Resilient Robust Finite-Time $L_2$-$L_{\infty}$ Controller Design for Uncertain Neutral System with Mixed Time-Varying Delays

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The delay-dependent resilient robust finite-time $L_2$-$L_{\infty}$ control problem of uncertain neutral time-delayed system is studied. The disturbance input is assumed to be energy bounded and the time delays are time-varying. Based on the Lyapunov function approach and linear matrix inequalities (LMIs) techniques, a state feedback controller is designed to guarantee that the resulted closed-loop system is finite-time bounded for all uncertainties and to satisfy a given $L_2$-$L_{\infty}$ constraint condition. Simulation results illustrate the validity of the proposed approach.

1. Introduction

Dynamical systems with time delays and uncertain parameters have been of considerable interest over the past decades. In fact, time delays are always the important source of system instability and poor performance [1–4]. As a special class of time-delay systems, the neutral type time-delayed system has also received some attention in recent years. This time-delayed system contains time delays both in its state and in the derivative of its states. Moreover, neutral time-delayed systems are frequently encountered in many dynamics, such as automatic control, distributed network system containing lossless transmission line, heat exchangers, and population ecology. Various analysis approaches have been utilized to find stability criteria and control design conditions for asymptotic stability of neutral time delays [5–10].

It is now worth pointing out that the control performances mentioned above concern the desired behavior of control dynamics over an infinite-time interval and it always deals with the asymptotic property of system trajectories. For controlling a dynamical system, it can meet the requirements of asymptotic stability, but it will not reflect the transient characteristics. Asymptotic stability is unable to satisfy the transient requirements of industrial production if there exists large amount of overshoot, oscillation change, and nonlinear disturbance within a finite-time interval. To deal with this transient performance of control dynamics, Dorato gave the concept of finite-time stability [11] (or short-time stability) in the early 1960s. Then, the relevant concepts of finite-time bounded (FTB) [12], finite-time stabilization [13], finite-time $H_{\infty}$ control [14], and finite-time $L_2$-$L_{\infty}$ [15] control have been revisited in form of linear matrix inequalities (LMIs) techniques. And this transient performance is widely applied to time-delay systems, uncertain systems, nonlinear systems, stochastic systems, and so forth. However, to the best of our knowledge, very few results in the literature consider the related control problems of neutral time-varying delays in the finite-time interval.

On the other hand, the $L_2$-$L_{\infty}$ performance has attracted considerable attention as an important performance evaluation index when it was first proposed in 1989 [16]. In engineering practice, although the study of the impact of noise and delay on the system performance is important, the extremum problem of the controlled output cannot be ignored, because the controlled output should be controlled within a certain range. In control theory and engineering application, the $L_2$-$L_{\infty}$ control has very important significance that lies in its performance index which can control the output value minimization. Unfortunately, up to now, the theme of $L_2$-$L_{\infty}$ control design of uncertain neutral systems with time-varying delays has received little attention.

Motivated by the above discussion, this paper focuses on the problem of finite-time $L_2$-$L_{\infty}$ controller design for a class
of neutral systems with mixed time-varying delays and uncertainties. By constructing a suitable Lyapunov function, the sufficient conditions are derived that closed-loop controlled system is FTBB and satisfies the given finite-time interval induced $L_2$-$L_{\infty}$ norm of the operator from the unknown disturbance to the output. We also show that the $L_2$-$L_{\infty}$ controller designing problem can be dealt with by solving a set of coupled LMsIs. Finally, a numerical example illustrates the effectiveness of the developed techniques.

2. Problem Statement

Consider the following neutral time-delayed system with uncertainties:

$$
\Sigma_0 : \quad \begin{cases}
\dot{x}(t) - (C + \Delta C(t)) \dot{x}(t - \tau(t)) = (A + \Delta A(t)) x(t) + (A_d + \Delta A_d(t)) x(t - h(t)) + Bu(t) \\
\quad + (D + \Delta D(t)) w(t) \\
\quad y(t) = (F + \Delta F(t)) x(t) + Gu(t) \\
x(t_0 + \theta) = \phi(\theta), \quad \theta \in [-\max\{h, \tau\}, 0], t_0 = 0,
\end{cases}
$$

(1)

