{3\alpha}}{\Gamma(1+3\alpha)}\frac{\partial^{4\alpha}}{\partial x^{4\alpha}} f(\varphi_0)\varepsilon^{4\alpha} + O\left(\varepsilon^{4\alpha}\right), \tag{96}$$

which leads to

$$\begin{aligned}
&\frac{\partial^\alpha \eta}{\partial t^\alpha} + a\left(\frac{\partial^\alpha f(\varphi_0)}{\partial x^\alpha} - \frac{(1+a\eta)^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{3\alpha}}{\partial x^{3\alpha}} f(\varphi_0)\varepsilon^{2\alpha} + \frac{(1+a\eta)^{4\alpha}}{\Gamma(1+4\alpha)}\frac{\partial^{5\alpha}}{\partial x^{5\alpha}} f(\varphi_0)\varepsilon^{4\alpha}\right)\frac{\partial^\alpha \eta}{\partial x^\alpha} \\
&- \frac{1}{\varepsilon^{2\alpha}}\left(-\frac{(1+a\eta)^\alpha}{\Gamma(1+\alpha)}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}} f(\varphi_0)\varepsilon^{2\alpha} + \frac{(1+a\eta)^{3\alpha}}{\Gamma(1+3\alpha)}\frac{\partial^{4\alpha}}{\partial x^{4\alpha}} f(\varphi_0)\varepsilon^{4\alpha}\right) \\
&+ O\left(\varepsilon^{4\alpha}\right) = 0,
\end{aligned} \tag{97}$$

$$\begin{aligned}
&\frac{\partial^\alpha f(\varphi_0)}{\partial t^\alpha} - \frac{(1+a\eta)^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\frac{\partial^\alpha f(\varphi_0)}{\partial t^\alpha}\varepsilon^{2\alpha} \\
&+ \frac{(1+a\eta)^{4\alpha}}{\Gamma(1+4\alpha)}\frac{\partial^{4\alpha}}{\partial x^{4\alpha}}\frac{\partial^\alpha}{\partial t^\alpha} f(\varphi_0)\varepsilon^{4\alpha} \\
&+ \frac{a}{2}\left(\frac{\partial^\alpha f(\varphi_0)}{\partial x^\alpha} + \frac{(1+a\eta)^{4\alpha}}{\Gamma(1+4\alpha)}\frac{\partial^{5\alpha}}{\partial x^{5\alpha}} f(\varphi_0)\varepsilon^{4\alpha} - \frac{(1+a\eta)^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{3\alpha}}{\partial x^{3\alpha}} f(\varphi_0)\varepsilon^{2\alpha}\right)^2 \\
&+ \frac{1}{2}\frac{a}{\varepsilon^{2\alpha}}\left(-\frac{(1+a\eta)^\alpha}{\Gamma(1+\alpha)}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}} f(\varphi_0)\varepsilon^{2\alpha} + \frac{(1+a\eta)^{3\alpha}}{\Gamma(1+3\alpha)}\frac{\partial^{4\alpha}}{\partial x^{4\alpha}} f(\varphi_0)\varepsilon^{4\alpha}\right)^2 \\
&+ \eta + O\left(\varepsilon^{4\alpha}\right) = 0.
\end{aligned} \tag{98}$$

Hence, from (97) and (98), we obtain

$$\begin{aligned}
&\frac{\partial^\alpha \eta}{\partial t^\alpha} + \left(a\frac{\partial^\alpha f(\varphi_0)}{\partial x^\alpha}\frac{\partial^\alpha \eta}{\partial x^\alpha} + \frac{(1+a\eta)^\alpha}{\Gamma(1+\alpha)}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}} f(\varphi_0)\right) \\
&- \left\{\frac{(1+a\eta)^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{3\alpha}}{\partial x^{3\alpha}} f(\varphi_0)\, a\frac{\partial^\alpha \eta}{\partial x^\alpha} + \frac{(1+a\eta)^{3\alpha}}{\Gamma(1+3\alpha)}\frac{\partial^{4\alpha}}{\partial x^{4\alpha}} f(\varphi_0)\right\}\varepsilon^{2\alpha} \\
&+ \frac{(1+a\eta)^{4\alpha}}{\Gamma(1+4\alpha)}\frac{\partial^{5\alpha}}{\partial x^{5\alpha}} f(\varphi_0)\frac{\partial^\alpha \eta}{\partial x^\alpha} a\varepsilon^{4\alpha} + O\left(\varepsilon^{4\alpha}\right) = 0,
\end{aligned}$$

