Research Article

Implicit Vector Integral Equations Associated with Discontinuous Operators

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Let $I := [0, 1]$. We consider the vector integral equation $h(u(t)) = f\left(t, \int_I g(t, z)u(z)dz\right)$ for a.e. $t \in I$, where $f : I \times J \rightarrow \mathbf{R}$, $g : I \times I \rightarrow [0, +\infty[$, and $h : X \rightarrow \mathbf{R}$ are given functions and X, J are suitable subsets of $\mathbf{R}^n$. We prove an existence result for solutions $u \in L^s(I, \mathbf{R}^n)$, where the continuity of f with respect to the second variable is not assumed. More precisely, f is assumed to be a.e. equal (with respect to second variable) to a function $f^* : I \times J \rightarrow \mathbf{R}$ which is almost everywhere continuous, where the involved null-measure sets should have a suitable geometry. It is easily seen that such a function f can be discontinuous at each point $x \in J$. Our result, based on a very recent selection theorem, extends a previous result, valid for scalar case $n = 1$.

1. Introduction

Let $I := [0, 1]$. Recently, in the paper [1], the following integral equation was studied: given $\lambda > 0$, $f : I \times [0, \lambda] \rightarrow \mathbf{R}$, $g : I \times I \rightarrow [0, +\infty[$, $h :]0, +\infty[\rightarrow \mathbf{R}$, and $s > 1$; find $u \in L^s(I)$ such that

$$h(u(t)) = f\left(t, \int_I g(t, z)u(z)dz\right) \quad \text{for a.e. } t \in I. \quad (1)$$

In [1], the following existence result was proved, where, unlike other results in the field (see, for instance, the papers [2–5] and references therein, to which we also refer for motivations for studying (1)), the continuity of f with respect to the second variable was not assumed.

Theorem 1 (Theorem 1 of [1]). *Let $\lambda > 0$, $A \subseteq]0, +\infty[$ a closed interval, $h : A \rightarrow \mathbf{R}$ a continuous function, $f : I \times [0, \lambda] \rightarrow \mathbf{R}$, and $g : I \times I \rightarrow [0, +\infty[$ two given functions. Let $s \in]1, +\infty[$, $\phi_0 \in L^j(I)$, with $j > 1$ and $j \geq s'$ (the conjugate exponent of s), $\phi_1 \in L^{s'}(I)$, and $\beta \in L^s(I)$. Assume that*

- (i) *there exists a function $f^* : I \times [0, \lambda] \rightarrow \mathbf{R}$, two negligible sets $E_1, E_2 \subseteq [0, \lambda]$, with E_2 closed, and a countable dense subset D of $[0, \lambda]$, such that for all $x \in D$ the function $f^*(\cdot, x)$ is measurable, and for a.e. $t \in I$ one has*

$$\begin{aligned} \{x \in [0, \lambda] : f(t, x) \neq f^*(t, x)\} &\subseteq E_1, \\ \{x \in [0, \lambda] : f^*(t, \cdot) \text{ is discontinuous at } x\} &\subseteq E_2; \end{aligned} \quad (2)$$

- (ii) *for all $z \in \text{int}(h(A))$ (the interior of $h(A)$), one has $\text{int } h^{-1}(z) = \emptyset$;*

- (iii) *if one puts, for all $t \in I$,*

$$v(t) := \text{ess inf}_{x \in [0, \lambda]} f(t, x), \quad z(t) := \text{ess sup}_{x \in [0, \lambda]} f(t, x), \quad (3)$$

then for a.e. $t \in I$ one has

$$[v(t), z(t)] \subseteq h(A), \quad \sup h^{-1}([v(t), z(t)]) \leq \beta(t); \quad (4)$$

(iv) one has

$$0 < \|\phi_0\|_{L^{s'}(I)} \leq \frac{\lambda}{\|\beta\|_{L^s(I)}}; \quad (5)$$

(v) for all $t \in I$, the function $g(t, \cdot)$ is measurable;

(vi) for a.e. $z \in I$, the function $g(\cdot, z)$ is continuous in I , differentiable in $]0, 1[$ and

$$g(t, z) \leq \phi_0(z), \quad 0 < \frac{\partial g}{\partial t}(t, z) \leq \phi_1(z) \quad \forall t \in]0, 1[. \quad (6)$$

Then, there exists a solution $u \in L^s(I)$ to (1).

