Research Article

A New Numerical Algorithm for Two-Point Boundary Value Problems

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We present a new numerical algorithm for two-point boundary value problems. We first present the exact solution in the form of series and then prove that the n -term numerical solution converges uniformly to the exact solution. Furthermore, we establish the numerical stability and error analysis. The numerical results show the effectiveness of the proposed algorithm.

1. Introduction

It is well known that many problems can be presented by the following two-point boundary value problems:

$$\begin{aligned} [x^\alpha y'(x)]' &= f(x, y), \quad x \in (0, 1), \\ y(0) &= a, \quad y(1) = b, \end{aligned} \quad (1)$$

where $\alpha \in [0, 1]$, a , and b are finite constants.

Problem (1) arises from many fields of applied mathematics and physics, such as nuclear physics, economical system, chemical engineering, and underground water flow. Therefore, this problem has attracted considerable attention. For example, Aziz and Kumar [1, 2] presented a finite difference method based on nonuniform mesh to solve this problem. Kumar [3, 4] presented a second order spline finite difference method to solve (1) by using a spline function. Rashidinia et al. [5] presented a parametric spline method for (1). For these references, please see [6–10].

In this paper, we propose a new numerical algorithm to solve (1) by using the reproducing kernel theory. By homogenizing the boundary value conditions, (1) is converted into a nonlinear operator equation. We show that the solution of (1) is equivalent to the solution of the operator equation, and its exact solution $y(x)$ can be represented in the form of series. Furthermore, we prove that the n -term numerical solution $y_n(x)$ converges uniformly to the exact solution.

Then, numerical stability and error analysis of the method are presented. Numerical results show that this method has high accuracy.

The paper is organized as follows. In Section 2, fundamental definitions and theorems of the reproducing kernel theory are given. In Section 3, the nonlinear operator equation is constructed. The new numerical algorithm is presented in Section 4. In Section 5, we apply our method to linear and nonlinear numerical examples and illustrate the applicability of the presented method. Section 6 ends this paper with a brief conclusion.

2. Fundamental Definitions and Theorems

In this section, we show some fundamental theories of the reproducing kernel space [11, 12].

Definition 1. Let H denote a Hilbert space, which is composed of functions defined on an abstract set D and admits a reproducing kernel $K(x, y)$. That is, for each fixed $y \in D$, $K(x, y)$ belongs to H as a function in x and for any $f \in H$,

$$\langle f(x), K(x, y) \rangle_H = f(y). \quad (2)$$

We call (2) the reproducing property of $K(x, y)$.

Theorem 2. Let H be a Hilbert space; let $\{\varphi_i(x)\}_{i=1}^{\infty}$ be a complete function system; that is,

$$\langle \varphi_i(x), \varphi_j(x) \rangle_H = \begin{cases} 0, & i \neq j, \\ 1, & i = j; \end{cases} \quad (3)$$

then $K(x, y) = \sum_{i=1}^{\infty} \varphi_i(x) \overline{\varphi_i(y)}$ is the reproducing kernel of H .

Proof. In fact, $\forall f(x) \in H$; $f(x) = \sum_{i=1}^{\infty} a_i \varphi_i(x)$, $a_i \in \mathbb{C}$. In view of (2) and (3), we have

$$\begin{aligned} \langle f(x), K(x, y) \rangle_H &= \left\langle \sum_{i=1}^{\infty} a_i \varphi_i(x), \sum_{i=1}^{\infty} \varphi_i(x) \overline{\varphi_i(y)} \right\rangle_H \\ &= \sum_{i=1}^{\infty} a_i \varphi_i(y) = f(y). \end{aligned} \quad (4)$$

Definition 3. The reproducing kernel space $W_2^m[0, 1]$ is defined as follows.

$W_2^m[0, 1] = \{f(x) \mid f^{(m-1)}(x) \text{ is an absolutely continuous function, } f^{(m)}(x) \in L^2[0, 1], x \in [0, 1]\}$.

The inner product and norm are defined as, respectively, $\forall f(x), g(x) \in W_2^m[0, 1]$,

$$\begin{aligned} \langle f(x), g(x) \rangle_{W_2^m} &= \sum_{i=0}^{m-1} f^{(i)}(0) g^{(i)}(0) + \int_0^1 f^{(m)}(x) g^{(m)}(x) dx, \\ \|f(x)\| &= \sqrt{\langle f(x), f(x) \rangle_{W_2^m}}. \end{aligned} \quad (5)$$

Theorem 4. $W_2^m[0, 1]$ is a complete space with respect to $\|\cdot\|_{W_2^m}$.

