Research Article

Identities of Symmetry for Higher-Order Generalized q -Euler Polynomials

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We investigate the properties of symmetry in two variables related to multiple Euler q - l -function which interpolates higher-order q -Euler polynomials at negative integers. From our investigation, we can derive many interesting identities of symmetry in two variables related to generalized higher-order q -Euler polynomials and alternating generalized q -power sums.

1. Introduction

Throughout this paper, the notations $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of positive integers, integers, real numbers, and complex numbers, respectively, and $\mathbb{Z}_{\geq 0} := \mathbb{N} \cup \{0\}$. Let χ be a Dirichlet character with $d \in \mathbb{N}$ with conductor $d \equiv 1 \pmod{2}$. Then the generalized Euler polynomials attached to χ are defined by the following generating function (see [1–3]):

$$2 \sum_{a=0}^{d-1} \frac{\chi(a) (-1)^a e^{(a+x)t}}{e^{dt} + 1} = \sum_{n=0}^{\infty} E_{n,\chi}(x) \frac{t^n}{n!}. \quad (1)$$

The generalized Euler polynomials of order $r \in \mathbb{N}$ attached to χ are also defined by the generating function:

$$\left(2 \sum_{a=0}^{d-1} \frac{\chi(a) (-1)^a e^{(a+x)t}}{e^{dt} + 1} \right)^r = \sum_{n=0}^{\infty} E_{n,\chi}^{(r)}(x) \frac{t^n}{n!}. \quad (2)$$

When $x = 0$, $E_{n,\chi}^{(r)} = E_{n,\chi}^{(r)}(0)$ are called the generalized Euler numbers attached to χ (see [2, 4]).

Assume that $q \in \mathbb{C}$ with $|q| < 1$ and define q -numbers by (see [2–15])

$$[x]_q = \frac{1 - q^x}{1 - q}. \quad (3)$$

Note that $\lim_{q \rightarrow 1} [x]_q = x$.

In [4, 8], Kim initiated to consider various q -extensions (or (h, q) -extensions) of Euler numbers and polynomials and constructed analytic continuations which interpolate his q -numbers and polynomials. Until recently, many authors have studied q -Euler or (h, q) -Euler polynomials due to him (see [1–21]). In [4], Kim defined the (h, q) -extension of generalized higher-order Euler polynomials attached to χ which is given by the generating function:

$$F_{q,\chi}^{(h,r)}(t, x) = [2]_q^r \sum_{m_1, \dots, m_r=0}^{\infty} q^{\sum_{j=1}^r (h-j+1)m_j} (-1)^{\sum_{j=1}^r m_j} \times \left(\prod_{j=1}^r \chi(m_j) \right) e^{[x + \sum_{l=1}^r m_l]_q t}$$

$$= \sum_{n=0}^{\infty} E_{n,\chi,q}^{(h,r)}(x) \frac{t^n}{n!}, \quad (4)$$

where $h \in \mathbb{Z}$ and $r \in \mathbb{N}$.

Note that

$$\begin{aligned} \lim_{q \rightarrow 1} F_q^{(h,r)}(t, x) &= \left(2 \sum_{a=0}^{d-1} \frac{\chi(a) (-1)^a e^{(a+x)t}}{e^{dt} + 1} \right)^r \\ &= \sum_{n=0}^{\infty} E_{n,\chi}^{(r)}(x) \frac{t^n}{n!}. \end{aligned} \quad (5)$$

When $x = 0$, $E_{n,\chi,q}^{(h,r)} = E_{n,\chi,q}^{(h,r)}(0)$ are called the (h, q) -extension of generalized higher-order Euler numbers attached to χ .

We find from (4) that

$$\begin{aligned} E_{n,\chi,q}^{(h,r)}(x) &= \sum_{l=0}^n \binom{n}{l} q^{lx} E_{l,\chi,q}^{(h,r)} [x]_q^{n-l} \\ &= (q^x E_{\chi,q}^{(h,r)} + [x]_q)^n, \end{aligned} \quad (6)$$

with the usual convention about replacing $(E_{\chi,q}^{(h,r)})^n$ by $E_{n,\chi,q}^{(h,r)}$.