Of course, the main peculiarity of Theorem 1 resides in the kind of discontinuity that is allowed for f . Indeed, it is easy to construct examples of functions f, g , and h satisfying the assumptions of Theorem 1 and such that for all $t \in I$ the function $f(t, \cdot)$ is discontinuous at all points $x \in [0, \lambda]$.

Theorem 1 extends a previous result (Theorem 1 of [6]), valid for the case where f does not depend on t explicitly. At this point, it is natural to ask if Theorem 1 above can be extended to the more general case where the function u takes its values in the space $\mathbf{R}^n$. In this direction, we note that some results exist for the vector explicit equation

$$u(t) = f\left(t, \int_I g(t, z) u(z) dz\right) \quad (7)$$

(see [7, 8]), while for the implicit equation (1) the problem is still unsolved.

The aim of this note is exactly to provide such an extension. In the following, if $n \in \mathbf{N}$ and $i \in \{1, \dots, n\}$, we will denote by $P_i : \mathbf{R}^n \rightarrow \mathbf{R}$ the projection over the i th axis. Moreover, we will denote by m_n the n -dimensional Lebesgue measure over $\mathbf{R}^n$. If $\lambda := (\lambda_1, \dots, \lambda_n) \in \mathbf{R}^n$, with $\lambda_i > 0$ for all $i = 1, \dots, n$, we will put $J_\lambda := \prod_{i=1}^n [0, \lambda_i]$. Finally, if n and λ are as above, we will denote by $\mathcal{F}_{n,\lambda}$ the family of all subsets $F \subseteq J_\lambda$ such that there exist sets $F_1, F_2, \dots, F_n \subseteq \mathbf{R}^n$, with $m_1(P_i(F_i)) = 0$ for all $i = 1, \dots, n$, such that $F := \bigcup_{i=1}^n F_i$. The following is our main result (where $\mathbf{R}_+^n$ denotes the positive open orthant of $\mathbf{R}^n$, and $\text{int}_A(B)$ is the interior of B in A).

Theorem 2. Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{R}_+^n$, and let $X \subseteq \mathbf{R}_+^n$ be a nonempty, closed, connected, and locally connected subset of $\mathbf{R}^n$, with $\inf P_i(X) > 0$ for all $i = 1, \dots, n$. Let $h : X \rightarrow \mathbf{R}$ be a continuous function and $f : I \times J_\lambda \rightarrow \mathbf{R}$ and $g : I \times I \rightarrow [0, +\infty[$ two given functions. Let $s \in]1, +\infty]$, $\phi_0 \in L^j(I)$, with $j > 1$ and $j \geq s'$, $\phi_1 \in L^{s'}(I)$, and $\beta \in L^s(I, \mathbf{R}^n)$. Finally, let D be a countable dense subset of J_λ .

Assume that there exists a function $f^* : I \times J_\lambda \rightarrow \mathbf{R}$ and two sets $E, F \in \mathcal{F}_{n,\lambda}$, with F closed, such that

- (i) for all $x \in D$ the function $f^*(\cdot, x)$ is measurable;
- (ii) for a.e. $t \in I$ one has

$$\begin{aligned} \{x \in J_\lambda : f(t, x) \neq f^*(t, x)\} &\subseteq E, \\ \{x \in J_\lambda : f^*(t, \cdot) \text{ is discontinuous at } x\} &\subseteq F. \end{aligned} \quad (8)$$

Moreover, assume that

(iii) $\text{int}_X(h^{-1}(t)) = \emptyset$, for all $t \in \text{int}_{\mathbf{R}}(h(X))$;

