The Existence of Solutions to the Nonhomogeneous $A$-Harmonic Equations with Variable Exponent

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We first discuss the existence and uniqueness of weak solution for the obstacle problem of the nonhomogeneous $A$-harmonic equation with variable exponent, and then we obtain the existence of the solutions of the equation $d^*A(x,d\omega) = B(x,d\omega)$ in the weighted variable exponent Sobolev space $W_0^{p(x)}(\Omega, \Lambda^I, \mu)$.

1. Introduction

In [1–5], the nonhomogeneous $A$-harmonic equation $d^*A(x,d\omega) = B(x,d\omega)$ for differential forms has received much investigation. In [6], the obstacle problem of the $A$-harmonic equation for differential forms has been discussed. However, most of these results are developed in the $L^p(\Omega, \Lambda^I)$ space or $W_1^{1,p}(\Omega, \Lambda^I)$ space. Meanwhile, in the past few years the subject of variable exponent space has undergone a vast development; see [7–11]. For example, [8–10] discuss the weighted $L^{p(x)}$ and $W^{k,p(x)}$ spaces and the weak solution for obstacle problem with variable growth has been studied in [10, 11].

In this paper, we are interested in the following obstacle problem:

$$\int_{\Omega} (A(x,du) \cdot d(v-u) + B(x,du) \cdot (v-u)) \, dx \geq 0$$

for $v$ belonging to

$$\mathcal{K}_{\psi,\theta} = \{ v \in W_0^{p(x)}(\Omega, \Lambda^I, \mu) : v \geq \psi, \text{ a.e. } x \in \Omega, v - \theta \in W_0^{p(x)}(\Omega, \Lambda^I, \mu) \},$$

where $\psi(x) = \sum n_i(x)dx_i \in \Lambda^I(\mathbb{R}^n)$, $\psi(x) = \sum v_i(x)dx_i \in \Lambda^I(\mathbb{R}^n)$, $\psi_i, \Omega_i : \Omega \rightarrow [-\infty, +\infty]$; $v \geq \psi$, a.e. $x \in \Omega$ means that, for any $I$, we have $v_I \geq \psi_I$, a.e. $x \in \Omega$; $\theta \in W_0^{p(x)}(\Omega, \Lambda^I, \mu), l = 0, 1, \ldots, n - 1$, and the variable exponent $p(x) \in \mathcal{P}(\Omega)$ satisfies

$$1 < p^- \leq p(x) \leq p^+ < \infty \text{ for a.e. } x \in \Omega.$$ (3)

The operators $A(x,\xi) : \Omega \times \Lambda^I(\mathbb{R}^n) \rightarrow \Lambda^I(\mathbb{R})$ and $B(x,\xi) : \Omega \times \Lambda^I(\mathbb{R}^n) \rightarrow \Lambda^{I-1}(\mathbb{R})$ satisfy the following growth conditions on a bounded domain $\Omega$:

(H1) $A(x,\xi)$ and $B(x,\xi)$ are measurable for all $\xi$ with respect to $x$ and continuous for a.e. $x \in \Omega$ with respect to $\xi$,

(H2) $|A(x,\xi)| \leq C_1 w(x)|\xi|^{p(x)-1}$,

(H3) $A(x,\xi) \cdot \xi \geq C_2 w(x)|\xi|^{p(x)}$,

(H4) $|B(x,\xi)| \leq C_3 w(x)|\xi|^{p(x)-1}$,

(H5) $B(x,d\xi) \cdot \xi \geq C_4 w(x)|\xi|^{p(x)}$,

(H6) $(A(x,d\xi)-A(x,d\eta)) \cdot (d\xi-d\eta) + (B(x,d\xi)-B(x,d\eta)) \cdot (\xi-\eta) \geq 0$ for $\xi \neq \eta$,

where $C_1, C_2, C_3, C_4$ are nonnegative constants. $w(x) \in L^1(\Omega)$ nonnegative and $w(x)^{1/(p(x)-1)} \in L^1(\Omega)$. We will discuss the existence and uniqueness of the solution $u \in \mathcal{K}_{\psi,\theta}$ for the abovementioned obstacle problem.