In [4], Dirichlet-type multiple (h, q) - l -function is defined by Kim to be

$$\begin{aligned} I_{q,r}^{(h)}(s, x | \chi) &= \frac{1}{\Gamma(s)} \int_0^\infty F_{q,\chi}^{(h,r)}(-t, x) t^{s-1} dt \\ &= [2]_q^r \sum_{m_1, \dots, m_r=0}^{\infty} \frac{q^{\sum_{l=1}^r (h-l+1)m_l} (\prod_{l=1}^r \chi(m_l)) (-1)^{\sum_{l=1}^r m_l}}{[m_1 + \dots + m_r + x]_q^s}, \end{aligned} \quad (7)$$

where $s, h \in \mathbb{C}$ and $x \in \mathbb{R}$, with $x \neq 0, -1, -2, \dots$.

By using Cauchy residue theorem, we get

$$I_{q,r}^{(h)}(-n, x | \chi) = E_{n,\chi,q}^{(h,r)}(x), \quad n \in \mathbb{Z}_{\geq 0}. \quad (8)$$

In this paper, we investigate certain properties of symmetry in two variables related to Dirichlet-type multiple (h, q) -function which interpolates the (h, q) -extension of generalized higher-order Euler polynomials attached to χ at negative integers. From our investigation, we can derive many interesting identities of symmetry in two variables related to (h, q) -extension of generalized higher-order Euler polynomials and alternating generalized q -power sums.

2. Identities for the (h, q) -Extension of Generalized Higher-Order Euler Polynomials

In this section, we assume that χ is a Dirichlet character with conductor $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$.

Let $w_1, w_2, r \in \mathbb{N}$ with $w_1 \equiv 1 \pmod{2}$ and $w_2 \equiv 1 \pmod{2}$ and $h \in \mathbb{Z}$. First, we observe that

$$\begin{aligned} &\frac{1}{[2]_{q^{w_1}}^r} I_{q^{w_1},r}^{(h)} \left(s, w_2 x + \frac{w_2}{w_1} \sum_{l=1}^r j_l | \chi \right) \\ &= \sum_{m_1, \dots, m_r=0}^{\infty} \left((-1)^{m_1 + \dots + m_r} q^{w_1 \sum_{l=1}^r (h-l+1)m_l} \left(\prod_{l=1}^r \chi(m_l) \right) \right. \\ &\quad \times \left(\left[m_1 + \dots + m_r + w_2 x \right. \right. \\ &\quad \left. \left. + \frac{w_2}{w_1} (j_1 + \dots + j_r) \right]_{q^{w_1}}^s \right)^{-1} \Bigg) \\ &= \sum_{m_1, \dots, m_r=0}^{\infty} \left(q^{w_1 \sum_{l=1}^r (h-l+1)m_l} (-1)^{\sum_{l=1}^r m_l} \left(\prod_{l=1}^r \chi(m_l) \right) [w_1]_q^s \right. \\ &\quad \times \left([w_2 (j_1 + \dots + j_r) + w_1 w_2 x \right. \\ &\quad \left. \left. + w_1 (m_1 + \dots + m_r) \right]_q^s \right)^{-1} \Bigg) \\ &= [w_1]_q^s \\ &\quad \times \sum_{n_1, \dots, n_r=0}^{\infty} \sum_{i_1, \dots, i_r=0}^{dw_2-1} \left((-1)^{\sum_{l=1}^r (i_l + n_l)} q^{w_1 \sum_{l=1}^r (h-l+1)(i_l + n_l)db} \right. \\ &\quad \times \left(\prod_{l=1}^r \chi(i_l) \right) \\ &\quad \times \left(\left[w_1 w_2 \left(x + d \sum_{l=1}^r n_l \right) \right. \right. \\ &\quad \left. \left. + w_2 \sum_{l=1}^r j_l + w_1 \sum_{l=1}^r i_l \right]_q^s \right)^{-1} \Bigg) \\ &= [w_1]_q^s \\ &\quad \times \sum_{n_1, \dots, n_r=0}^{\infty} \sum_{i_1, \dots, i_r=0}^{w_2 d-1} \left((-1)^{\sum_{l=1}^r (i_l + n_l)} q^{w_1 \sum_{l=1}^r (h-l+1)(i_l + n_l)w_2 d} \right. \\ &\quad \times \left(\prod_{l=1}^r \chi(i_l) \right) \\ &\quad \times \left(\left[w_1 w_2 \left(x + d \sum_{l=1}^r n_l \right) \right. \right. \\ &\quad \left. \left. + w_2 \sum_{l=1}^r j_l + w_1 \sum_{l=1}^r i_l \right]_q^s \right)^{-1} \Bigg). \end{aligned} \quad (9)$$