(iv) if one puts, for all $t \in I$;

$$v(t) := \text{ess inf}_{x \in J_\lambda} f(t, x), \quad z(t) := \text{ess sup}_{x \in J_\lambda} f(t, x), \quad (9)$$

then for a.e. $t \in I$ and all $i = 1, \dots, n$ one has

$$\begin{aligned} [v(t), z(t)] &\subseteq h(X), \\ \sup P_i(h^{-1}([v(t), z(t)])) &\leq \beta_i(t), \end{aligned} \quad (10)$$

(where $\beta_i : I \rightarrow \mathbf{R}$ denotes the i th component of the function β);

(v) one has

$$0 < \|\phi_0\|_{L^{s'}(I)} \leq \min_{1 \leq i \leq n} \frac{\lambda_i}{\|\beta_i\|_{L^s(I)}}; \quad (11)$$

(vi) for all $t \in I$, the function $g(t, \cdot)$ is measurable;

(vii) for a.e. $z \in I$, the function $g(\cdot, z)$ is continuous in I , differentiable in $]0, 1[$ and

$$g(t, z) \leq \phi_0(z), \quad 0 < \frac{\partial g}{\partial t}(t, z) \leq \phi_1(z) \quad \forall t \in]0, 1[. \quad (12)$$

Then, there exists $u \in L^s(I, \mathbf{R}^n)$ such that

$$h(u(t)) = f\left(t, \int_I g(t, z) u(z) dz\right) \quad \text{for a.e. } t \in I. \quad (13)$$

Theorem 2 will be proved as an application of the following selection theorem, recently proved in [9], which we now state for the reader's convenience (in the following, if S is a topological space, we will denote by $\mathcal{B}(S)$ the Borel family of S).

Theorem 3 (Theorem 2.2 of [9]). Let T and $X_1, X_2, \dots, X_k$ be complete separable metric spaces, with $k \in \mathbf{N}$, and let $X := \prod_{j=1}^k X_j$ (endowed with the product topology). Let $\mu, \psi_1, \dots, \psi_k$ be positive regular Borel measures over $T, X_1, X_2, \dots, X_k$, respectively, with μ finite and $\psi_1, \dots, \psi_k$ σ -finite.

Let S be a separable metric space, and let $F : T \times X \rightarrow 2^S$ be a multifunction with nonempty complete values. Finally, let $E \subseteq X$ be a given set, and, for each $i \in \{1, \dots, k\}$, let $P_{*,i} : X \rightarrow X_i$ be the projection over X_i . Assume that

- (i) the multifunction F is $\mathcal{T}_\mu \otimes \mathcal{B}(X_1) \otimes \dots \otimes \mathcal{B}(X_k)$ -measurable (where $\mathcal{T}_\mu$ denotes the completion of the Borel σ -algebra $\mathcal{B}(T)$ of T with respect to the measure μ);
- (ii) for a.e. $t \in T$, one has

$$\begin{aligned} \{x := (x_1, \dots, x_k) \in X : F(t, \cdot) \\ \text{is not lower semicontinuous at } x\} &\subseteq E. \end{aligned} \quad (14)$$

Then, there exist sets $Q_1, \dots, Q_k$, with $Q_i \in \mathcal{B}(X_i)$ and $\psi_i(Q_i) = 0$ for all $i = 1, \dots, k$, and a function $\phi : T \times X \rightarrow S$ such that

- (a) $\phi(t, x) \in F(t, x)$ for all $(t, x) \in T \times X$;
 - (b) for all $x := (x_1, x_2, \dots, x_k) \in X \setminus [(\bigcup_{i=1}^k P_{*,i}^{-1}(Q_i)) \cup E]$, the function $\phi(\cdot, x)$ is $\mathcal{T}_\mu$ -measurable over T ;
 - (c) for a.e. $t \in T$, one has
- $$\{x := (x_1, x_2, \dots, x_k) \in X : \phi(t, \cdot) \text{ is discontinuous at } x\} \subseteq E \cup \left[\bigcup_{i=1}^k P_{*,i}^{-1}(Q_i) \right]. \quad (15)$$

The proof of Theorem 2 will be given in Section 2. Further, we will point out some counterexamples to possible improvements of Theorem 2.

