Research Article

On the Strong Convergence of a Sufficient Descent Polak-Ribière-Polyak Conjugate Gradient Method

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Recently, Zhang et al. proposed a sufficient descent Polak-Ribière-Polyak (SDPRP) conjugate gradient method for large-scale unconstrained optimization problems and proved its global convergence in the sense that $\liminf_{k \to \infty} \|\nabla f(x_k)\| = 0$ when an Armijo-type line search is used. In this paper, motivated by the line searches proposed by Shi et al. and Zhang et al., we propose two new Armijo-type line searches and show that the SDPRP method has strong convergence in the sense that $\lim_{k \to \infty} \|\nabla f(x_k)\| = 0$ under the two new line searches. Numerical results are reported to show the efficiency of the SDPRP with the new Armijo-type line searches in practical computation.

1. Introduction

In this paper, we are concerned with the following unconstrained minimization problem:

$$\min f(x), \quad x \in \mathbb{R}^n,$$

(1)

where $f: \mathbb{R}^n \to \mathbb{R}^1$ is a smooth function whose gradient $\nabla f(x)$ is often denoted by $g(x)$. The related problem is called large-scale minimization problem when its dimension $n$ is very large (e.g., $n > 10^6$). For solving large-scale minimization problems, the matrices-free methods are quite efficient. Among such methods, the conjugate gradient method is very famous for its excellent numerical performance in the practical computation. Much progress has been achieved in the study of global convergence of the various conjugate gradient methods, such as the Polak-Ribiére-Polyak (PRP) [1, 2], the Fletcher-Reeves (FR) [3], the Hestenes-Stiefel (HS) [4, 5], and the Dai-Yuan (DY) [6] conjugate gradient methods, et al.

Recently, Zhang et al. [7] presented a sufficient descent Polak-Ribière-Polyak (SDPRP) conjugate gradient method for solving large-scale problem (1), whose most important property is that its generated direction is always a sufficient descent direction for the objective function. Moreover, this property is independent of the line search used, and it reduces to the classical PRP method when the exact line search is used. The iterative process of the SDPRP method is given by

$$x_{k+1} = x_k + \alpha_k d_k, \quad k = 0, 1, \ldots,$$

(2)

where $x_k$ is the current iterate, $\alpha_k > 0$ is called the stepsize which can be obtained by some line search techniques, such as the Armijo line search, the Goldstein line search, and the (strong) Wolfe line search, and $d_k$ is the search direction determined by

$$d_k = \begin{cases} -g_k, & \text{if } k = 0, \\ -g_k + \beta_k^{\text{PRP}} d_{k-1} - \theta_k y_{k-1}, & \text{if } k \geq 1, \end{cases}$$

(3)

with

$$\beta_k^{\text{PRP}} = \frac{g_k y_{k-1}}{\|g_{k-1}\|^2}, \quad \theta_k = \frac{\|g_k d_{k-1}\|}{\|g_{k-1}\|^2},$$

(4)

where $y_{k-1} = g_k - g_{k-1}$. It is easy to deduce from (3) and (4) that

$$g_k d_k = -\|g_k\|^2,$$

(5)
which indicates that $d_k$ is a sufficient descent direction of $f(x)$ at the current iterate $x_k$ if $\|g_k\| \neq 0$; that is, $x_k$ is not a stationary point of the objective function $f(x)$. It has been proved that SDPRP method has global convergence under an Armijo-type line search \cite{7} in the sense that
\[
\lim \inf_{k \to \infty} \|g(x_k)\| = 0, \tag{6}
\]
which means that at least one cluster point of the sequence $\{x_k\}$ is a stationary point if it is bounded.

In another recent paper, Shi and Shen \cite{8} showed that the classical PRP method in \cite{1} has strong convergence and linear convergence rate under a customized Armijo-type line search, which is somewhat complicated. The new Armijo-type line search ensures that the search direction generated by the classical PRP method possesses the sufficient descent property, which is helpful to prove the global convergence.