Thus, by (9), we get

$$\begin{aligned}
 & \frac{[w_2]_q^s}{[2]_{q^{w_1}}^r} \sum_{j_1, \dots, j_r=0}^{dw_1-1} (-1)^{\sum_{l=1}^r j_l} \left(\prod_{l=1}^r \chi(j_l) \right) \\
 & \quad \times q^{b \sum_{l=1}^r (h-l+1) j_l} l_{q^{w_1}, r}^{(h)} \left(s, w_2 x + \frac{w_2}{w_1} \sum_{l=1}^r j_l \mid \chi \right) \\
 &= [w_1]_q^s [w_2]_q^s \\
 & \quad \times \sum_{i_1, \dots, i_r=0}^{dw_2-1} \sum_{j_1, \dots, j_r=0}^{dw_1-1} \sum_{n_1, \dots, n_r=0}^{\infty} \left(\left((-1)^{\sum_{l=1}^r (i_l + j_l + n_l)} \right. \right. \\
 & \quad \times \left(\prod_{l=1}^r \chi(j_l) \right) \left(\prod_{l=1}^r \chi(i_l) \right) \\
 & \quad \times q^{w_2 \sum_{l=1}^r (h-l+1) j_l + w_1 \sum_{l=1}^r (h-l+1) i_l} \\
 & \quad \times \left(\left[w_1 w_2 \left(x + d \sum_{l=1}^r n_l \right) \right. \right. \\
 & \quad \left. \left. + w_2 \sum_{l=1}^r j_l + w_1 \sum_{l=1}^r i_l \right]^s \right)^{-1} \Bigg) \\
 & \quad \times q^{w_1 w_2 d \sum_{l=1}^r (h-l+1) n_l}. \tag{10}
 \end{aligned}$$

By using the same method as (10), we get

$$\begin{aligned}
 & \frac{[w_1]_q^s}{[2]_{q^{w_2}}^s} \sum_{j_1, \dots, j_r=0}^{dw_2-1} (-1)^{\sum_{l=1}^r j_l} \left(\prod_{l=1}^r \chi(j_l) \right) \\
 & \quad \times q^{w_1 \sum_{l=1}^r (h-l+1) j_l} l_{q^{w_2}, r}^{(h)} \left(s, w_1 x + \frac{w_1}{w_2} \sum_{l=1}^r j_l \mid \chi \right) \\
 &= [w_1]_q^s [w_2]_q^s \\
 & \quad \times \sum_{j_1, \dots, j_r=0}^{dw_2-1} \sum_{i_1, \dots, i_r=0}^{dw_1-1} \sum_{n_1, \dots, n_r=0}^{\infty} \left(\left((-1)^{\sum_{l=1}^r (i_l + j_l + n_l)} \right. \right. \\
 & \quad \times \left(\prod_{l=1}^r \chi(j_l) \right) \left(\prod_{l=1}^r \chi(i_l) \right) \\
 & \quad \times q^{w_1 \sum_{l=1}^r (h-l+1) j_l + w_2 \sum_{l=1}^r (h-l+1) i_l} \\
 & \quad \times \left(\left[w_1 w_2 \left(x + d \sum_{l=1}^r n_l \right) \right. \right. \\
 & \quad \left. \left. + w_1 \sum_{l=1}^r j_l + w_2 \sum_{l=1}^r i_l \right]^s \right)^{-1} \Bigg) \\
 & \quad \times q^{w_1 w_2 d \sum_{l=1}^r (h-l+1) n_l}. \tag{11}
 \end{aligned}$$

Therefore, by (10) and (11), we obtain the following theorem.

Theorem 1. For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod{2}$ and $w_2 \equiv 1 \pmod{2}$, one has

$$\begin{aligned}
 & [2]_{q^{w_2}}^r [w_2]_q^s \sum_{j_1, \dots, j_r=0}^{w_1 d-1} (-1)^{\sum_{l=1}^r j_l} \left(\prod_{l=1}^r \chi(j_l) \right) \\
 & \quad \times q^{w_2 \sum_{l=1}^r (h-l+1) j_l} l_{q^{w_1}, r}^{(h)} \\
 & \quad \times \left(s, w_2 x + \frac{w_2}{w_1} \sum_{l=1}^r j_l \mid \chi \right) \\
 &= [2]_{q^{w_1}}^r [w_1]_q^s \sum_{j_1, \dots, j_r=0}^{w_2 d-1} (-1)^{\sum_{l=1}^r j_l} \left(\prod_{l=1}^r \chi(j_l) \right) \\
 & \quad \times q^{w_1 \sum_{l=1}^r (h-l+1) j_l} l_{q^{w_2}, r}^{(h)} \\
 & \quad \times \left(s, w_1 x + \frac{w_1}{w_2} \sum_{l=1}^r j_l \mid \chi \right). \tag{12}
 \end{aligned}$$

By (8) and Theorem 1, we obtain the following theorem.