$$\begin{aligned}
&\left\{\frac{\partial^\alpha f(\varphi_0)}{\partial t^\alpha} + \frac{a}{2}\left(\frac{\partial^\alpha f(\varphi_0)}{\partial x^\alpha}\right)^2\right\} \\
&- \left\{\frac{(1+a\eta)^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\frac{\partial^\alpha f(\varphi_0)}{\partial t^\alpha} + a\frac{(1+a\eta)^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^\alpha f(\varphi_0)}{\partial x^\alpha}\frac{\partial^{3\alpha} f(\varphi_0)}{\partial x^{3\alpha}} - \frac{a}{2}\left(\frac{(1+a\eta)^\alpha}{\Gamma(1+\alpha)}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}} f(\varphi_0)\right)^2\right\}\varepsilon^{2\alpha} \\
&+ \left\{\frac{(1+a\eta)^{4\alpha}}{\Gamma(1+4\alpha)}\frac{\partial^{4\alpha}}{\partial x^{4\alpha}}\frac{\partial^\alpha}{\partial t^\alpha} f(\varphi_0) + a\frac{\partial^\alpha f(\varphi_0)}{\partial x^\alpha}\frac{(1+a\eta)^{4\alpha}}{\Gamma(1+4\alpha)}\frac{\partial^{5\alpha}}{\partial x^{5\alpha}} f(\varphi_0) + \frac{a}{2}\left(\frac{(1+a\eta)^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{3\alpha}}{\partial x^{3\alpha}} f(\varphi_0)\right)^2 - a\frac{(1+a\eta)^\alpha}{\Gamma(1+\alpha)}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\right. \\
&\left.\times f(\varphi_0)\frac{(1+a\eta)^{3\alpha}}{\Gamma(1+3\alpha)}\frac{\partial^{4\alpha}}{\partial x^{4\alpha}} f(\varphi_0)\right\}\varepsilon^{4\alpha} \\
&+ \eta + O\left(\varepsilon^{4\alpha}\right) = 0.
\end{aligned} \tag{99}$$

To this end, let us consider the following relations:

$$\begin{aligned}
\varepsilon &= \frac{h}{\lambda} \ll 1, \\
a &= \frac{\varsigma}{h^\alpha} \ll 1.
\end{aligned} \tag{100}$$

Then, from (99) and (103), we have

$$\begin{aligned}
&\frac{\partial^\alpha \eta}{\partial t^\alpha} + \left(a\frac{\partial^\alpha f(\varphi_0)}{\partial x^\alpha}\frac{\partial^\alpha \eta}{\partial x^\alpha} + \frac{(1+a\eta)^\alpha}{\Gamma(1+\alpha)}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}} f(\varphi_0)\right) \\
&- \left\{\frac{(1+a\eta)^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{3\alpha}}{\partial x^{3\alpha}} f(\varphi_0)\, a\frac{\partial^\alpha \eta}{\partial x^\alpha} + \frac{(1+a\eta)^{3\alpha}}{\Gamma(1+3\alpha)}\frac{\partial^{4\alpha}}{\partial x^{4\alpha}} f(\varphi_0)\right\}\varepsilon^{2\alpha} + O\left(\varepsilon^{4\alpha}\right) = 0,
\end{aligned}$$

$$\frac{\partial^\alpha f(\varphi_0)}{\partial t^\alpha} + \frac{a}{2}\left(\frac{\partial^\alpha f(\varphi_0)}{\partial x^\alpha}\right)^2 - \frac{(1+a\eta)^{2\alpha}}{\Gamma(1+2\alpha)}$$