Proof. If the reproducing kernel $K(x, y)$ of the space $W_2^m[0, 1]$ exists, in view of (2) and Cauchy-Schwartz's inequality, we have

$$\begin{aligned} f(y) &= \langle f(x), K(x, y) \rangle_{W_2^m} \leq \|f\|_{W_2^m} \|K(x, y)\|_{W_2^m} \\ &= \|f\|_{W_2^m} \sqrt{\langle K(x, y), K(x, y) \rangle_{W_2^m}} \\ &= \|f\|_{W_2^m} \sqrt{K(y, y)} \end{aligned} \quad (6)$$

which shows that $f(y)$ is a bounded linear function on $W_2^m[0, 1]$. Hence, there exists a Cauchy sequence $\{f_n(x)\} \in W_2^m[0, 1]$. By (2) and Cauchy-Schwartz's inequality, we obtain

$$\begin{aligned} \|f_n(y) - f_m(y)\| &= \left\| \langle f_n(x) - f_m(x), K(x, y) \rangle_{W_2^m} \right\| \\ &\leq \|f_n(x) - f_m(x)\|_{W_2^m} \|K(x, y)\|_{W_2^m} \\ &= \|f_n(x) - f_m(x)\|_{W_2^m} \sqrt{\langle K(x, y), K(x, y) \rangle_{W_2^m}} \\ &\leq \|f_n(x) - f_m(x)\|_{W_2^m} \sqrt{K(y, y)}. \end{aligned} \quad (7)$$

Therefore, there exists $f(x) \in W_2^m[0, 1]$ such that $\lim_{n \rightarrow \infty} f_n(x) = f(x)$. Furthermore, we have $\lim_{n \rightarrow \infty} \|f_n\|_{W_2^m} = \|f\|_{W_2^m}$ and $\lim_{n \rightarrow \infty} \langle f_n, g_n \rangle_{W_2^m} = \langle f, g \rangle_{W_2^m}$. So the proof of Theorem 4 is complete. $\square$

3. Structure of the Nonlinear Operator

Now, we show the method to solve (1). By transformation, we have

$$x^\alpha y''(x) + \alpha x^{\alpha-1} y'(x) = [x^\alpha y'(x)]' = f(x, y). \quad (8)$$

That is,

$$\begin{aligned} y''(x) + \alpha x^{-1} y'(x) &= x^{-\alpha} f(x, y), \\ y''(x) + a(x) y'(x) &= g(x, y), \end{aligned} \quad (9)$$

where $a(x) = \alpha x^{-1}$ and $g(x, y) = x^{-\alpha} f(x, y)$. Let

$$Ly = y'' + a(x) y', \quad 0 < x < 1, \quad (10)$$

with $L : W_2^3[0, 1] \rightarrow W_2^1[0, 1]$. By homogenizing the boundary value conditions, (1) can be converted into the equivalent nonlinear operator equation:

$$\begin{aligned} Ly &= g(x, y), \quad 0 < x < 1, \\ y(0) &= y(1) = 0. \end{aligned} \quad (11)$$

For any $y(x) \in W_2^3[0, 1]$ and each fixed point $x \in [0, 1]$,

$$\varphi_i(x) = K(x_i, y), \quad \psi_i(x) = L^* \varphi_i(x), \quad (12)$$

where $\{x_i\}_{i=1}^{\infty}$ is a different dense point set on $[0, 1]$, L^* is the conjugate operator of L , and $K(x, y)$ is the reproducing kernel of $W_2^3[0, 1]$. In terms of the property of (2) and the inner product, we obtain

$$\begin{aligned} \langle y(x), \psi_i(x) \rangle_{W_2^3} &= \langle y(x), L^* \varphi_i(x) \rangle_{W_2^3} \\ &= \langle Ly(x), \varphi_i(x) \rangle_{W_2^1} \\ &= \langle Ly(x), K(x_i, y) \rangle_{W_2^1} \\ &= Ly(x_i), \quad i = 1, 2, \dots \end{aligned} \quad (13)$$

Therefore, we can see that the solution of (1) is equivalent to the solution of (11).