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the controlled input, and $w(t) \in \mathbb{R}^p$ is the disturbance input that belongs to $L_2[0, +\infty)$ and for a given positive number $\delta$ and constant time $T$, the following form is satisfied:

$$
\int_0^T w^T(t) w(t) dt \leq \delta, \quad \delta \geq 0.
$$

(2)

$h(t)$ and $\tau(t)$ are time-varying delays and satisfy

$$
0 \leq h(t) \leq h, \quad 0 \leq \tau(t) \leq \tau, \quad 0 \leq h(t) \leq h_d, \quad 0 \leq \tau(t) \leq \tau_d < 1,
$$

(3)

where $h$, $\tau$, $h_d$, and $\tau_d$ are constant scalars. $\phi(\theta) \in L_2[-\max\{h, \tau\}, 0]$ is the continuous initial function. $A, A_d, C, D$ and $F$ are known constant matrices, and $\Delta A(t), \Delta A_d(t), \Delta C(t), \Delta D(t),$ and $\Delta F(t)$ are unknown time-varying matrices representing the norm-bounded parameter uncertainties and satisfy the following form:

$$
[\Delta A(t) \quad \Delta A_d(t) \quad \Delta C(t) \quad \Delta D(t)] = M_1 \sigma(t) [H_1 \quad H_2 \quad H_3 \quad H_4],
$$

(4)

$$
\Delta F(t) = M_2 \sigma(t) H_1,
$$

(5)

where $M_1, M_2, H_1, H_2, H_3,$ and $H_4$ are known real matrices with suitable dimension and $\sigma(t)$ is an unknown real and possibly time-varying matrix with Lebesgue measurable elements satisfying

$$
\sigma^T(t) \sigma(t) \leq I.
$$

(6)

In this paper, we consider the state feedback controller as follows:

$$
u(t) = (K + \Delta K(t)) x(t),
$$

(7)

where $K$ is the unknown controller gain and $\Delta K(t)$ is the time-varying controller gain which satisfies

$$
\Delta K(t) = N \eta(t) S, \quad \eta^T(t) \eta(t) \leq I.
$$

(8)

Then, we can get the following closed-loop control system:

$$
\begin{align*}
\Sigma : & \quad \dot{x}(t) - \overline{C} x(t - \tau(t)) = \hat{A} x(t) + \hat{A}_d x(t - h(t)) + B u(t) \\
& \quad + \hat{D} w(t) \\
& \quad y(t) = F x(t) \\
& \quad x(t_0 + \theta) = \phi(\theta), \quad \theta \in [-\max\{h, \tau\}, 0], t_0 = 0,
\end{align*}
$$

(9)

where $\hat{A} = \bar{A} + \Delta \bar{A}(t), \hat{A}_d = \bar{A}_d + \Delta \bar{A}_d(t), \bar{A} = A + B K, \Delta \bar{A}(t) = \Delta A(t) + B \Delta K(t), \bar{A}_d = A_d + \Delta A_d(t), \bar{C} = C + \Delta C(t), \bar{D} = D + \Delta D(t), \bar{F} = F + \Delta F(t), F = F + G K,$ and $\Delta \bar{F}(t) = \Delta F(t) + G \Delta K(t)$.

The main purpose of this paper is to design an appropriate resilient state feedback controller (7), such that the closed-loop control system $\Sigma$ is finite-time bounded and satisfies the given performance index constraints.

Before proceeding with the study, we give the relevant definitions and lemmas first.

**Definition 1.** For given positive scalars $c_1, \delta,$ and $T$ and a symmetrical positive determined matrix $R$, the closed-loop system $\Sigma$ is robust finite-time bounded (FTB) with respect to $(c_1, c_2, \delta, R, T)$, if there exists a positive constant $c_2$ with $c_2 > c_1$, such that, for all the external disturbances $w(t)$ satisfying condition (2), the following formula is satisfied:

$$
\phi^T(\theta) R \phi(\theta) \leq c_1 \implies x^T(t) R x(t) < c_2, \quad \forall t \in [0, T].
$$

(10)

**Remark 2.** If the disturbance input is not present in the closed-loop system, that is, $w(t) = 0$, the concept of FTB will reduce into finite-time stability (FTS). It is worth mentioning that Lyapunov stability and finite-time stability are two different concepts. The former is largely known to the control characteristic in infinite-time interval, but the latter concerns the boundedness analysis of the controlled states within a finite-time interval. Obviously, a finite-time stable system may not be Lyapunov stochastically stable and vice versa.