$$\times\left\{\frac{\partial^{2\alpha}}{\partial x^{3\alpha}}\frac{\partial^{\alpha}f(\varphi_0)}{\partial t^{\alpha}}+a\frac{\partial^{\alpha}f(\varphi_0)}{\partial x^{\alpha}}\frac{\partial^{3\alpha}f(\varphi_0)}{\partial x^{3\alpha}}\right.$$
$$\left.-\frac{a}{2}\frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}\left(\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}f(\varphi_0)\right)^2\right\}\varepsilon^{2\alpha}$$
$$+\eta+O\left(\varepsilon^{4\alpha}\right)=0. \tag{101}$$

From (101), we get

$$\frac{\partial^{\alpha}\eta}{\partial t^{\alpha}}+a\frac{\partial^{\alpha}f(\varphi_0)}{\partial x^{\alpha}}\frac{\partial^{\alpha}\eta}{\partial x^{\alpha}}+\frac{(1+a\eta)^{\alpha}}{\Gamma(1+\alpha)}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}f(\varphi_0)=0,$$
$$\frac{\partial^{\alpha}f(\varphi_0)}{\partial t^{\alpha}}+\frac{a}{2}\left(\frac{\partial^{\alpha}f(\varphi_0)}{\partial x^{\alpha}}\right)^2+\eta=0 \tag{102}$$

which yield

$$\frac{\partial^{\alpha}\eta}{\partial t^{\alpha}}+a\sigma\frac{\partial^{\alpha}\eta}{\partial x^{\alpha}}+\frac{(1+a\eta)^{\alpha}}{\Gamma(1+\alpha)}\frac{\partial^{\alpha}\sigma}{\partial x^{\alpha}}=0,$$
$$\frac{\partial^{\alpha}\sigma}{\partial t^{\alpha}}+a\sigma\frac{\partial^{\alpha}\sigma}{\partial x^{\alpha}}+\frac{\partial^{\alpha}\eta}{\partial x^{\alpha}}=0, \tag{103}$$

where

$$\sigma=\frac{\partial^{\alpha}f(\varphi_0)}{\partial x^{\alpha}}. \tag{104}$$

If the terms of $O(a\varepsilon^{2\alpha},\varepsilon^{4\alpha})$ are omitted, then from (101), we obtain

$$\frac{\partial^{\alpha}\eta}{\partial t^{\alpha}}+a\sigma\frac{\partial^{\alpha}\eta}{\partial x^{\alpha}}+\frac{(1+a\eta)^{\alpha}}{\Gamma(1+\alpha)}\frac{\partial^{\alpha}\sigma}{\partial x^{\alpha}}-\frac{(1+a\eta)^{3\alpha}}{\Gamma(1+3\alpha)}\frac{\partial^{3\alpha}}{\partial x^{3\alpha}}\sigma\varepsilon^{2\alpha}$$
$$+O\left(a\varepsilon^{2\alpha},\varepsilon^{4\alpha}\right)=0, \tag{105}$$

$$\frac{\partial^{\alpha}\sigma}{\partial t^{\alpha}}+a\sigma\frac{\partial^{\alpha}\sigma}{\partial x^{\alpha}}-\frac{(1+a\eta)^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\frac{\partial^{\alpha}\sigma}{\partial t^{\alpha}}+\frac{\partial^{\alpha}\eta}{\partial x^{\alpha}}$$
$$+O\left(a\varepsilon^{2\alpha},\varepsilon^{4\alpha}\right)=0. \tag{106}$$