In this paper, motivated by the Armijo-type line search in \cite{8}, we first propose a similar but simple line search, which can ensure that the SDPRP method has strongly global convergence in the sense that
\[
\lim_{k \to \infty} \|g(x_k)\| = 0, \tag{7}
\]
that is, any cluster point of the sequence $\{x_k\}$ is a stationary point of the objective function $f(x)$. Noting that the above new line search needs to estimate the Lipschitz constant, which is not easy even for linear function, we present another Armijo-type line search, which is motivated by the line search in \cite{7}. This new line search can also guarantee the global convergence of the SDPRP method in the above sense.

The remainder of the paper is organized as follows. In Section 2 we introduce the two new Armijo-type line searches and present the strongly convergent SDPRP method. The global convergence is established under the above two new Armijo-type line searches in Section 3. Some numerical results are presented in Section 4, and in the last section, we conclude the paper with some remarks.

\section*{2. Strongly Convergent SDPRP Method}

First, we give the following basic assumptions on the objective function $f(x)$.

\textbf{Assumption 1.} Consider the following.

(H1) The objective function $f(x)$ has a lower bound on the level set $L_0 = \{x \in \mathbb{R}^n \mid f(x) \leq f(x_0)\}$, where $x_0$ is the starting point.

(H2) In some neighborhood $N$ of $L_0$, the gradient $g(x)$ is Lipschitz continuous on an open convex set $B$ that contains $L_0$, that is, there exists a constant $L > 0$ such that
\[
\|g(x) - g(y)\| \leq L \|x - y\|, \quad \text{for any } x, y \in B. \tag{8}
\]

(H3) The level set $L_0 = \{x \in \mathbb{R}^n \mid f(x) \leq f(x_0)\}$ is bounded.

Although $g(x)$ is Lipschitz continuous, the Lipschitz constant $L$ is usually unknown in practice, even for the linear function $g(x)$. Therefore, we need to estimate the Lipschitz constant $L$. Here, we adopt one of three estimating approaches proposed in \cite{9}. More specifically, if $k \geq 1$, then we can set
\[
L \equiv L_k = \max \left\{ L_{k-1}, \frac{\|y_{k-1}\|}{\|g_{k-1}\|} \right\}, \tag{9}
\]
with $L_0 > 0$, and $s_{k-1} = x_k - x_{k-1}$.

\textbf{Armijo-Type Line Search I.} Set $\mu \in (0, 1), \rho \in (0, 1), c \in (0, 1), \rho = 0, \ldots, \mu \rho \in (0, 1)$, and the initial stepsize $\delta_k = (1 - c)\|g_k\|^2/(L_k \|d_k\|^2)$, where $L_k$ is determined by (9). Let $\alpha_k$ be the largest $\alpha$ in $\{\delta_k, \rho \delta_k, \rho^2 \delta_k, \ldots\}$ such that
\[
f(x_k + \alpha d_k) - f(x_k) \leq -\mu \alpha \|g_k\|^2. \tag{10}
\]

\textbf{Armijo-Type Line Search II.} Set $\mu > 0, \rho \in (0, 1)$. Let $\alpha_k$ be the largest $\alpha$ in $\{1, \rho, \rho^2, \ldots\}$ such that
\[
f(x_k + \alpha d_k) \leq f(x_k) - \mu \alpha^2 \|d_k\|^4. \tag{11}
\]

Now we begin to describe the strongly convergent SDPRP method.

\textbf{Algorithm 2 (strongly convergent SDPRP method).}

\textbf{Step 0.} Given an initial point $x_0 \in \mathbb{R}^n$, $\mu \in (0, 1/2)$ and $\rho \in (0, 1), c \in (0, 1)$ and set $d_0 = -g_0, k := 0$.

\textbf{Step 1.} If $\|g_k\| = 0$ then stop; otherwise go to Step 2.

\textbf{Step 2.} Compute the descent direction $d_k$ by (3) and (4). Determine the stepsize $\alpha_k$ by the Armijo-type line search (10) or (11).