4. Solving the Problem

Through the normal orthogonal process, $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$ of $W_2^m[0, 1]$ can be derived from $\{\psi_i(x)\}_{i=1}^{\infty}$. That is,

$$\bar{\psi}_i(x) = \sum_{k=1}^i \beta_{ik} \psi_k(x), \quad (14)$$

where β_{ik} are the orthogonalization coefficients.

Theorem 5. If $\{x_i\}_{i=1}^{\infty}$ is the different dense point set on $[0, 1]$ and $y(x)$ is the exact solution of (11) in $W_2^3[0, 1]$, then

$$y(x) = \sum_{i=1}^{\infty} \sum_{k=1}^i \bar{\beta}_{ik} g(x_k, y(x_k)) \bar{\psi}_i(x). \quad (15)$$

Proof. Since $y(x) \in W_2^3[0, 1]$ and $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$ is a normal complete orthogonal system, then $y(x)$ can be expanded by Fourier series with the normal orthogonal basis; namely,

$$y(x) = \sum_{i=1}^{\infty} \langle y(x), \bar{\psi}_i(x) \rangle \bar{\psi}_i(x). \quad (16)$$

Because $W_2^3[0, 1]$ is complete, $y(x)$ is uniformly convergent in the sense of $\|\cdot\|_{W_2^3}$. Note that $y(x) \in W_2^3[0, 1]$; $y(x)$ is absolutely continuous, in terms of (2) and (14); we obtain

$$\begin{aligned} y(x) &= \sum_{i=1}^{\infty} \langle y(x), \bar{\psi}_i(x) \rangle \bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty} \left\langle y(x), \sum_{k=1}^i \beta_{ik} \psi_k(x) \right\rangle \bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^i \bar{\beta}_{ik} \langle y(x), \psi_k(x) \rangle \bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^i \bar{\beta}_{ik} \langle y(x), L^* \varphi_k(x) \rangle \bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^i \bar{\beta}_{ik} \langle Ly(x), \varphi_k(x) \rangle \bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^i \bar{\beta}_{ik} \langle Ly(x), K(x_k, y) \rangle \bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^i \bar{\beta}_{ik} Ly(x_k) \bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^i \bar{\beta}_{ik} g(x_k, y(x_k)) \bar{\psi}_i(x). \end{aligned} \quad (17)$$

The proof is complete. $\square$

By truncating the right hand of (15), we obtain the approximate solution of (11); namely,

$$y_n(x) = \sum_{i=1}^n \sum_{k=1}^i \bar{\beta}_{ik} g(x_k, y(x_k)) \bar{\psi}_i(x), \quad (18)$$

where $y_n(x)$ is the n -term intercept of $y(x)$ in (15). In view of the completeness of the reproducing kernel space, $y_n(x) \rightarrow y(x)$ as $n \rightarrow \infty$.

Next, in order to discuss the uniform convergence of the approximate solution, for any fixed $y_0(x) \in W_2^3[0, 1]$, $y_n(x) = \sum_{i=1}^n a_i \bar{\psi}_i(x)$ with $a_i = \sum_{k=1}^i \bar{\beta}_{ik} g(x_k, y(x_k))$; we construct an iterative sequence $\{y_n(x)\}$. That is,

$$\begin{aligned} a_1 &= \bar{\beta}_{11} g(x_1, y_0(x_1)), \\ a_2 &= \sum_{k=1}^2 \bar{\beta}_{2k} g(x_k, y_{k-1}(x_k)) \\ &= \bar{\beta}_{21} g(x_1, y_0(x_1)) + \bar{\beta}_{22} g(x_2, y_1(x_2)), \\ &\vdots \\ a_n &= \sum_{k=1}^n \bar{\beta}_{nk} g(x_k, y_{n-1}(x_k)). \end{aligned} \quad (19)$$

Theorem 6. Assume the following.

- (a) $\{x_i\}_{i=1}^{\infty}$ is a different dense point set on $[0, 1]$.
- (b) $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$ is a normal orthogonal system.
- (c) $\|y_n(x)\|_{W_2^3}$ is bounded.
- (d) $g(x, y(x)) \in W_2^1[0, 1]$, for any $y(x) \in W_2^3[0, 1]$, $x \in [0, 1]$.

Then the iterative formula $y_n(x)$ converges uniformly to the exact solution $y(x)$ of (11).