Now, we introduce the existing results and related definitions.

Throughout this paper, we assume that $\Omega$ is a bounded domain in $\mathbb{R}^n$. Let $\Lambda^I = \Lambda^I(\mathbb{R}^n)$ be the set of all $I$-forms in
A differential 1-form $u(x)$ is generated by $\{dx_1 \wedge dx_2 \wedge \cdots \wedge dx_l\}$, $l = 1, 2, \ldots, n$; that is, $u(x) = \sum_{i=1}^n u_i(x)dx_i = \sum u_{ij}dx_j \wedge dx_k$, where $u_{ij}$ is differential function, $I = (i_1, i_2, \ldots, i_l)$, and $1 \leq i_1 < i_2 < \cdots < i_l \leq n$. Let $D'(\Omega, \Lambda^l)$ be the space of all differential 1-forms on $\Omega$. For $\alpha(x)$, $\Sigma x_i dx_i = \Lambda$ and $\beta(x) = \Sigma \beta_i dx_i = \Lambda'$, then the inner product is obtained by $\alpha \cdot \beta = + (\alpha \wedge \beta) = \sum \alpha_i(x) \beta_i(x)$. We write $|u| = (u \cdot u)^{1/2} = (\sum |u_i(x)|^2)^{1/2}$. We denote the exterior derivative by $du = \sum_{i=1}^n (\partial u_i(x)/\partial x_i)dx_j \wedge dx_i : D'(\Omega, \Lambda^l) \rightarrow D'(\Omega, \Lambda^{l+1})$ for $l = 0, 1, \ldots, n$ - 1. Its formal adjoint operator $d^* : D'(\Omega, \Lambda^{l+1}) \rightarrow D'(\Omega, \Lambda^l)$ is given by $d^* = -(1)^{l+1}d*$. Next, we will introduce some basic properties of the weighted variable exponent Lebesgue spaces $L^{p(x)}(\Omega, \mu)$ and weighted variable exponent Sobolev spaces $W^{1,p(x)}(\Omega, \mu)$, and define $\mathcal{P}(\Omega)$ to be the set of all n-dimensional Lebesgue measurable functions $\rho : \Omega \rightarrow [1, \infty)$. Functions $\rho \in \mathcal{P}(\Omega)$ are called variable exponents on $\Omega$. We define $\rho^{-} := \inf_{x \in \Omega} \rho(x)$, $\rho^{+} := \sup_{x \in \Omega} \rho(x)$. If $\rho^{-} < \infty$, then we call $\rho$ a bounded variable exponent. If $\rho \in \mathcal{P}(\Omega)$, then we define $\rho := (1/\rho^{-}(x)) + (1/\rho^{+}(x)) = 1$, where $1/\infty := 0$. The property $\rho^{-}$ is called the dual variable exponent of $\rho$. We denote $\omega$ as a weight by $w \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $w > 0$ a.e.; also in general $du = wdx$. From [7, 10], we know that if $\rho \in \mathcal{P}(\Omega)$ satisfies (3), the weighted variable exponent Lebesgue spaces $L^{p(x)}(\Omega, \mu) = \{f : \int_{\Omega} |f(x)|^{p(x)}d\mu < \infty, \lambda > 0\}$ with the norm $\|f\|_{L^{p(x)}(\Omega, \mu)} = \inf\{\lambda > 0 : \int_{\Omega} |f(x)/\lambda|^{p(x)}d\mu < 1\}$ and the weighted variable exponent Sobolev spaces $W^{1,p(x)}(\Omega, \mu) = \{f \in L^{p(x)}(\Omega, \mu) : \forall \varphi \in L^{p(x)}(\Omega, \mu) \text{ with the norm } \|f\|_{W^{1,p(x)}(\Omega, \mu)} = \|f\|_{L^{p(x)}(\Omega, \mu)} + \|\nabla f\|_{L^{p(x)}(\Omega, \mu)}\}$ are Banach spaces and reflexive and uniformly convex. On the set of all differential forms on $\Omega$, we define the weighted variable exponent Lebesgue spaces of differential 1-forms $L^{p(x)}(\Omega, \Lambda^1, \mu)$ and the weighted variable exponent Sobolev spaces of differential 1-forms $W^{1,p(x)}(\Omega, \Lambda^1, \mu)$.