\textbf{Step 3.} Set $x_{k+1} = x_k + \alpha_k d_k, \text{ and } k := k + 1; \text{ go to Step 1}.$

\textbf{Lemma 3.} Assume that (H1) and (H2) hold, then there exist $m_0 > 0$ and $M_0 > 0$ such that for any $k \geq 0$, one has
\[
m_0 \leq L_k \leq M_0, \tag{12}
\]
where $L_k$ is defined by (9).

\textbf{Proof.} See [9, Lemma 2.1].

\textbf{Lemma 4.} Assume that (H1) and (H2) hold. If $\|g_k\| > 0$, then the new Armijo-type line search I is well-defined for the index $k$.

\textbf{Proof.} The proof is easy; for completeness, we give the proof here. In fact, we can prove this lemma by contradiction. Suppose that the conclusion does not hold; then for $k$, the inequality (10) does not hold for any nonnegative integer $m$; that is,
\[
f(x_k) - f(x_k + \delta_k \rho^m d_k) < \mu \delta_k \rho^m \|g_k\|^2, \quad \forall m. \tag{13}
\]
Thus,
\[
\frac{f(x_k) - f(x_k + \delta_k \rho^m d_k)}{\delta_k \rho^m} < \mu \|g_k\|^2, \quad \forall m. \tag{14}
\]

Letting \( m \to +\infty \), by the continuity of \( f(x) \) and \( -g_k^T d_k = \|g_k\|^2 \), we can obtain
\[
\|g_k\|^2 \leq \mu \|g_k\|^2. \tag{15}
\]

This and \( \mu \in (0, 1) \) yield that
\[
\|g_k\| = 0, \tag{16}
\]

which contradicts to \( \|g_k\| > 0 \). The proof is completed. \( \square \)

**Lemma 5.** Assume that (H2) and (H3) hold. If \( \|g_k\| > 0 \), then the new Armijo-type line search II is well-defined for the index \( k \).

**Proof.** The lemma is also proved by contradiction. Suppose that the conclusion does not hold; then, for \( k \), the inequality (11) does not hold for any nonnegative integer \( m \); that is,
\[
f(x_k + \rho^m d_k) > f(x_k) - \mu \rho^m \|d_k\|^2, \quad \forall m. \tag{17}
\]

That is,
\[
\frac{f(x_k + \rho^m d_k) - f(x_k)}{\rho^m} > -\mu \rho^m \|d_k\|^2, \quad \forall m. \tag{18}
\]

Letting \( m \to +\infty \), by the continuity of \( f(x) \) and \( -g_k^T d_k = \|g_k\|^2 \), we can obtain
\[
-\|g_k\|^2 \geq 0, \tag{19}
\]

that is,
\[
\|g_k\| = 0, \tag{20}
\]

which contradicts to \( \|g_k\| > 0 \). The proof is completed. \( \square \)

**3. Strongly Global Convergence**

Throughout this section, we assume that \( \|g_k\| > 0 \), for all \( k \geq 0 \); otherwise a stationary point of the objective function \( f(x) \) has been found.

3.1. Global Convergence of SDPRP Method with the Line Search I. We first prove the global convergent of SDPRP method with the Armijo-type line search I.

**Lemma 6.** For all \( k \geq 0 \), one has
\[
\|d_k\| \leq \left(1 + \frac{2L(1-c)}{m_0}\right) \|g_k\|, \quad \forall k, \tag{21}
\]

where \( m_0 \) is defined in Lemma 3.