Proof. By $y_n(x) = \sum_{i=1}^n a_i \bar{\psi}_i(x)$, we have $y_{n+1}(x) = y_n(x) + a_{n+1} \bar{\psi}_{n+1}(x)$. In view of the orthonormality of $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$, we have

$$\begin{aligned} \|y_{n+1}(x)\|^2 &= \|y_n(x) + a_{n+1} \bar{\psi}_{n+1}(x)\|^2 \\ &= \|y_n(x)\|^2 + \|a_{n+1} \bar{\psi}_{n+1}(x)\|^2 \\ &= \|y_n(x)\|^2 + a_{n+1}^2 \\ &= \|y_{n-1}(x)\|^2 + a_n^2 + a_{n+1}^2 \\ &\vdots \\ &= a_1^2 + \cdots + a_n^2 + a_{n+1}^2 = \sum_{i=1}^{n+1} a_i^2. \end{aligned} \quad (20)$$

In view of the boundedness of $\|y_n(x)\|_{W_2^3}$, we have $\sum_{i=1}^{\infty} a_i^2 < \infty$. Let $n > m$ and $m, n \rightarrow \infty$, owing to $(y_n - y_{n-1}) \perp (y_{n-1} - y_{n-2}) \perp \cdots \perp (y_{m+1} - y_m)$; we obtain

$$\begin{aligned} \|y_n - y_m\|^2 &= \|y_n - y_{n-1} + y_{n-1} - y_{n-2} + \cdots + y_{m+1} - y_m\|^2 \\ &= \|y_n - y_{n-1}\|^2 + \|y_{n-1} - y_{n-2}\|^2 + \cdots + \|y_{m+1} - y_m\|^2 \\ &= \|a_n \bar{\psi}_n(x)\|^2 + \|a_{n-1} \bar{\psi}_{n-1}(x)\|^2 + \cdots + \|a_{m+1} \bar{\psi}_{m+1}(x)\|^2 \\ &= a_n^2 + a_{n-1}^2 + \cdots + a_{m+1}^2 = \sum_{i=m+1}^n a_i^2. \end{aligned} \quad (21)$$

FIGURE 1: R.M.S of the exact solution $y^k(x)$ and the approximate solution $y_n^k(x)$ ($k = 0, 1, 2$).

By the completeness of $W_2^3[0, 1]$, there exists $y(x) \in W_2^3[0, 1]$, such that $y_n(x) \rightarrow y(x)$ in the sense of $\|\cdot\|_{W_2^3}$ as $n \rightarrow \infty$. The proof is complete. $\square$

Theorem 7. Error Analysis. Assume that the conditions of Theorem 6 are satisfied; then the error of the numerical solution is monotonically decreasing with the increasing of nodes; that is, $\varepsilon_n(x) \rightarrow 0$, as $n \rightarrow \infty$.

Proof. It is easy to see that

$$\begin{aligned} \|\varepsilon_n(x)\| &= \|y(x) - y_{n-1}(x)\| \\ &= \left\| \sum_{i=1}^{\infty} a_i \bar{\psi}_i(x) - \sum_{i=1}^{n-1} a_i \bar{\psi}_i(x) \right\| \\ &= \left\| \sum_{i=n}^{\infty} a_i \bar{\psi}_i(x) \right\| \geq \left\| \sum_{i=n+1}^{\infty} a_i \bar{\psi}_i(x) \right\| \\ &= \|y(x) - y_n(x)\| = \|\varepsilon_{n+1}(x)\|. \end{aligned} \quad (22)$$

Thus we complete the proof. $\square$

Now, we show the stability of the proposed method.

Theorem 8. For (11), if $g(x, y)$ has a small perturbation δ , then the proposed method is stable.

Proof. For the problem (11), if $g(x, y)$ has a small perturbation δ , then $Ly = g(x, y) + \delta$. Let $\tilde{y}_n(x)$ be the numerical solution of $L\tilde{y} = g(x, y) + \delta$; in view of Theorems 5 and 6, we have

$$\begin{aligned} y_n(x) &= \sum_{i=1}^n \sum_{k=1}^i \bar{\beta}_{ik} g(x_k, y(x_k)) \bar{\psi}_i(x), \\ \tilde{y}_n(x) &= \sum_{i=1}^n \sum_{k=1}^i \bar{\beta}_{ik} [g(x_k, y(x_k)) + \delta] \bar{\psi}_i(x). \end{aligned} \quad (23)$$

Hence,

$$\|y_n(x) - \tilde{y}_n(x)\| = |\delta| \cdot \left\| \sum_{i=1}^n \sum_{k=1}^i \bar{\beta}_{ik} \bar{\psi}_i(x) \right\| \leq |\delta| \cdot M < \varepsilon. \quad (24)$$

TABLE 1: The M.A. error of Example 1.