**Definition 3.** The state feedback controller in the form of (7) is considered as a robust finite-time $L_2$-$L_{\infty}$ controller for the closed-loop system $\Sigma$. If the system $\Sigma$ is FTB with respect to $(c_1, c_2, \delta, R, T)$ and under the zero initial condition, there exist two positive scalars $\gamma$ and $T$ for all disturbance which satisfy condition (2), such that

$$
\|y(t)\|_2^2 \leq \gamma^2 \|w(t)\|_2^2,
$$

(11)

where $\|y(t)\|_2^2 = \sup_{t \in [0, T]} y^T(t) y(t)$, $\|w(t)\|_2^2 = \int_0^T w^T(t) w(t) dt$. 

Lemma 4 (see [17]). For any real positive scalars \( \alpha, \beta \) (where \( \alpha > \beta \)) and a positive definite symmetric matrix \( S \), then the following inequality holds for a vector function \( \omega : [\beta, \alpha] \to \mathbb{R}^n \) which can let the integrals converge:

\[
\left( \int_{\beta}^{\alpha} \omega(\sigma) \, d\sigma \right)^T S \left( \int_{\beta}^{\alpha} \omega(\sigma) \, d\sigma \right) 
\leq (\alpha - \beta) \left( \int_{\beta}^{\alpha} \omega^T(\sigma) S \omega(\sigma) \, d\sigma \right).
\]

Lemma 5 (see [17]). For any positive scalar \( \tau \) and positive definite symmetric matrix \( S \), the following inequality is satisfied:

\[
\frac{2}{\tau^2} \left( \int_{-\tau}^{0} \int_{-\tau}^{t} \omega(\sigma) \, d\sigma \, d\theta \right)^T S \left( \int_{-\tau}^{0} \int_{-\tau}^{t} \omega(\sigma) \, d\sigma \, d\theta \right) 
\leq \int_{-\tau}^{0} \int_{-\tau}^{t} \omega^T(\sigma) S \omega(\sigma) \, d\sigma \, d\theta.
\]

Lemma 6 (see [15]). For given appropriate dimension matrix \( H \) and \( E \), if there exists a matrix \( W(t) \) which satisfies \( W^T(t)W(t) \leq 1 \) and a scalar \( \varepsilon > 0 \), then

\[
HW(t)E + E^TW^T(t)H^T \leq \varepsilon^{-1}HH^T + \varepsilon EE^T.
\]

3. Main Results

In this section, our main purpose is to solve the design problem of a resilient robust finite-time \( L_2-L_{\infty} \) controller for a class of uncertain neutral systems with mixed time-varying delays.

Theorem 7. Given positive scalars \( c_j, \delta, T, \) and \( \alpha \), positive definite symmetric matrix \( R \), and time-delay parameters \( h, d > 0 \), \( h > 0, \tau > 0, \) and \( \tau_j > 0 \), the closed-loop system \( \Sigma \) is \( FTB \) with respect to \( (c_j, c_0, \delta, R, T) \), if there exist positive scalars \( \lambda, i = 1, 2, \ldots, 6 \), and symmetric positive definite matrices \( P_i, i = 1, 2, \ldots, 6 \), such that

\[
\Pi = \begin{bmatrix} \Pi_1 & \Pi_2 & \Pi_3 \\ * & \Pi_4 & \Pi_5 \\ * & * & \Pi_6 \end{bmatrix} < 0,
\]

\[
c_j [\lambda_2 + h\lambda_3 + h\lambda_4 + \lambda_5 + \tau\lambda_6] + \delta (1 - e^{-\alpha T}) < \lambda_1 e^\alpha e^{-\alpha T},
\]

where

\[
\Pi_1 = \frac{1}{\tau^2} \int_{\tau}^{\tau_0} \cdots \int_{\tau}^{\tau_0}
\]

\[
\Pi_{11} = A^T P_1 + P_1 A + P_2 + P_3 + P_4 + P_5 + W_1 + W_3 + W_4 + W_6 + W_1^T + W_3^T + W_4^T + W_6^T
\]