2. Proof of Theorem 2

Before giving the proof of Theorem 2, we fix some notations. If $n \in \mathbb{N}$, the space $\mathbf{R}^n$ will be considered with its Euclidean norm $\|\cdot\|_n$. Moreover, if $x \in \mathbf{R}^n$ and $r > 0$, we put

$$B(x, r) := \{v \in \mathbf{R}^n : \|v - x\|_n < r\}, \quad (16)$$

$$\bar{B}(x, r) := \{v \in \mathbf{R}^n : \|v - x\|_n \leq r\}.$$

If $p \in [1, +\infty]$, the space $L^p(I, \mathbf{R}^n)$ will be considered with the usual norm

$$\|u\|_{L^p(I, \mathbf{R}^n)} := \left(\int_I \|u(t)\|_n^p dt \right)^{1/p} \quad \text{if } p < +\infty, \quad (17)$$

$$\|u\|_{L^\infty(I, \mathbf{R}^n)} := \operatorname{ess\,sup}_{t \in I} \|u(t)\|_n \quad \text{if } p = +\infty.$$

As usual, we put $L^p(I) := L^p(I, \mathbf{R})$. For the basic definitions and facts about multifunctions, we refer to [10].

Proof of Theorem 2. Without loss of generality, we can assume that (8) and (10) hold for all $t \in I$. Moreover, we can suppose that $j < +\infty$. Firstly, we prove that the functions v and z are measurable. Observe that, by assumption (ii), for all $t \in I$, one has

$$v(t) = \inf_{x \in J_\lambda \setminus F} f^*(t, x), \quad z(t) = \sup_{x \in J_\lambda \setminus F} f^*(t, x). \quad (18)$$

To see this, fix $t \in I$, and let $\psi(t) := \sup_{x \in J_\lambda \setminus F} f^*(t, x)$. Since $m_n(E \cup F) = 0$ we get

$$z(t) \leq \sup_{x \in J_\lambda \setminus (E \cup F)} f(t, x) = \sup_{x \in J_\lambda \setminus (E \cup F)} f^*(t, x) \leq \psi(t). \quad (19)$$

Now, assume that $z(t) < \psi(t)$. Hence, there is $x^* \in J_\lambda \setminus F$ such that $f^*(t, x^*) > z(t)$. Since the function $f^*(t, \cdot)$ is continuous at x^* , there exist $\delta, \varepsilon > 0$ such that

$$f^*(t, x) > z(t) + \varepsilon \quad \forall x \in J_\lambda \cap B(x^*, \delta). \quad (20)$$

Since $m_n(J_\lambda \cap B(x^*, \delta)) > 0$, we get

$$z(t) := \operatorname{ess\,sup}_{x \in J_\lambda} f(t, x) = \operatorname{ess\,sup}_{x \in J_\lambda} f^*(t, x) \geq z(t) + \varepsilon, \quad (21)$$

which is absurd. Therefore, the second equality in (18) is proved. The first one can be checked in analogous way.

Since F is closed, it can be easily checked that the set $D \cap (J_\lambda \setminus F)$ is nonempty, countable, and dense in $(J_\lambda \setminus F)$. Consequently, by Lemma at page 198 of [11], the function $f^*|_{I \times (J_\lambda \setminus F)}$ is $\mathcal{L}(I) \otimes \mathcal{B}(J_\lambda \setminus F)$ -measurable (where $\mathcal{L}(I)$ denotes the family of all Lebesgue-measurable subsets of I). By (18) and Lemma III.39 of [12], it follows that the functions v and z are measurable over I , as claimed.