Definition 1. We denote the weighted variable exponent Lebesgue spaces of differential 1-forms by $L^{p(x)}(\Omega, \Lambda^1, \mu) = \{\varphi = \sum_{i=1}^n \varphi_i(x)dx_i \in \Lambda^1 : \varphi_i(x) \in L^{p(x)}(\Omega, \mu)\}$, $l = 0, 1, 2 \ldots, n$ and we endow $L^{p(x)}(\Omega, \Lambda^1, \mu)$ with the following norm:
\[
\|\varphi\|_{L^{p(x)}(\Omega, \Lambda^1, \mu)} = \inf\{\lambda > 0 : \int_{\Omega} \left| \frac{\varphi(x)}{\lambda} \right|^{p(x)}d\mu \leq 1\}.
\]

And the spaces $W^{1,p(x)}(\Omega, \Lambda^1, \mu) = \{\varphi \in L^{p(x)}(\Omega, \Lambda^1, \mu) : du \in L^{p(x)}(\Omega, \Lambda^{1+1}, \mu)\}$ with the norm
\[
\|\varphi\|_{W^{1,p(x)}(\Omega, \Lambda^1, \mu)} = \|\varphi\|_{L^{p(x)}(\Omega, \Lambda^1, \mu)} + \|du\|_{L^{p(x)}(\Omega, \Lambda^{1+1}, \mu)}.
\]

are the weighted variable exponent Sobolev spaces of differential 1-forms; $l = 0, 1, 2, \ldots, n - 1$. $W^{1,p(x)}(\Omega, \Lambda^1, \mu)$ is the completion of $C_0^\infty(\Omega, \Lambda^1, \mu)$ in $W^{1,p(x)}(\Omega, \Lambda^1, \mu)$. We need the following Hölder inequalities; see [7, 10].

**Proposition 2.** Let $p, q \in \mathcal{P}(\Omega)$ be such that $1 = (1/p(x)) + (1/q(x))$ for $\mu$-almost every $x \in \Omega$. Then
\[
\int_\Omega |fg| d\mu \leq \left( \frac{1}{p(x)} \right)^{1/2} \|f\|_{L^{p(x)}(\Omega, \mu)} + \left( \frac{1}{q(x)} \right)^{1/2} \|g\|_{L^{q(x)}(\Omega, \mu)},
\]

for all $f \in L^{p(x)}(\Omega, \mu)$ and $g \in L^{q(x)}(\Omega, \mu)$.

**Lemma 3** (see [7]). Let $(D, \Sigma, \mu)$ be a σ-finite, complete measure space; if $f \in L^p(D, \mu)$, $g \in L^q(D, \mu)$, and $0 \leq |g| \leq |f|$ $\mu$-almost everywhere, then $g \in L^t(D, \mu)$ and $\|g\|_{L^{t}(D, \mu)} \leq \|f\|_{L^{t}(D, \mu)}$.

By the inequality \((\sum_{i=1}^n a_i^q)^{1/q} \leq \sum_{i=1}^n a_i^{q_i} \leq n^{1/2} \sum_{i=1}^n (a_i^{q_i})^{1/2}\) for any $a_i \geq 0$, using Lemma 3, we can easily have the following lemma.

**Lemma 4.** If $u = \sum_{i=1}^n u_i(x)dx_i \in D'(\Omega, \Lambda^1)$ and $|u| = (\sum_{i=1}^n |u_i|^q)^{1/q}$, then $u \in L^{p(x)}(\Omega, \Lambda^1, \mu)$, and $\|u\|_{L^{p(x)}(\Omega, \Lambda^1, \mu)}$ are equivalent, and $\|u\|_{L^{p(x)}(\Omega, \Lambda^1, \mu)} = \|u\|_{L^{p(x)}(\Omega, \Lambda^1, \mu)}$.