\[-\alpha P_1 - \frac{\alpha}{\tau} P_6 - 2\alpha Q_1 - 2\alpha Q_2 - 2\alpha Q_3 - 2\alpha Q_4.
\]

\[
\Pi_{12} = P_1 \tilde{A}_d - W_1 + W_2 + W_1^T + W_3^T + W_4^T + W_6^T,
\]

\[
\Pi_{13} = P_1 C + W_3^T + W_5^T + W_6^T,
\]

\[
\Pi_{14} = -W_4 + W_5 + W_1^T + W_3^T + W_4^T + W_6 + \frac{\alpha}{\tau} P_6,
\]

\[
\Pi_{15} = -W_2 - W_3 + W_1^T + W_3^T + W_4^T + W_6^T,
\]

\[
\Pi_{16} = -W_5 - W_6 + W_1^T + W_3^T + W_4^T + W_6^T,
\]

\[
\Pi_{17} = P_1 \tilde{D},
\]

\[
\Pi_{22} = -(1 - h_d) P_2 - W_1 + W_2 - W_1^T + W_2^T,
\]

\[
\Pi_{23} = -W_1^T + W_2^T,
\]

\[
\Pi_{24} = -W_4 + W_5 - W_1^T + W_2^T,
\]

\[
\Pi_{25} = -W_2 - W_3 - W_1^T + W_2^T,
\]

\[
\Pi_{26} = -W_5 - W_6 - W_1^T + W_2^T,
\]

\[
\Pi_{27} = 0,
\]

\[
\Pi_{33} = -(1 - \tau_d) P_6,
\]

\[
\Pi_{34} = -W_4 + W_5,
\]

\[
\Pi_{35} = -W_2 - W_3,
\]

\[
\Pi_{36} = -W_5 - W_6,
\]

\[
\Pi_{37} = 0,
\]

\[
\Pi_{44} = -(1 - \tau_d) P_4 - W_4 - W_5 - W_4^T + W_5^T - \frac{\alpha}{\tau} P_6,
\]

\[
\Pi_{45} = -W_2 - W_3 - W_4^T + W_5^T,
\]

\[
\Pi_{46} = -W_5 - W_6 + W_4^T + W_5^T,
\]

\[
\Pi_{47} = 0,
\]

\[
\Pi_{55} = P_3 - W_2 - W_3 - W_2^T + W_3^T,
\]

\[
\Pi_{56} = -W_5 - W_6 - W_2^T - W_3^T,
\]

\[
\Pi_{57} = 0,
\]

\[
\Pi_{66} = -P_5 - W_5 - W_6 - W_5^T - W_6^T,
\]

\[
\Pi_{67} = 0,
\]

\[
\Pi_{77} = -\alpha I,
\]

\[
\Pi_2 = \begin{bmatrix} hW_1 & hW_2 & hW_3 & \tau W_4 & \tau W_5 & \tau W_6 \\ hW_1 & hW_2 & hW_3 & \tau W_4 & \tau W_5 & \tau W_6 \\ hW_1 & hW_2 & hW_3 & \tau W_4 & \tau W_5 & \tau W_6 \\ hW_1 & hW_2 & hW_3 & \tau W_4 & \tau W_5 & \tau W_6 \\ hW_1 & hW_2 & hW_3 & \tau W_4 & \tau W_5 & \tau W_6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},
\]
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\[ \Pi_\delta = \begin{bmatrix} A^T P_6 & h A^T Q_1 & h A^T Q_2 & \tau A^T Q_3 & \tau A^T Q_4 \\ A^T P_6 & h A^T Q_1 & h A^T Q_2 & \tau A^T Q_3 & \tau A^T Q_4 \\ C^T P_6 & h C^T Q_1 & h C^T Q_2 & \tau C^T Q_3 & \tau C^T Q_4 \\ D^T P_6 & h D^T Q_1 & h D^T Q_2 & \tau D^T Q_3 & \tau D^T Q_4 \end{bmatrix}, \]

\[ \Pi_0 = \text{diag} \{-h Q_1, -h Q_2, -\tau Q_3, -\tau Q_3, -\tau Q_3\}, \]

\[ \Pi_5 = [0]_{6\times 5}, \]

\[ \Pi_6 = \text{diag} \{-P_6, -h Q_1, -h Q_2, -\tau Q_3, -\tau Q_3\}. \]