By assumption (iii) and Theorem 2.4 di [13], there exists a set $Y \subseteq X$ such that $h(Y) = h(X)$ and the function $h|_Y$ is open (it carries open subsets of Y onto open subsets of $h(X) = h(Y)$). Consequently, the multifunction $T : h(X) \rightarrow 2^Y$ defined by putting, for each $s \in h(X)$,

$$T(s) := h^{-1}(s) \cap Y, \quad (22)$$

is lower semicontinuous in $h(X)$ with nonempty values. Let $f_0 : I \times J_\lambda \rightarrow \mathbf{R}$ be defined by putting, for all $(t, x) \in I \times J_\lambda$,

$$f_0(t, x) = \begin{cases} f^*(t, x) & \text{if } x \notin F \\ z(t) & \text{if } x \in F. \end{cases} \quad (23)$$

Clearly, the function f_0 is $\mathcal{L}(I) \otimes \mathcal{B}(J_\lambda)$ -measurable and, by (18), one has

$$v(t) \leq f_0(t, x) \leq z(t) \quad \forall (t, x) \in I \times J_\lambda. \quad (24)$$

Moreover, assumption (ii) and the closedness of F imply that for all $t \in I$ one has

$$\{x \in J_\lambda : f_0(t, \cdot) \text{ is discontinuous at } x\} \subseteq F. \quad (25)$$

Let $G : I \times J_\lambda \rightarrow 2^Y$ be the multifunction defined by setting, for each $(t, x) \in I \times J_\lambda$,

$$G(t, x) := T(f_0(t, x)) = h^{-1}(f_0(t, x)) \cap Y. \quad (26)$$

Observe that G is well-defined since for all $(t, x) \in I \times J_\lambda$ one has

$$f_0(t, x) \in [v(t), z(t)] \subseteq h(X). \quad (27)$$

Moreover, by the lower semicontinuity of T and by (25), for all $t \in I$, we get

$$\{x \in J_\lambda : G(t, \cdot) \text{ is not lower semicontinuous at } x\} \subseteq F. \quad (28)$$

Let $\Psi : I \times J_\lambda \rightarrow 2^{\mathbf{R}^n}$ (more precisely, $\Psi : I \times J_\lambda \rightarrow 2^{\bar{Y}}$) be the multifunction defined by putting, for each $(t, x) \in I \times J_\lambda$, $\Psi(t, x) := \bar{G}(t, x)$. By (28), for all $t \in [0, 1]$, we get

$$\{x \in J_\lambda : \Psi(t, \cdot) \text{ is not lower semicontinuous at } x\} \subseteq F. \quad (29)$$

Moreover, the values of Ψ are closed (in $\mathbf{R}^n$) subsets of X .

Since f_0 is $\mathcal{L}(I) \otimes \mathcal{B}(J_\lambda)$ -measurable and T is lower semi-continuous, by Proposition 13.2.1 of [10] the multifunction G is $\mathcal{L}(I) \otimes \mathcal{B}(J_\lambda)$ -weakly measurable. That is, for each set $\Omega \subseteq Y$, with Ω open in the relative topology of Y , the set

$$G^-(\Omega) := \{(t, x) \in I \times J_\lambda : G(t, x) \cap \Omega \neq \emptyset\} \quad (30)$$

belongs to $\mathcal{L}(I) \otimes \mathcal{B}(J_\lambda)$. By Proposition 2.6 and Theorem 3.5 of [14], the multifunction Ψ is $\mathcal{L}(I) \otimes \mathcal{B}(J_\lambda)$ -measurable.

Since $F \in \mathcal{F}_{n,\lambda}$, there exist sets $F_1, F_2, \dots, F_n \subseteq \mathbf{R}^n$ such that $F = \bigcup_{i=1}^n F_i$ and $m_1(P_i(F_i)) = 0$ for all $i = 1, \dots, n$. By Theorem 3, there exist sets $Q_1, Q_2, \dots, Q_n \subseteq \mathbf{R}$, with $Q_i \in \mathcal{B}([0, \lambda_i])$ and $m_1(Q_i) = 0$ for all $i = 1, \dots, n$, and a function $\psi : [0, 1] \times J_\lambda \rightarrow \mathbf{R}^n$ such that