**Proof.** If \( k = 0 \) then
\[
\|d_k\| = \|g_k\| \leq \left(1 + \frac{2L(1-c)}{m_0}\right) \|g_k\|. \tag{22}
\]

If \( k \geq 1 \) then, from (3), (4), and (H2), we can get that
\[
\|d_k + g_k\| = \left\|\beta_k^{\text{PRP}} d_{k-1} - \theta_k y_{k-1}\right\|
\leq \frac{2 \|g_k - g_{k-1}\| \|d_{k-1}\| \|g_k\|}{\|g_{k-1}\|^2}
\leq \frac{2L\delta_{k-1} \|d_{k-1}\|^2}{\|g_{k-1}\|^2} \|g_k\|
\leq \frac{2L(1-c)}{m_0} \|g_k\|, \tag{23}
\]

which together with the triangular inequality implies that
\[
\|d_k\| \leq \|d_k + g_k\| + \|g_k\| \leq \left(1 + \frac{2L(1-c)}{m_0}\right) \|g_k\|. \tag{24}
\]

This completes the proof. \( \square \)

The following lemma shows that the stepsize sequence \( \{\alpha_k\} \) generated by the Armijo-type line search I is bounded from below.

**Lemma 7.** For all \( k \geq 0 \), there exists a constant \( C > 0 \), such that
\[
\alpha_k \geq C, \tag{25}
\]

in which \( \alpha_k \) is generated by the Armijo-type line search I.

**Proof.** We divide the proof into two cases: \( \alpha_k = \delta_k \) and \( \alpha_k < \delta_k \). For the first case, by (12) and (21), we get
\[
\alpha_k \geq \frac{(1-c)}{M_0} \left(1 + \frac{2L(1-c)}{m_0}\right)^{-2}. \tag{26}
\]

For the second case, that is \( \alpha_k < \delta_k \), which indicates that \( \alpha_k / \rho \) does not satisfy (10); that is,
\[
f(x_k + \alpha_k d_k / \rho) > f(x_k) - \frac{\rho \alpha_k \|g_k\|^2}{\rho}. \tag{27}
\]

Using the mean value theorem in the above inequality, we obtain \( \theta_k \in (0, 1) \), such that
\[
\left[g(x_k + \theta_k \alpha_k d_k / \rho) - g_k\right]^T d_k > (1-\mu) \|g_k\|^2. \tag{28}
\]
This inequality and (H2), (21) show that
\[
\frac{L\alpha_k}{\rho} \geq \frac{\|g(x_k + \theta_k\alpha_k d_k/\rho) - g_k\|}{\|d_k\|}
\]
\[
= \frac{\|g(x_k + \theta_k\alpha_k d_k/\rho) - g_k\| \cdot \|d_k\|^2}{\|d_k\|^2}
\]
\[
\geq \frac{[g(x_k + \theta_k\alpha_k d_k/\rho) - g_k]^\top d_k}{\|d_k\|^2}
\]
\[
\geq (1 - \mu) \frac{\|g_k\|^2}{\|d_k\|^2}
\]
\[
\geq (1 - \mu) \left(1 + \frac{2L(1 - c)}{m_0}\right)^{-2}.
\]
Therefore, we have that
\[
\alpha_k \geq \frac{(1 - \mu) \rho}{L} \left(1 + \frac{2L(1 - c)}{m_0}\right)^{-2}.
\]
Obviously, (26) and (30) show that (25) holds with
\[
C = \min \left\{ \frac{(1 - c)}{M_0}, \frac{(1 - \mu) \rho}{L} \right\} \left(1 + \frac{2L(1 - c)}{m_0}\right)^{-2}.
\]
This completes the proof. \( \square \)

We are now ready to establish the strong convergence of SDPRP method using the Armijo-type line search I.

**Theorem 8.** Suppose that (H1) and (H2) hold. Then
\[
\lim_{k \to \infty} \|g_k\| = 0.
\]

**Proof.** Since the generated sequence \( \{x_k\} \subseteq L_0 \) and the objection function \( f(x) \) is bounded below on the level set \( L_0 \), by (10) and (25), we have
\[
\sum_{k=0}^{\infty} C_k \|g_k\|^2 \leq \sum_{k=0}^{\infty} (f_k - f_{k+1}) < f_0.
\]
Thus
\[
\lim_{k \to \infty} \|g_k\| = 0.
\]
This completes the proof. \( \square \)