Theorem 2. For $n \in \mathbb{Z}_{\geq 0}$ and $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod{2}$ and $w_2 \equiv 1 \pmod{2}$, one has

$$\begin{aligned}
 & [2]_{q^{w_2}}^r [w_1]_q^n \sum_{j_1, \dots, j_r=0}^{w_1 d-1} (-1)^{\sum_{l=1}^r j_l} \left(\prod_{l=1}^r \chi(j_l) \right) \\
 & \quad \times q^{w_2 \sum_{l=1}^r (h-l+1) j_l} E_{n, \chi, q^{w_1}}^{(h, r)} \left(w_2 x + \frac{w_2}{w_1} \sum_{l=1}^r j_l \right) \\
 &= [2]_{q^{w_1}}^r [w_2]_q^n \\
 & \quad \times \sum_{j_1, \dots, j_r=0}^{w_2 d-1} (-1)^{\sum_{l=1}^r j_l} \left(\prod_{l=1}^r \chi(j_l) \right) \\
 & \quad \times q^{w_1 \sum_{l=1}^r (h-l+1) j_l} E_{n, \chi, q^{w_2}}^{(h, r)} \left(w_1 x + \frac{w_1}{w_2} \sum_{l=1}^r j_l \right). \tag{13}
 \end{aligned}$$

From (6), we note that

$$\begin{aligned}
 E_{n, \chi, q}^{(h, r)}(x + y) &= (q^{x+y} E_{\chi, q}^{(h, r)} + [x + y]_q)^n \\
 &= (q^{x+y} E_{\chi, q}^{(h, r)} + q^x [y]_q + [x]_q)^n \\
 &= \sum_{i=0}^n \binom{n}{i} q^{xi} (q^y E_{\chi, q}^{(h, r)} + [y]_q)^i [x]_q^{n-i} \\
 &= \sum_{i=0}^n \binom{n}{i} q^{xi} E_{i, \chi, q}^{(h, r)}(y) [x]_q^{n-i}. \tag{14}
 \end{aligned}$$

By (14), we get

$$\begin{aligned}
& \sum_{j_1, \dots, j_r=0}^{dw_1-1} (-1)^{\sum_{l=1}^r j_l} q^{w_2 \sum_{l=1}^r (h-l+1)j_l} \left(\prod_{l=1}^r \chi(j_l) \right) \\
& \quad \times E_{n, \chi, q^{w_1}}^{(h,r)} \left(w_2 x + \frac{w_2}{w_1} \sum_{l=1}^r j_l \right) \\
& = \sum_{j_1, \dots, j_r=0}^{dw_1-1} (-1)^{\sum_{l=1}^r j_l} q^{w_2 \sum_{l=1}^r (h-l+1)j_l} \left(\prod_{l=1}^r \chi(j_l) \right) \\
& \quad \times \sum_{i=0}^n \binom{n}{i} q^{i w_2 (j_1 + \dots + j_r)} E_{i, \chi, q^{w_1}}^{(h,r)} (w_2 x) \\
& \quad \times \left[\frac{w_2 (j_1 + \dots + j_r)}{w_1} \right]_{q^{w_1}}^{n-i} \\
& = \sum_{j_1, \dots, j_r=0}^{dw_1-1} (-1)^{\sum_{l=1}^r j_l} q^{w_2 \sum_{l=1}^r (h-l+1)j_l} \left(\prod_{l=1}^r \chi(j_l) \right) \\
& \quad \times \sum_{i=0}^n \binom{n}{i} q^{(n-i)w_2 \sum_{l=1}^r j_l} E_{n-i, \chi, q^{w_1}}^{(h,r)} (w_2 x) \left[\frac{w_2}{w_1} \sum_{l=1}^r j_l \right]_{q^{w_1}}^i \\
& = \sum_{i=0}^n \binom{n}{i} \left(\frac{[w_2]_q}{[w_1]_q} \right)^i E_{n-i, \chi, q^{w_1}}^{(h,r)} (w_2 x) \\
& \quad \times \sum_{j_1, \dots, j_r=0}^{dw_1-1} (-1)^{\sum_{l=1}^r j_l} q^{w_2 \sum_{l=1}^r (h-l+n-i+1)j_l} \\
& \quad \times \left(\prod_{l=1}^r \chi(j_l) \right) [j_1 + \dots + j_r]_{q^{w_2}}^i \\
& = \sum_{i=0}^n \binom{n}{i} \left(\frac{[w_2]_q}{[w_1]_q} \right)^i E_{n-i, \chi, q^{w_1}}^{(h,r)} (w_2 x) S_{n,i, q^{w_2}}^{(h,r)} (w_1 d \mid \chi),
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
S_{n,i, q}^{(h,r)} (w \mid \chi) & = \sum_{j_1, \dots, j_r=0}^{w-1} (-1)^{\sum_{l=1}^r j_l} q^{\sum_{l=1}^r (h-l+n-i+1)j_l} \\
& \quad \times [j_1 + \dots + j_r]_q^i \left(\prod_{l=1}^r \chi(j_l) \right).
\end{aligned} \tag{16}$$