- (a) for all $(t, x) \in I \times J_\lambda$ one has $\psi(t, x) \in \Psi(t, x)$;
- (b) for all $x \in J_\lambda \setminus [\bigcup_{i=1}^n (P_i^{-1}(Q_i) \cup F_i)]$, the function $\psi(\cdot, x)$ is $\mathcal{L}(I)$ -measurable;
- (c) for a.e. $t \in I$, one has

$$\begin{aligned} \{x \in J_\lambda : \psi(t, \cdot) \text{ is discontinuous at } x\} \\ \subseteq \left[\bigcup_{i=1}^n (P_i^{-1}(Q_i) \cup F_i) \right] \cap J_\lambda. \end{aligned} \quad (31)$$

Since X is closed and h is continuous, for all $(t, x) \in I \times J_\lambda$, the set $h^{-1}(f_0(t, x))$ is closed in $\mathbf{R}^n$. Consequently, for all $(t, x) \in I \times J_\lambda$, we get

$$\begin{aligned} \psi(t, x) \in \Psi(t, x) = \overline{h^{-1}(f_0(t, x)) \cap Y} \subseteq \overline{h^{-1}(f_0(t, x))} \\ = h^{-1}(f_0(t, x)). \end{aligned} \quad (32)$$

Now, let

$$\alpha := \min_{1 \leq i \leq n} \inf P_i(X) > 0. \quad (33)$$

By (27), (32), and assumption (iv) we get

$$\psi(t, x) \in \prod_{i=1}^n [\alpha, \beta_i(t)] \quad \forall (t, x) \in I \times J_\lambda. \quad (34)$$

Let $\psi_1 : I \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ be defined by putting

$$\psi_1(t, x) = \begin{cases} \psi(t, x) & \text{if } x \in J_\lambda \\ \beta(t) & \text{if } x \in \mathbf{R}^n \setminus J_\lambda. \end{cases} \quad (35)$$

By (34) and (35) we easily get

$$\psi_1(t, x) \in \prod_{i=1}^n [\alpha, \beta_i(t)] \quad \forall (t, x) \in I \times \mathbf{R}^n. \quad (36)$$

Let

$$\Lambda := \left[\bigcup_{i=1}^n (P_i^{-1}(Q_i) \cup F_i) \right] \cap J_\lambda, \quad (37)$$

and let D_0 be any countable dense subset of $J_\lambda \setminus \Lambda$. Since $m_n(\Lambda) = 0$, it is easily seen that D_0 is dense in J_λ . Let D_1 be any

countable dense subset of $\mathbf{R}^n \setminus J_\lambda$. Then, the set $D_2 := D_0 \cup D_1$ is countable and dense in $\mathbf{R}^n$, and for all $x \in D_2$ the function $\psi_1(\cdot, x)$ is measurable by the above construction. Moreover, taking into account (31), for all $t \in I$, one has

$$\begin{aligned} \{x \in \mathbf{R}^n : \psi_1(t, \cdot) \text{ is discontinuous at } x\} \\ \subseteq \left[\bigcup_{i=1}^n (P_i^{-1}(Q_i) \cup P_i^{-1}(\{0, \lambda_i\}) \cup F_i) \right] \cap J_\lambda. \end{aligned} \quad (38)$$

Let $H : [0, 1] \times \mathbf{R}^n \rightarrow 2^{\mathbf{R}^n}$ be defined by putting, for all $(t, x) \in [0, 1] \times \mathbf{R}^n$,

$$H(t, x) = \bigcap_{m \in \mathbf{N}} \overline{\text{conv}} \left(\bigcup_{\substack{y \in D_2 \\ \|x-y\|_n \leq 1/m}} \{\psi_1(t, y)\} \right), \quad (39)$$

where “ $\overline{\text{conv}}$ ” stands for “closed convex hull.” By Proposition 2 of [8], taking into account (36) and (38), we have that