### 3.2. Global Convergence of SDPRP Method with the Line Search II

Then, we prove the strongly global convergent of SDPRP method with the Armijo-type line search II. It is obvious that \( x_k \in L_0 \) for all \( k \geq 0 \). Therefore, from the line search II, we have
\[
\lim_{k \to \infty} \alpha_k \|d_k\|^2 = 0.
\]
This together with (5) implies that
\[
\lim_{k \to \infty} \alpha_k \|g_k\|^2 \leq \lim_{k \to \infty} \|d_k\|^2 = 0.
\]
In addition, (H3) implies that there is a constant \( M > 0 \) such that
\[
\|g_k\| \leq M, \quad \forall k \geq 0.
\]

**Lemma 9.** Suppose that (H2) and (H3) hold. Then for all \( k \geq 0 \), one has
\[
\alpha_k \geq \min \left\{ 1, \frac{\rho \|g_k\|^2}{(L + \mu \|d_k\|^2) \|d_k\|^2} \right\}.
\]

**Proof.** If \( \alpha_k \neq 1 \), then \( \alpha_k' = \alpha_k/\rho \) does not satisfy (II); that is
\[
f(x_k + \alpha_k' d_k) > f(x_k) - \mu (\alpha_k')^2 \|d_k\|^4.
\]
From the mean value theorem and (H2), there exists a constant \( \theta_k \in (0, 1) \), such that
\[
f(x_k + \alpha_k' d_k) - f(x_k)
\]
\[
= \alpha_k' \frac{\rho}{L} (\theta_k \alpha_k' d_k)^\top d_k
\]
\[
= \alpha_k' \frac{\rho}{L} (\theta_k \alpha_k' d_k)^\top d_k
\]
\[
\leq -\alpha_k' \|g_k\|^2 + (\alpha_k')^2 L \|d_k\|^2,
\]
which together with (36) shows that (35) holds. This completes the proof. \( \square \)

We are now ready to establish the strong convergence of SDPRP method using the Armijo-type line search II. The proof is motivated by the proof of Theorem 2.2 in [10].

**Theorem 10.** Suppose that (H2) and (H3) hold. Then
\[
\lim_{k \to \infty} \|g_k\| = 0.
\]

**Proof.** For the sake of contradiction, we suppose that the conclusion is not right. Then there exist a constant \( \epsilon > 0 \) and an infinite index set \( K \) such that
\[
\|g_k\| \geq \epsilon, \quad \forall k \in K.
\]
Moreover, the fact \( \alpha_k \leq 1 \), (35) and (H2) imply that
\[
\|g_k - g_{k-1}\|^2 \leq L^2 \alpha_k'^2 \|d_{k-1}\|^2 \leq L^2 \alpha_k \|d_{k-1}\|^2 \to 0.
\]
This and (42) indicate that there exists a positive constant \( \epsilon_1 \) such that for sufficiently large \( k \in K \), we have
\[
\|g_k\| \geq \epsilon_1.
\]
Then by (36) and (44), we can get
\[
\lim_{k \to \infty} \alpha_k = 0.
\]

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This inequality and (H2), (21) show that
\[
\frac{L\alpha_k}{\rho} \geq \frac{\|g(x_k + \theta_k\alpha_k d_k/\rho) - g_k\|}{\|d_k\|}
\]
\[
= \frac{\|g(x_k + \theta_k\alpha_k d_k/\rho) - g_k\| \cdot \|d_k\|^2}{\|d_k\|^2}
\]
\[
\geq \frac{[g(x_k + \theta_k\alpha_k d_k/\rho) - g_k]^\top d_k}{\|d_k\|^2}
\]
\[
\geq (1 - \mu) \frac{\|g_k\|^2}{\|d_k\|^2}
\]
\[
\geq (1 - \mu) \left(1 + \frac{2L(1 - c)}{m_0}\right)^{-2}.
\]
Therefore, we have that
\[
\alpha_k \geq \frac{(1 - \mu) \rho}{L} \left(1 + \frac{2L(1 - c)}{m_0}\right)^{-2}.
\]
Obviously, (26) and (30) show that (25) holds with
\[
C = \min \left\{ \frac{(1 - c)}{M_0}, \frac{(1 - \mu) \rho}{L} \right\} \left(1 + \frac{2L(1 - c)}{m_0}\right)^{-2}.
\]
This completes the proof. \( \square \)

We are now ready to establish the strong convergence of SDPRP method using the Armijo-type line search II. The proof is motivated by the proof of Theorem 2.2 in [10].