From (15), we have

$$\begin{aligned}
& [2]_{q^{w_2}}^r [w_1]_q^n \\
& \quad \times \sum_{j_1, \dots, j_r=0}^{dw_1-1} (-1)^{\sum_{l=1}^r j_l} q^{w_2 \sum_{l=1}^r (h-l+1)j_l} \\
& \quad \times \left(\prod_{l=1}^r \chi(j_l) \right) E_{n, \chi, q^{w_1}}^{(h,r)}
\end{aligned}$$

$$\begin{aligned}
& \times \left(w_2 x + \frac{w_2}{w_1} (j_1 + \dots + j_r) \right) \\
& = [2]_{q^{w_2}}^r \sum_{i=0}^n \binom{n}{i} [w_1]_q^{n-i} [w_2]_q^i E_{n-i, \chi, q^{w_1}}^{(h,r)} (w_2 x) S_{n,i, q^{w_2}}^{(h,r)} \\
& \quad \times (w_1 d \mid \chi).
\end{aligned} \tag{17}$$

By using the same method as in (17), we get

$$\begin{aligned}
& [2]_{q^{w_1}}^r [w_2]_q^n \sum_{j_1, \dots, j_r=0}^{dw_2-1} (-1)^{\sum_{l=1}^r j_l} q^{w_1 \sum_{l=1}^r (h-l+1)j_l} \\
& \quad \times \left(\prod_{l=1}^r \chi(j_l) \right) E_{n, \chi, q^{w_2}}^{(h,r)} \left(w_1 x + \frac{w_1}{w_2} \sum_{l=1}^r j_l \right) \\
& = [2]_{q^{w_1}}^r \sum_{i=0}^n \binom{n}{i} [w_2]_q^{n-i} [w_1]_q^i E_{n-i, \chi, q^{w_2}}^{(h,r)} \\
& \quad \times (w_1 x) S_{n,i, q^{w_1}}^{(h,r)} (w_2 d \mid \chi).
\end{aligned} \tag{18}$$

Therefore, by (17) and (18), we obtain the following theorem.

Theorem 3. For $n \in \mathbb{Z}_{\geq 0}$ and $w_1, w_2 \in \mathbb{N}$, with $w_1 \equiv 1 \pmod{2}$ and $w_2 \equiv 1 \pmod{2}$, one has

$$\begin{aligned}
& [2]_{q^{w_2}}^r \sum_{i=0}^n \binom{n}{i} [w_1]_q^{n-i} [w_2]_q^i E_{n-i, \chi, q^{w_1}}^{(h,r)} (w_2 x) S_{n,i, q^{w_2}}^{(h,r)} (w_1 d \mid \chi) \\
& = [2]_{q^{w_1}}^r \sum_{i=0}^n \binom{n}{i} [w_2]_q^{n-i} [w_1]_q^i E_{n-i, \chi, q^{w_2}}^{(h,r)} (w_1 x) S_{n,i, q^{w_1}}^{(h,r)} \\
& \quad \times (w_2 d \mid \chi).
\end{aligned} \tag{19}$$