- (a) H has nonempty closed convex values;
- (b) for all $x \in \mathbf{R}^n$, the multifunction $H(\cdot, x)$ is measurable;
- (c) for all $t \in I$, the multifunction $H(t, \cdot)$ has closed graph;
- (d) for all $t \in I$, one has

$$H(t, x) = \{\psi_1(t, x)\}$$

$$\begin{aligned} \forall x \in \mathbf{R}^n \setminus \left(\left[\bigcup_{i=1}^n (P_i^{-1}(Q_i) \cup P_i^{-1}(\{0, \lambda_i\}) \right. \right. \\ \left. \left. \cup F_i) \right] \cap J_\lambda \right). \end{aligned} \quad (40)$$

Moreover, by (36), we have

$$H(t, x) \subseteq \prod_{i=1}^n [\alpha, \beta_i(t)] \quad \forall (t, x) \in I \times \mathbf{R}^n. \quad (41)$$

Now we want to apply Theorem 1 of [15], choosing $T = [0, 1]$, $X = Y = \mathbf{R}^n$, $p = s$, $q = j'$, $V = L^s(I, \mathbf{R}^n)$, $\Psi(u) = u$, $r = \|\beta\|_{L^s(I, \mathbf{R}^n)}$, $\varphi \equiv +\infty$, $F = H$, and

$$\Phi(u)(t) = \int_I g(t, z) u(z) dz. \quad (42)$$

To this aim, we can argue as in [8]. In particular, observe the following.

- (a) $\Phi(L^s(I, \mathbf{R}^n)) \subseteq C^0(I, \mathbf{R}^n)$. This follows easily from our assumptions (vi) and (vii) and the classical Lebesgue's dominated convergence theorem.
- (b) If $v \in L^s(I, \mathbf{R}^n)$ and $\{v^k\}$ is a sequence in $L^s(I, \mathbf{R}^n)$, weakly convergent to v in $L^{j'}(I, \mathbf{R}^n)$, then the sequence $\{\Phi(v^k)\}$ converges to $\Phi(v)$ strongly in $L^1(I, \mathbf{R}^n)$. This follows by Theorem 2 at page 359 of [16], since g is j th power summable in $I \times I$ (note that g is measurable on $I \times I$ by the classical Scorza-Draconi theorem; see [17] or also [11]).

(c) By (41), the function

$$h_0 : t \in I \longrightarrow \sup_{x \in \mathbf{R}^n} \inf_{y \in H(t,x)} \|y\|_n \quad (43)$$

belongs to $L^s(I)$ and $\|h_0\|_{L^s(I)} \leq \|\beta\|_{L^s(I, \mathbf{R}^n)}$.

Therefore, taking into account the above construction, all the assumptions of Theorem 1 of [15] are satisfied. Consequently, there exists a function $u^* \in L^s(I, \mathbf{R}^n)$ and a set $K_1 \subseteq I$, with $m_1(K) = 0$, such that

$$u^*(t) \in H(t, \Phi(u^*)(t)) \subseteq \prod_{i=1}^n [\alpha, \beta_i(t)] \quad \forall t \in I \setminus K_1. \quad (44)$$

That is,

$$u^*(t) \in H\left(t, \int_I g(t, z) u^*(z) dz\right) \subseteq \prod_{i=1}^n [\alpha, \beta_i(t)] \quad (45)$$

$$\forall t \in I \setminus K_1.$$

We now prove that the function u^* satisfies our conclusion. To this aim, observe that, since $E \in \mathcal{F}_{n,\lambda}$, there exist sets $E_1, E_2, \dots, E_n \subseteq \mathbf{R}^n$, with $m_1(P_i(E_i)) = 0$ for all $i = 1, \dots, n$, such that $E := \bigcup_{i=1}^n E_i$.