**Theorem 10.** Suppose that (H2) and (H3) hold. Then
\[
\lim_{k \to \infty} \|g_k\| = 0.
\]

**Proof.** For the sake of contradiction, we suppose that the conclusion is not right. Then there exist a constant \( \epsilon > 0 \) and an infinite index set \( K \) such that
\[
\|g_k\| \geq \epsilon, \quad \forall k \in K.
\]
Moreover, the fact \( \alpha_k \leq 1 \), (35) and (H2) imply that
\[
\|g_k - g_{k-1}\|^2 \leq L^2 \alpha_k'^2 \|d_{k-1}\|^2 \leq L^2 \alpha_k \|d_{k-1}\|^2 \to 0.
\]
This and (42) indicate that there exists a positive constant \( \epsilon_1 \) such that for sufficiently large \( k \in K \), we have
\[
\|g_k\| \geq \epsilon_1.
\]
Then by (36) and (44), we can get
\[
\lim_{k \to \infty} \alpha_k = 0.
\]
By (3), (4), and (H2), for all $k \in K$, we have
\[
\|d_k\| \leq \|g_k\| + \|\beta_k^{\text{PRP}}\| \|d_{k-1}\| + \|\theta_k\|\|y_{k-1}\| \\
\leq \|g_k\| + \frac{2L\alpha_{k-1}\|d_{k-1}\|^2}{\|g_{k-1}\|^2} \|g_k\| \\
\leq \left(1 + \frac{2L\alpha_{k-1}\|d_{k-1}\|^2}{\epsilon^2}\right) \|g_k\|.
\]
From (35), for all sufficiently large $k \in K$, there exist a constant $r > 0$, such that
\[
\alpha_{k-1}\|d_{k-1}\|^2 \leq r.
\]
Therefore $\|d_k\| \leq M_2\|g_k\|$ with $M_2 = 1 + (2Lr/\epsilon^2)$. Thus, for $k \in K$, this and (37) imply that
\[
\|d_k\| \leq MM_2.
\]
Thus, from (38) and (48), we get
\[
\alpha_k \geq \min \left\{1, \frac{\rho}{(L + \mu M^2 M_2^2)} \right\}, \quad \forall k \in K,
\]
which contradicts (45). The proof is then completed.

4. Numerical Results

In this section, we present some numerical results to compare the performance of SDPRP method with the two new Armijo-type line searches I and II and the three-term PRP method in [7].

(i) SDPRPI: the SDPRP method with the line search (10), with $\mu = 10^{-4}$, $\rho = 0.5$, $\epsilon = 0.2$;

(ii) SDPRPII: the SDPRP method with the line search (11), with $\mu = 10^{-4}$, $\rho = 0.5$.

(iii) TTPRP: the two-term PRP method with the following Armijo-type line search: let $\alpha_k$ be the largest $\alpha$ in $\{1, \rho, \rho^2, \ldots\}$ such that
\[
f(x_k + \alpha d_k) \leq f(x_k) - \mu \alpha^2 \|d_k\|^2,
\]
where $\mu = 10^{-4}$, $\rho = 0.5$.

All codes were written in Matlab 7.1 and run on a portable computer. We stopped the iteration if the number of iteration exceeds 10000 or $\|g_k\| < 10^{-5}$. Tables 1 and 2 list the numerical results for solving some test problems numbered from 1 to 30 in [11] with different dimension $n$. Our numerical results are listed in the form NI/NF/CPU, where the symbols NI, NF, and

Table 1: The results for the methods on the tested problems.