N	Method in [13]	Method in [2]	Method in [5]	Our method
16	1.15 (−2)	2.10 (−2)	7.64 (−4)	9.91 (−5)
32	2.90 (−3)	5.20 (−3)	2.15 (−4)	1.74 (−5)
64	7.28 (−4)	1.30 (−3)	5.55 (−5)	5.78 (−6)
128	1.82 (−4)	3.30 (−4)	1.39 (−5)	1.51 (−6)
256	—	—	—	3.55 (−7)

That is, $\forall \varepsilon > 0$, $\exists \delta = \varepsilon/M$, such that

$$\|y_n(x) - \tilde{y}_n(x)\| < \varepsilon. \quad (25)$$

Therefore, the method is stable. $\square$

5. Numerical Examples

5.1. Example 1. Consider the following linear two-point boundary value problem [5]:

$$\begin{aligned} x^{-\alpha} (x^\alpha y')' &= \beta x^{\beta-2} (\alpha + \beta - 1 + \beta x^\beta) y, \\ y(0) &= 1, \quad y(1) = e. \end{aligned} \quad (26)$$

The exact solution is $y(x) = \exp(x^\beta)$ with $\alpha = 0.5$ and $\beta = 4$. The maximum absolute errors (M.A. error) are tabulated in Table 1; the Root-mean-square errors (R.M.S) of the exact solution $y^k(x)$ and the approximate solution $y_n^k(x)$ ($k = 0, 1, 2$) are shown in Figure 1. From the numerical results, we can see that the present method produces better approximate solution than [5] and the error of the numerical solution is monotonically decreasing with the increase of nodes.

5.2. Example 2. Consider the nonlinear singular two-point boundary value problem [14, 15]:

$$\begin{aligned} y'' + \frac{1}{2x} y' &= e^y \left(\frac{1}{2} - e^y \right), \\ y(0) &= \ln 2, \quad y(1) = 0. \end{aligned} \quad (27)$$

The exact solution is $y(x) = \ln(2/(1 + x^2))$. When $N = 8, 16, 32, 64, 128, 256$, the maximum absolute errors

FIGURE 2: R.M.S of the exact solution $y^k(x)$ and the approximate solution $y_n^k(x)$ ($k = 0, 1, 2$).

TABLE 2: The M.A. error in solutions of Example 2.

x	$ y(x) - \text{Padé}[5/5](x) $ method in [16]	$ y(x) - \phi_{10}(x) $ method in [14, 15]	N	Our method
10^{-1}	$4.432E - 04$	$1.734E - 17$	8	3.956 (−6)
10^{-2}	$1.368E - 04$	$0.000E - 00$	16	9.822 (−7)
10^{-3}	$1.300E - 05$	$1.602E - 17$	32	1.244 (−7)
10^{-4}	$1.219E - 06$	$1.077E - 17$	64	6.325 (−8)
10^{-5}	$2.659E - 06$	$8.279E - 18$	128	2.333 (−8)
10^{-6}	$2.803E - 06$	$2.212E - 17$	256	9.012 (−9)

(M.A. error) are tabulated in Table 2; the Root-mean-square error (R.M.S) of the exact solution $y^k(x)$ and the approximate solution $y_n^k(x)$ ($k = 0, 1, 2$) are shown in Figure 2. From the numerical results, we can see that the error of the numerical solution is monotonically decreasing with the increase of nodes.

6. Conclusions

In this paper, we use the reproducing kernel iterative method to solve a class of two-point boundary value problems. By homogenizing the boundary conditions, the two-point boundary value problem is converted into the equivalent nonlinear operator equation. We prove that their solutions are equivalent, the exact solution $y(x)$ can be represented in the form of series, and the n -term numerical solution $y_n(x)$ converges uniformly to the exact solution $y(x)$. Furthermore, we show the analysis of error and stability for the method. At last, numerical results show the high accuracy and the validity of this method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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