2. **Main Results**

In this section, we will obtain the existence and uniqueness of weak solution for obstacle problem of the nonhomogeneous A-harmonic equation in space $W^{1,p(x)}(\Omega, \Lambda^1, \mu)$.

**Theorem 5.** Suppose $\mathfrak{R}_{\rho, \beta}$ is not empty, under conditions (H1)–(H6), and there exists a unique solution $u$ to the obstacle problem (1)-(2). That is, there is a differential form $u$ in $\mathfrak{R}_{\rho, \beta}$ such that
\[
\int_{\Omega} \left( A(x, du) \cdot d(v - u) + B(x, du) \cdot (v - u) \right) dx \geq 0,
\]

whenever $v \in \mathfrak{R}_{\rho, \beta}$.

We deduce Theorem 5 from a proposition of Kinderlehrer and Stampacchia.

**Proposition 6** (see [12]). Let $K$ be a nonempty closed convex subset of $X$ and let $\mathfrak{M} : K \rightarrow X'$ be monotone, coercive, and weakly continuous on $K$. Then there exists an element $u$ in $K$ such that $\langle \mathfrak{M}u, v - u \rangle \geq 0$ whenever $v \in K$.

Now let $X = W^{1,p(x)}(\Omega, \Lambda^1, \mu)$ and $(\cdot, \cdot)$ be the usual pairing between $X$ and $X'$, $(f, g) = \int_{\Omega} f \cdot g d\mu$, where $g$ is in $X$ and $f$ in $X' = W^{1,p(x)}(\Omega, \Lambda^1, \mu)$. We will take $\mathfrak{R}_{\rho, \beta}$ as $K$. We define a mapping $\mathfrak{M} : \mathfrak{R}_{\rho, \beta} \rightarrow X'$ by
\[
\langle \mathfrak{M}u, v \rangle = \int_{\Omega} (A(x, dv) \cdot du + B(x, dv) \cdot u) dx
\]

for $u \in W^{1,p(x)}(\Omega, \Lambda^1, \mu)$. 

Abstract and Applied Analysis
Lemma 7. If $p(x)$ satisfies (3), then spaces $L^p(x)(\Omega, \Lambda', \mu)$ and $W^p_d(x)(\Omega, \Lambda', \mu)$ are complete and convex.

Proof. From [7], we know that if $p$ satisfies (3) and $w(x)^{1/(p(x)-1)} \in L^1(\Omega)$, then let $L^p(x)(\Omega, \mu)$ be Banach space and uniformly convex. If $\omega_1$ and $\omega_2$ are two $l$-forms: $\omega_1 = \sum_i a_i dx_i$ and $\omega_2 = \sum_i b_i dx_i$, we can easily have $\omega_1 + \omega_2 = \sum_i (a_i + b_i) dx_i$ and $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$, so we can immediately obtain the convexity of spaces $L^p(x)(\Omega, \Lambda', \mu)$ and $W^p_d(x)(\Omega, \Lambda', \mu)$.

Let $u_j = \sum_i u_{ij} dx_i \in L^p(x)(\Omega, \Lambda', \mu)$ be a Cauchy sequence in $W^p_d(x)(\Omega, \Lambda', \mu)$. Then for any $I$, $u_j(x)$ converges in $L^p(x)(\Omega, \mu)$. Suppose that $u_j(x) \to u(x)$ in $L^p(x)(\Omega, \mu)$. Now let $u = \sum_i u_i dx_i \in L^p(x)(\Omega, \Lambda', \mu)$, we have

$$|u_j - u| = \left(\sum_i |u_{ij} - u_i|^2 \right)^{1/2} \leq \sum_i |u_{ij} - u_i|,$$

using Lemmas 3 and 4, and we know the sequence $u_j$ converges to $u$ in $L^p(x)(\Omega, \Lambda', \mu)$.