From (93) and (95), we have

$$\frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}=\sigma-\frac{1}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\sigma\varepsilon^{2\alpha}+O\left(\varepsilon^{4\alpha}\right),$$
$$\omega=\sigma-\frac{1}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\sigma\varepsilon^{2\alpha}+O\left(a\varepsilon^{2\alpha},\varepsilon^{4\alpha}\right), \tag{107}$$

which can alternatively be written as

$$\sigma=\omega+\frac{1}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}\omega}{\partial x^{2\alpha}}\varepsilon^{2\alpha}+O\left(a\varepsilon^{2\alpha},\varepsilon^{4\alpha}\right). \tag{108}$$

Substituting (108) in (97) and (106), we get

$$\frac{\partial^{\alpha}\eta}{\partial t^{\alpha}}+a\omega\frac{\partial^{\alpha}\eta}{\partial x^{\alpha}}+\frac{1}{\Gamma(1+\alpha)}\frac{\partial^{\alpha}\omega}{\partial x^{\alpha}}+O\left(a\varepsilon^{2\alpha},\varepsilon^{4\alpha}\right)=0,$$
$$\frac{\partial^{\alpha}\omega}{\partial t^{\alpha}}+a\omega\frac{\partial^{\alpha}\omega}{\partial x^{\alpha}}-\frac{1}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\frac{\partial^{\alpha}\omega}{\partial t^{\alpha}}\varepsilon^{2\alpha}$$
$$+\frac{\partial^{\alpha}\eta}{\partial x^{\alpha}}+O\left(a\varepsilon^{2\alpha},\varepsilon^{4\alpha}\right)=0, \tag{109}$$

where $\varepsilon=h/\lambda\ll 1$ and $a=\varsigma/h^{\alpha}\ll 1$.

In view of (109), we obtain

$$\frac{\partial^{\alpha}\eta}{\partial t^{\alpha}}+\frac{1}{\Gamma(1+\alpha)}\frac{\partial^{\alpha}\omega}{\partial x^{\alpha}}=0,$$
$$\frac{\partial^{\alpha}\omega}{\partial t^{\alpha}}+\frac{\partial^{\alpha}\eta}{\partial x^{\alpha}}=0, \tag{110}$$

which result into the form

$$\frac{\partial^{2\alpha}\omega}{\partial t^{2\alpha}}=\frac{1}{\Gamma(1+\alpha)}\frac{\partial^{2\alpha}\eta}{\partial x^{2\alpha}}. \tag{111}$$

We notice that (111) is the linear local fractional wave equation for water waves when $\omega=\eta$.

We may also transform (110) into the following forms:

$$\frac{\partial^{\alpha}\eta}{\partial t^{\alpha}}+\frac{\partial^{\alpha}\omega}{\partial x^{\alpha}}=0,$$
$$\frac{\partial^{\alpha}\omega}{\partial t^{\alpha}}+\frac{\partial^{\alpha}\eta}{\partial x^{\alpha}}=0, \tag{112}$$

which yield

$$\frac{\partial^{\alpha}\eta}{\partial t^{\alpha}}+\frac{\partial^{\alpha}\eta}{\partial x^{\alpha}}=0, \tag{113}$$

where $\omega=\eta$ and $\alpha\to 1$.

For developing the problem further, by (113), (105), and (106) can be expressed as

$$\frac{\partial^{\alpha}\eta}{\partial t^{\alpha}}+a\sigma\frac{\partial^{\alpha}\eta}{\partial x^{\alpha}}+\frac{\partial^{\alpha}\sigma}{\partial x^{\alpha}}-\frac{1}{\Gamma(1+3\alpha)}\frac{\partial^{3\alpha}}{\partial x^{3\alpha}}\sigma\varepsilon^{2\alpha}$$
$$+O\left(a\varepsilon^{2\alpha},\varepsilon^{4\alpha}\right)=0,$$
$$\frac{\partial^{\alpha}\sigma}{\partial t^{\alpha}}+a\sigma\frac{\partial^{\alpha}\sigma}{\partial x^{\alpha}}-\frac{1}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\frac{\partial^{\alpha}\sigma}{\partial t^{\alpha}}+\frac{\partial^{\alpha}\eta}{\partial x^{\alpha}}$$
$$+O\left(a\varepsilon^{2\alpha},\varepsilon^{4\alpha}\right)=0 \tag{114}$$