Now, we observe that

$$\begin{aligned}
& e^{[x]_{q^u}} \sum_{m_1, \dots, m_r=0}^{\infty} q^{\sum_{l=1}^r (h-l+1)m_l} (-1)^{\sum_{l=1}^r m_l} \\
& \quad \times \left(\prod_{l=1}^r \chi(m_l) \right) e^{[y + \sum_{l=1}^r m_l]_q q^x (u+v)} \\
& = e^{-[x]_{q^u}} \sum_{m_1, \dots, m_r=0}^{\infty} q^{\sum_{l=1}^r (h-l+1)m_l} (-1)^{\sum_{l=1}^r m_l} \\
& \quad \times \left(\prod_{l=1}^r \chi(m_l) \right) e^{[x+y + \sum_{l=1}^r m_l]_q (u+v)}.
\end{aligned} \tag{20}$$

The left hand side of (20) multiplied by $[2]_q^r$ is given by

$$\begin{aligned}
& [2]_q^r e^{[x]_{q^u}} \\
& \quad \times \sum_{m_1, \dots, m_r=0}^{\infty} q^{\sum_{l=1}^r (h-l+1)m_l} (-1)^{\sum_{l=1}^r m_l} e^{[y + \sum_{l=1}^r m_l]_q q^x (u+v)}
\end{aligned}$$

$$\begin{aligned}
 & \times \left(\prod_{l=1}^r \chi(m_l) \right) \\
 &= e^{[x]_q u} \sum_{n=0}^{\infty} q^{nx} E_{n,\chi,q}^{(h,r)}(y) \frac{(u+v)^n}{n!} \\
 &= \left(\sum_{l=0}^{\infty} [x]_q^l \frac{u^l}{l!} \right) \\
 & \quad \times \left(\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} q^{(k+n)x} E_{k+n,\chi,q}^{(h,r)}(y) \frac{u^k}{k!} \frac{v^n}{n!} \right) \\
 &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\sum_{k=0}^m \binom{m}{k} q^{(k+n)x} E_{k+n,\chi,q}^{(h,r)}(y) [x]_q^{m-k} \right) \\
 & \quad \times \frac{u^m}{m!} \frac{v^n}{n!}.
 \end{aligned} \tag{21}$$

The right hand side of (20) multiplied by $[2]_q^r$ is given by

$$\begin{aligned}
 & [2]_q^r e^{-[x]_q v} \sum_{m_1, \dots, m_r=0}^{\infty} (-1)^{\sum_{l=1}^r m_l} q^{\sum_{l=1}^r (h-l+1)m_l} \\
 & \quad \times \left(\prod_{l=1}^r \chi(m_l) \right) e^{[x+\sum_{l=1}^r m_l]_q (u+v)} \\
 &= e^{-[x]_q v} \sum_{n=0}^{\infty} E_{n,\chi,q}^{(h,r)}(x+y) \frac{(u+v)^n}{n!} \\
 &= \left(\sum_{l=0}^{\infty} \frac{(-[x]_q)^l}{l!} v^l \right) \\
 & \quad \times \left(\sum_{m=0}^{\infty} \sum_{k=0}^{\infty} E_{m+k,\chi,q}^{(h,r)}(x+y) \frac{u^m}{m!} \frac{v^k}{k!} \right) \\
 &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left(\sum_{k=0}^n \binom{n}{k} E_{m+k,\chi,q}^{(h,r)}(x+y) (-[x]_q)^{n-k} \right) \\
 & \quad \times \frac{u^m}{m!} \frac{v^n}{n!} \\
 &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left(\sum_{k=0}^n \binom{n}{k} E_{m+k,\chi,q}^{(h,r)}(x+y) q^{(n-k)x} [-x]_q^{n-k} \right) \\
 & \quad \times \frac{u^m}{m!} \frac{v^n}{n!}.
 \end{aligned} \tag{22}$$

Therefore, by (21) and (22), we obtain the following theorem.

Theorem 4. For $m, n \in \mathbb{Z}_{\geq 0}$, one has

$$\begin{aligned}
 & \sum_{k=0}^m \binom{m}{k} q^{kx} E_{n+k,\chi,q}^{(h,r)}(y) [x]_q^{m-k} \\
 &= \sum_{k=0}^n \binom{n}{k} q^{-kx} E_{m+k,\chi,q}^{(h,r)}(x+y) [-x]_q^{n-k}.
 \end{aligned} \tag{23}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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