Fix $i \in \{1, \dots, n\}$. Let $\gamma_i : I \rightarrow \mathbf{R}$ be the function

$$\gamma_i(t) := P_i(\Phi(u^*)(t)) = \int_I g(t, z) u_i^*(z) dz. \quad (46)$$

By (45) we get

$$u_i^*(t) \in [\alpha, \beta_i(t)] \quad \forall t \in I \setminus K_1; \quad (47)$$

hence for all $t \in I$ we get the inequality

$$\begin{aligned} 0 \leq \gamma_i(t) &\leq \|\phi_0\|_{L^s(I)} \|u^*\|_{L^s(I)} \\ &\leq \frac{\lambda_i}{\|\beta_i\|_{L^s(I)}} \cdot \|\beta_i\|_s = \lambda_i, \end{aligned} \quad (48)$$

hence $\gamma_i(I) \subseteq [0, \lambda_i]$. By (vi), (vii), and (45) we have that γ_i is strictly increasing. Moreover, by Lemma 2.2. at page 226 of [18] we get

$$\gamma_i'(t) = \int_I \frac{\partial g}{\partial t}(t, z) u_i^*(z) dz > 0 \quad \forall t \in]0, 1[. \quad (49)$$

Consequently, by Theorem 2 of [19], the function γ_i^{-1} is absolutely continuous. By Theorem 18.25 of [20], the set

$$W_i := \gamma_i^{-1}[(P_i(E_i \cup F_i) \cup Q_i \cup \{0, \lambda_i\}) \cap \gamma_i(I)] \quad (50)$$

has null Lebesgue measure. Now, put

$$\Omega := \left(\bigcup_{i=1}^n W_i \right) \cup K_1. \quad (51)$$

Of course, $m_1(\Omega) = 0$. Choose any point $t^* \in I \setminus \Omega$. Since $t^* \notin K_1$, by (45) we get

$$u^*(t^*) \in H(t^*, \Phi(u^*)(t^*)). \quad (52)$$

For each $i \in \{1, \dots, n\}$, since $t^* \notin W_i$, taking into account (48), we have

$$\gamma_i(t^*) \in [0, \lambda_i] \setminus [P_i(E_i \cup F_i) \cup \{0, \lambda_i\} \cup Q_i]. \quad (53)$$

Therefore, $\Phi(u^*)(t^*) \in J_\lambda$ and for all $i \in \{1, \dots, n\}$ we have

$$\Phi(u^*)(t^*) \notin [(E_i \cup F_i) \cup P_i^{-1}(\{0, \lambda_i\}) \cup P_i^{-1}(Q_i)]. \quad (54)$$

Consequently, we get

$$\begin{aligned} \Phi(u^*)(t^*) \\ \in J_\lambda \setminus \bigcup_{i=1}^n [(E_i \cup F_i) \cup P_i^{-1}(\{0, \lambda_i\}) \cup P_i^{-1}(Q_i)]. \end{aligned} \quad (55)$$

By (40) we get

$$\begin{aligned} u^*(t^*) \in H(t^*, \Phi(u^*)(t^*)) &= \{\psi_1(t^*, \Phi(u^*)(t^*))\} \\ &= \{\psi(t^*, \Phi(u^*)(t^*))\}. \end{aligned} \quad (56)$$

By (32) and (56) we get

$$u^*(t^*) \in h^{-1}(f_0(t^*, \Phi(u^*)(t^*))), \quad (57)$$

hence

$$h(u^*(t^*)) = f_0(t^*, \Phi(u^*)(t^*)). \quad (58)$$

Since $\Phi(u^*)(t^*) \notin E \cup F$, we get

$$\begin{aligned} h(u^*(t^*)) &= f^*(t^*, \Phi(u^*)(t^*)) \\ &= f(t^*, \Phi(u^*)(t^*)) \\ &= f\left(t^*, \int_I g(t^*, z) u^*(z) dz\right). \end{aligned} \quad (59)$$

This completes the proof. $\square$

Remark 4. Of course, a function f satisfying the assumptions of Theorem 2 can be discontinuous at each point $x \in J_\lambda$. The Example at the end of [8] shows that, in the statement of Theorem 2, none of the sets E and F can be assumed to depend on t . Moreover, the Example at the end of [6] shows that the second inequality in assumption (vii) cannot be weakened by assuming that

$$0 \leq \frac{\partial g}{\partial t}(t, z) \leq \phi_1(z). \quad (60)$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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