which lead to

$$\frac{\partial^{\alpha}\eta}{\partial t^{\alpha}}+a\sigma\frac{\partial^{\alpha}\eta}{\partial x^{\alpha}}+\frac{\partial^{\alpha}\sigma}{\partial x^{\alpha}}-\frac{1}{\Gamma(1+3\alpha)}\frac{\partial^{3\alpha}}{\partial x^{3\alpha}}\sigma\varepsilon^{2\alpha}=0, \tag{115}$$

$$\frac{\partial^{\alpha}\sigma}{\partial t^{\alpha}}+a\sigma\frac{\partial^{\alpha}\sigma}{\partial x^{\alpha}}-\frac{1}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\frac{\partial^{\alpha}\sigma}{\partial t^{\alpha}}+\frac{\partial^{\alpha}\eta}{\partial x^{\alpha}}=0, \tag{116}$$

where $\varepsilon=h/\lambda\ll 1$ and $a=\varsigma/h^{\alpha}\ll 1$.

In view of (108), (115), and (116), we obtain

$$\frac{\partial^\alpha \eta}{\partial t^\alpha} + a\omega \frac{\partial^\alpha \eta}{\partial x^\alpha} + \frac{\partial^\alpha \omega}{\partial x^\alpha} = 0,$$
$$\frac{\partial^\alpha \omega}{\partial t^\alpha} + a\omega \frac{\partial^\alpha \omega}{\partial x^\alpha} - \frac{1}{\Gamma(1+2\alpha)} \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} \frac{\partial^\alpha \omega}{\partial t^\alpha} \varepsilon^{2\alpha} + \frac{\partial^\alpha \eta}{\partial x^\alpha} = 0. \tag{117}$$

Now, from (113), we may look for a solution of the form

$$\sigma = \eta + aA + \varepsilon^{2\alpha} B + O\left(a^2 + \varepsilon^{4\alpha}\right), \tag{118}$$

where A and B are functions of η and its local fractional derivatives.

Utilizing (115), (116), and (118), we obtain

$$\frac{\partial^\alpha \eta}{\partial t^\alpha} + a\left(\frac{\partial^\alpha A}{\partial x^\alpha} + \eta \frac{\partial^\alpha \eta}{\partial x^\alpha}\right) + \varepsilon^{2\alpha}\left(\frac{\partial^\alpha B}{\partial x^\alpha} - \frac{1}{\Gamma(1+3\alpha)} \frac{\partial^{3\alpha}}{\partial x^{3\alpha}} \eta\right) + \frac{\partial^\alpha \eta}{\partial x^\alpha} + O\left(a^2 + \varepsilon^{4\alpha}\right) = 0, \tag{119}$$

$$\frac{\partial^\alpha \eta}{\partial t^\alpha} + a\left(\frac{\partial^\alpha A}{\partial t^\alpha} + \eta \frac{\partial^\alpha \eta}{\partial t^\alpha}\right) + \varepsilon^{2\alpha}\left(\frac{\partial^\alpha B}{\partial t^\alpha} - \frac{1}{\Gamma(1+2\alpha)} \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} \frac{\partial^\alpha \sigma}{\partial t^\alpha}\right) + \frac{\partial^\alpha \eta}{\partial x^\alpha} + O\left(a^2 + \varepsilon^{4\alpha}\right) = 0, \tag{120}$$