<table>
<thead>
<tr>
<th>P</th>
<th>n</th>
<th>SDPRPI</th>
<th>SDPRPII</th>
<th>TTPRP</th>
</tr>
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<tbody>
<tr>
<td>FREUROTH</td>
<td>50</td>
<td>45/265/0.0313</td>
<td>152/1778/0.1719</td>
<td>62/633/0.0625</td>
</tr>
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<td>30/112/0.2188</td>
<td>218/2870/4.4375</td>
<td>26/122/0.2344</td>
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<td>SROSENBR</td>
<td>3000</td>
<td>72/116/0.8906</td>
<td>F</td>
<td>82/227/1.1875</td>
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<tr>
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<td>125/485/0.3906</td>
<td>170/2170/1.2031</td>
<td>71/760/0.4531</td>
</tr>
<tr>
<td>Extended penalty</td>
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<td>130/453/0.8906</td>
<td>F</td>
<td>26/250/0.0781</td>
</tr>
<tr>
<td>Perturbed quadratic</td>
<td>3000</td>
<td>29/328/0.2813</td>
<td>F</td>
<td>26/277/0.2500</td>
</tr>
<tr>
<td>Raydan1</td>
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<td>26/250/0.0781</td>
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<tr>
<td>Perturbed quadratic</td>
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<td>26/277/0.2500</td>
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<td>49/282/0.1406</td>
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<tr>
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<td>26/250/0.0781</td>
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<td>Extended penalty</td>
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<td>116/370/1.1875</td>
<td>41/233/0.6250</td>
<td>55/329/0.7813</td>
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<td>Hager</td>
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<td>49/282/0.1406</td>
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<td>6755/6756/10.9531</td>
<td>6753/6754/10.5625</td>
<td>6753/6754/10.4063</td>
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<tr>
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<td>100</td>
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<td>103/678/0.1250</td>
<td>103/678/0.1250</td>
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\(\|f\| = 5\) and (H2), for all $k \in K$, we have
\[
\|d_k\| \leq \|g_k\| + \|\beta_k^{\text{PRP}}\| \|d_{k-1}\| + \|\theta_k\|\|y_{k-1}\| \\
\leq \|g_k\| + \frac{2L\alpha_{k-1}\|d_{k-1}\|^2}{\|g_{k-1}\|^2} \|g_k\| \\
\leq \left(1 + \frac{2L\alpha_{k-1}\|d_{k-1}\|^2}{\epsilon^2}\right) \|g_k\|.
\]
Table 2: The results for the methods on the tested problems.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$n$</th>
<th>SDPRPI</th>
<th>SDPRPII</th>
<th>TTPRP</th>
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<tr>
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<td>66/533/3.8125</td>
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Figure 1: Performance profiles of three methods about the number of function evaluations.

Figure 2: Performance profiles of three methods about CPU time.

CPU mean the number of iterations, the number of function evaluations, and the CPU time in seconds, respectively.

Figures 1 and 2 show the performance of these methods relative to the number of function evaluations and CPU time, respectively, which are evaluated using the profiles of Dolan and Moré [12]. That is, for each method, we plot the fraction $P$ of problems for which the method is within a factor $\tau$ of the best time. The left side of the figure gives the percentage of the test problems for which a method is fastest; while the right side gives the percentage of the test problems that are
successfully solved by each of the methods. The top curve is the method that solved most problems in a time that was within a factor $\tau$ of the best time. Figures 1 and 2 show that SDPRP-I method performs a little better than TTPRP method and obviously better than SDPRP-II method. It solves about 72\% and 63\% of the problems with the smallest number of function evaluations and CPU time, respectively. Obviously, the performance of SDPRP-II method is not so good, and, in the future, we will further study the corresponding line search. Of course, more numerical experiments should be carried out to test our proposed methods.

5. Conclusion

In this paper, we have proposed two new Armijo-type line searches and proved that the sufficient descent PRP method proposed by Zhang et al. is strongly global convergent with the two new line searches. Numerical results show that the SDPRP method with the proposed line searches is efficient for the test problems.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

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