(17)

Proof. Construct a positive definite Lyapunov function as follows:

\[ V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) + V_5(t), \]

where

\[ V_1(t) = x^T(t) P_1 x(t), \]

\[ V_2(t) = \int_{t-h(t)}^{t} x^T(s) P_2 x(s) ds + \int_{t-h(t)}^{t} x^T(s) P_1 x(s) ds, \]

\[ V_3(t) = \int_{t-\tau(t)}^{t} x^T(s) P_4 x(s) ds + \int_{t-\tau(t)}^{t} x^T(s) P_3 x(s) ds, \]

\[ V_4(t) = \int_{t-\tau(t)}^{t} \dot{x}^T(s) P_6 \dot{x}(s) ds, \]

\[ V_5(t) = \int_{t-h(t)}^{t} \int_{t-h(t)}^{t} x^T(s) (Q_1 + Q_2) \dot{x}(s) ds d\theta \]

\[ + \int_{t-\tau(t)}^{t} \int_{t-\tau(t)}^{t} x^T(s) (Q_3 + Q_3) \dot{x}(s) ds d\theta. \]

We take the time derivative of \( V(t) \) along the trajectory of system \( \Sigma \) and it yields the following:

\[ \dot{V}_1(t) = x^T(t) \left( P_1 \dot{A} + A^T P_1 \right) x(t) \]

\[ + x^T(t) P_1 \dot{A} x(t-h(t)) + x^T(t) P_1 \dot{C} x(t-\tau(t)) \]

\[ + x^T(t) P_1 \dot{D} \dot{w}(t) + x^T(t-h(t)) A^T P_1 x(t) \]

\[ + x^T(t-\tau(t)) C^T P_1 x(t) + \omega^T(t) D^T P_1 x(t), \]

\[ \dot{V}_2(t) \leq x^T(t) (P_2 + P_3) x(t) \]

\[ (21) \]

where

\[ \zeta(t) = \begin{bmatrix} x^T(t) & x^T(t-h(t)) & x^T(t-\tau(t)) & x^T(t-\tau(t)) & x^T(t-h) & x^T(t-\tau) \end{bmatrix}^T, \]

\[ \xi(t) = \left[ \zeta^T(t) w^T(t) \right]^T. \]

For any symmetric positive definite matrices \( W_i, i = 1, 2, \ldots, 6 \), the following equations are satisfied according to Leibniz-Newton lemma:

\[ 2\zeta^T(t) W_1 \left[ x(t) - x(t-h(t)) - \int_{t-h(t)}^{t} \dot{x}(s) ds \right] = 0, \]

\[ 2\zeta^T(t) W_2 \left[ x(t-h(t)) - x(t-h) - \int_{t-h}^{t-h(t)} \dot{x}(s) ds \right] = 0, \]

\[ 2\zeta^T(t) W_3 \left[ x(t) - x(t-h(t)) - \int_{t-h}^{t-h(t)} \dot{x}(s) ds \right] = 0, \]

\[ 2\zeta^T(t) W_4 \left[ x(t-h(t)) - x(t-h(t)) - \int_{t-h(t)}^{t-h(t)} \dot{x}(s) ds \right] = 0, \]

\[ 2\zeta^T(t) W_5 \left[ x(t-\tau(t)) - x(t-\tau) - \int_{t-\tau(t)}^{t-\tau(t)} \dot{x}(s) ds \right] = 0, \]

\[ 2\zeta^T(t) W_6 \left[ x(t-\tau(t)) - x(t-\tau(t)) - \int_{t-\tau(t)}^{t-\tau(t)} \dot{x}(s) ds \right] = 0, \]

(22)
According to (20)-(21), we can obtain
\[
\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) + \dot{V}_4(t) + \ldots Q_1 + Q_2 + \tau(Q_3 + Q_4) C
- W_1 + W_2 + W_3 + W_4 + W_5 + W_6 + W_7 + W_8 + W_9 + W_{10},
\[
\Omega_{24} = -W_4 + W_5 - W_1 + W_2,
\[
\Omega_{25} = -W_2 + W_3 - W_1 + W_2,
\[
\Omega_{26} = -W_5 + W_6 - W_1 + W_2,
\]