which lead to

$$\frac{\partial^\alpha \eta}{\partial t^\alpha} + a\left(\frac{\partial^\alpha A}{\partial x^\alpha} + \eta \frac{\partial^\alpha \eta}{\partial x^\alpha}\right) + \varepsilon^{2\alpha}\left(\frac{\partial^\alpha B}{\partial x^\alpha} - \frac{1}{\Gamma(1+3\alpha)} \frac{\partial^{3\alpha} \eta}{\partial x^{3\alpha}}\right) + \frac{\partial^\alpha \eta}{\partial x^\alpha} = 0,$$
$$\frac{\partial^\alpha \eta}{\partial t^\alpha} + a\left(\frac{\partial^\alpha A}{\partial t^\alpha} + \eta \frac{\partial^\alpha \eta}{\partial t^\alpha}\right) + \varepsilon^{2\alpha}\left(\frac{\partial^\alpha B}{\partial t^\alpha} - \frac{1}{\Gamma(1+2\alpha)} \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} \frac{\partial^\alpha \sigma}{\partial t^\alpha}\right) + \frac{\partial^\alpha \eta}{\partial x^\alpha} = 0. \tag{121}$$

Moreover, from (121), we have

$$A = -\chi \eta^{2\alpha},$$
$$B = \vartheta \frac{\partial^{2\alpha} \eta}{\partial x^{2\alpha}}, \tag{122}$$

such that (119) and (120) are valid with χ and ϑ as constants.

Therefore, from (115), (119), and (122), we have

$$\sigma = \eta - a\chi\eta^{2\alpha} + \varepsilon^{2\alpha} \vartheta \frac{\partial^{2\alpha} \eta}{\partial x^{2\alpha}} + O\left(a^2 + \varepsilon^{4\alpha}\right), \tag{123}$$

such that

$$\frac{\partial^\alpha \eta}{\partial t^\alpha} - aM\eta \frac{\partial^\alpha \eta}{\partial x^\alpha} + \varepsilon^{2\alpha} N \frac{\partial^{3\alpha} \eta}{\partial x^{3\alpha}} + \frac{\partial^\alpha \eta}{\partial x^\alpha} = 0, \tag{124}$$

where

$$M = 2\chi - 1 \neq 0,$$
$$N = \vartheta - \frac{1}{\Gamma(1+3\alpha)} \neq 0. \tag{125}$$

From (124), we arrive at the following formula:

$$\frac{\partial^\alpha \eta}{\partial t^\alpha} - R\eta \frac{\partial^\alpha \eta}{\partial x^\alpha} + S \frac{\partial^{3\alpha} \eta}{\partial x^{3\alpha}} + \frac{\partial^\alpha \eta}{\partial x^\alpha} = 0, \tag{126}$$

which leads to

$$\frac{\partial^\alpha \eta}{\partial t^\alpha} - \eta \frac{\partial^\alpha \eta}{\partial x^\alpha} + \frac{\partial^{3\alpha} \eta}{\partial x^{3\alpha}} + \frac{\partial^\alpha \eta}{\partial x^\alpha} = 0, \tag{127}$$

or (neglecting the low term)

$$\frac{\partial^\alpha \eta}{\partial t^\alpha} - R\eta \frac{\partial^\alpha \eta}{\partial x^\alpha} + S \frac{\partial^{3\alpha} \eta}{\partial x^{3\alpha}} = 0, \tag{128}$$

where $\eta(x,t)$ is a nondifferentiable function, $R = aM$ and $S = \varepsilon^{2\alpha} N$.

We notice that (126) is the local fractional Korteweg-de Vries equation. When there are coefficient relations, namely, $R = 1$ and $S = 1$, we obtain a new local fractional Korteweg-de Vries equation. When neglecting the nonlinear term of (127), we obtain the linear local fractional Korteweg-de Vries equation as follows:

$$\frac{\partial^\alpha \eta}{\partial t^\alpha} + \frac{\partial^{3\alpha} \eta}{\partial x^{3\alpha}} + \frac{\partial^\alpha \eta}{\partial x^\alpha} = 0, \tag{129}$$

where $\eta(x,t)$ is a nondifferentiable function.

4. Conclusions

In this work, we have derived the local fractional Korteweg-de Vries equation related to fractal waves on shallow water surfaces from the local fractional calculus view point. The linear and nonlinear theories for fractal water wave are presented and the linear and nonlinear local fractional Korteweg-de Vries equations are also obtained.

Conflict of Interests

The authors declare that they have no conflict of interests regarding this paper.

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