For the sequence $\{du_j\}$, we suppose $du_j \to v$ in $L^p(x)(\Omega, \Lambda'^{i+1}, \mu)$, and then $v \in L^p(x)(\Omega, \Lambda'^{i+1}, \mu)$. So $(u_j, du_j)$ converges to $(u, v)$ in the normed space $L^p(x)(\Omega, \Lambda', \mu) \times L^p(x)(\Omega, \Lambda'^{i+1}, \mu)$. From $du_j - du = \sum_i (du_{ij} - du_i) \wedge dx_i$, we have

$$|du_j - du| = \left|\sum_i \frac{\partial u_{ij}}{\partial x_i} - \frac{\partial u_i}{\partial x_i} \wedge dx_i \right| \leq \left(\sum_i \left|\sum_{j=1}^n \frac{\partial u_{ij}}{\partial x_i} - \frac{\partial u_i}{\partial x_i}\right|^2 \right)^{1/2} \leq \sum_i \frac{\partial u_{ij}}{\partial x_i} - \frac{\partial u_i}{\partial x_i} \leq n^{1/2} \left(\sum_i \left|\sum_{j=1}^n \frac{\partial u_{ij}}{\partial x_i} - \frac{\partial u_i}{\partial x_i}\right|^2 \right)^{1/2} = n \left|\nabla u_{ij} - \nabla u_i\right|.$$
Proof. It follows from (H6) that \( \mathcal{A} \) is monotone. To show that \( \mathcal{A} \) is coercive on \( K_{\phi, \theta} \), fix \( \phi \in K_{\phi, \theta} \) and using the conditions (H2)–(H5), (12), (13), and (6), then

\[
\langle \mathcal{A}u - \mathcal{A}\phi, u - \phi \rangle \geq C_2 \int_{\Omega} |du|^p(x) \, d\mu - C_3 \int_{\Omega} |\phi|^p(x) \, d\mu

\]

where \( \| \cdot \| \) is the \( L^p(x)(\Omega, \lambda', \mu) \) norm; taking \( \epsilon = C_2(p') / 2(C_1 + C_3) \), we have

\[
\langle \mathcal{A}u - \mathcal{A}\phi, u - \phi \rangle \geq \frac{C_2}{2} \int_{\Omega} |du|^p(x) \, d\mu + C_4 \int_{\Omega} |u|^p(x) \, d\mu

\]

as \( \| u - \phi \|_W^{p(x)}(\Omega, \lambda', \mu) \to \infty \). If follow that \( \mathcal{A} \) is coercive on \( K_{\phi, \theta} \). This completes the proof of Lemma 10.
Lemma 11. \( \mathcal{A} \) is weakly continuous on \( \mathcal{R}_{\psi, \beta} \).

Proof. Let \( u_i \in \mathcal{R}_{\psi, \beta} \) be a sequence that converges to an element \( u \in \mathcal{R}_{\psi, \beta} \) in \( W^{p(x)}_d(\Omega, \Lambda^{l-1}, \mu) \). Pick a subsequence \( u_{ij} \) such that \( u_{ij} \to u \) a.e. in \( \Omega \). Since the mapping \( \xi \to A(x, \xi) \) and \( \xi \to B(x, \xi) \) are continuous for a.e. \( x \) in \( \Omega \), we have \( A(x, u_{ij}(x))w^{-1/p(x)} \to A(x, u(x))w^{-1/p(x)} \) a.e. in \( \Omega \). Under the conditions (H2) and (H4), we know that \( A(x, u_{ij})w^{-1/p(x)} \) and \( B(x, u_{ij})w^{-1/p(x)} \) are uniformly bounded in \( L^{p(x)}(\Omega, \Lambda^{l-1}) \), and we have \( A(x, du_{ij})w^{-1/p(x)} \to A(x, du)w^{-1/p(x)} \) weakly in \( L^{p(x)}(\Omega, \Lambda^{l-1}, \mu) \) and \( B(x, du_{ij})w^{-1/p(x)} \to B(x, du)w^{-1/p(x)} \) weakly in \( L^{p(x)}(\Omega, \Lambda^{l-1}, \mu) \).

Since the weak limit is independent of the choice of the subsequence, it follows that

\[
\begin{align*}
A(x, du)w^{-1/p(x)} & \to A(x, du)w^{-1/p(x)}, \\
B(x, du)w^{-1/p(x)} & \to B(x, du)w^{-1/p(x)}
\end{align*}
\]

for all \( u \in L^{p(x)}(\Omega, \Lambda^l, \mu) \), \( du^{1/p(x)} \in L^{p(x)}(\Omega, \Lambda^{l-1}) \). Then we have

\[
\langle \mathcal{A}u_i, v \rangle = \int_{\Omega} \left( A(x, du_i)w^{-1/p(x)} \cdot dvu^{1/p(x)} \right) dx + \int_{\Omega} \left( B(x, du_i)w^{-1/p(x)} \cdot vwu^{1/p(x)} \right) dx
\]

\[
- \int_{\Omega} \left( A(x, du)w^{-1/p(x)} \cdot dvu^{1/p(x)} \right) dx + \int_{\Omega} \left( B(x, du)w^{-1/p(x)} \cdot vwu^{1/p(x)} \right) dx = \langle \mathcal{A}u, v \rangle.
\]

Hence \( \mathcal{A} \) is weakly continuous on \( \mathcal{R}_{\psi, \beta} \). This ends the proof of Lemma 11.

Proof of Theorem 5. We can apply Proposition 6 and the above lemmas to obtain the existence. If there are two weak solutions \( u_1, u_2 \in \mathcal{R}_{\psi, \beta} \) to obstacle problem (1)-(2), then we have

\[
\int_{\Omega} (A(x, du_1) \cdot du_1 - A(x, du_2) \cdot du_2) dx \geq 0,
\]

so

\[
\int_{\Omega} ((A(x, du_2) - A(x, du_1)) \cdot du_2 - A(x, du_1) \cdot du_1) dx \leq 0.
\]

In view of (H6), we can further infer that

\[
\int_{\Omega} ((A(x, du_2) - A(x, du_1)) \cdot du_2 - A(x, du_1) \cdot du_1) dx = 0.
\]

Thus

\[
\int_{\Omega} (A(x, du_2) \cdot du_2 - A(x, du_1) \cdot du_1) dx = 0.
\]

Corollary 12. Let \( \Omega \) be a bounded domain and \( \theta \in W^{p(x)}_d(\Omega, \Lambda^{l-1}, \mu) \). Under the conditions (H1)-(H6), there is a differential form \( u \in W^{p(x)}(\Omega, \Lambda^{l-1}, \mu) \) with \( u - \theta \in W^{p(x)}_d(\Omega, \Lambda^{l-1}, \mu) \) such that

\[
d^* A(x, du) = B(x, du), \quad \text{weakly in } \Omega;
\]

that is, \( \int_{\Omega} (A(x, du) \cdot dp + B(x, du) \cdot \varphi) dx = 0 \), whenever \( \varphi \in W^{p(x)}_d(\Omega, \Lambda^{l-1}, \mu), l = 1, 2, \ldots, n \).

Proof. Choose \( \psi \equiv \infty \) and let \( u \) be the solution to the obstacle problem (1)-(2) in \( \mathcal{R}_{\psi, \beta} \). For any \( \varphi \in W^{p(x)}_d(\Omega, \Lambda^{l-1}, \mu) \), since \( u + \varphi \) and \( u - \varphi \) both belong to \( \mathcal{R}_{\psi, \beta} \), we have

\[
\int_{\Omega} (A(x, du) \cdot dp + B(x, du) \cdot \varphi) dx \geq 0.
\]

Thus

\[
\int_{\Omega} (A(x, du) \cdot dp + B(x, du) \cdot \varphi) dx = 0.
\]

as desired.

Remark 13. If \( p(x) = p \), then \( \| u \|_{L^p(\Omega, \Lambda^l, \mu)} = \| u \|_{L^p(\Omega, \Lambda^l, \mu)} = \| u \|_{L^p(\Omega, \Lambda^l, \mu)} \). 

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

References