Since \(Q_1, Q_2, Q_3,\) and \(Q_4\) are positive definite symmetric matrices, we have
\[
\dot{V}(t) \leq \xi^T(t) \Omega_1 \xi(t) + \xi^T(t) \Omega_2 \xi(t),
\]
where
\[
\Omega_1 = [\Omega_{ij}]_{j<k},
\]
\[
\Omega_{111} = \tilde{A}^T P_1 + P_1 \tilde{A} + P_2 + P_3 + P_4 + P_5 \quad + \tilde{A}^T (P_6 + h(Q_1 + Q_2) + \tau(Q_3 + Q_4)) \tilde{A}
+ W_1 + W_2 + W_3 + W_4 + W_5 + W_6 + W_7 + W_8 + W_9 + W_{10},
\]
\[
\Omega_{112} = P_1 \tilde{C} + \tilde{A}^T (P_6 + h(Q_1 + Q_2) + \tau(Q_3 + Q_4)) \tilde{C}
+ W_1 + W_2 + W_3 + W_4 + W_5 + W_6 + W_7 + W_8 + W_9 + W_{10},
\]
\[
\Omega_{113} = -W_4 + W_5 + W_1 + W_2 + W_3 + W_4 + W_5 + W_6,
\]
\[
\Omega_{114} = -W_2 + W_3 + W_1 + W_2 + W_3 + W_4 + W_5 + W_6 + W_7 + W_8 + W_9 + W_{10},
\]
\[
\Omega_{115} = -W_5 - W_6 + W_1 + W_2 + W_3 + W_4 + W_5 + W_6 + W_7 + W_8 + W_9 + W_{10},
\]
\[
\Omega_{116} = -W_5 + W_6 + W_1 + W_2 + W_3 + W_4 + W_5 + W_6 + W_7 + W_8 + W_9 + W_{10},
\]
\[
\Omega_{117} = P_1 \tilde{D} + \tilde{A}^T (P_6 + h(Q_1 + Q_2) + \tau(Q_3 + Q_4)) \tilde{D},
\]
\[
\Omega_{22} = -(1 - h_d) P_2 \quad + \tilde{A}^T (P_6 + h(Q_1 + Q_2) + \tau(Q_3 + Q_4)) \tilde{A}
- W_1 + W_2 - W_1 + W_2 + W_3 + W_4 + W_5 + W_6 + W_7 + W_8 + W_9 + W_{10},
\]
\[
\Omega_{23} = \tilde{A}^T (P_6 + h(Q_1 + Q_2) + \tau(Q_3 + Q_4)) \tilde{C}
- W_1 + W_2 - W_1 + W_2 + W_3 + W_4 + W_5 + W_6 + W_7 + W_8 + W_9 + W_{10},
\]
\[
\Omega_{24} = -W_4 + W_5 - W_1 + W_2 + W_3 + W_4 + W_5 + W_6 + W_7 + W_8 + W_9 + W_{10},
\]
\[
\Omega_{25} = -W_2 - W_3 - W_1 + W_2 + W_3 + W_4 + W_5 + W_6 + W_7 + W_8 + W_9 + W_{10},
\]
\[
\Omega_{26} = -W_5 - W_6 - W_1 + W_2 + W_3 + W_4 + W_5 + W_6 + W_7 + W_8 + W_9 + W_{10},
\]
Considering condition (2), (29) can be simplified as

\[ V(t) < e^{\alpha t} \left[ V(0) + \alpha \int_0^t e^{-\alpha \tau} w^T(\tau) w(\tau) d\tau \right] < e^{\alpha t} \left[ V(0) + \delta \left(1 - e^{-\alpha T}\right) \right]. \]  

On the other hand,

\[ V(t) \geq V(0) = x^T(t) P x(t) \geq \lambda_{\min}(\tilde{P}) x^T(0) Rx(t), \]

\[ V(0) \leq \phi^T(\theta) P_1 \phi(\theta) + h \phi^T(\theta) P_2 \phi(\theta) + \tau \phi^T(\theta) P_3 \phi(\theta) \]

\[ \leq \lambda_{\max}(\tilde{P}_1) \phi^T(\theta) R \phi(\theta) + h \lambda_{\max}(\tilde{P}_2) \phi^T(\theta) R \phi(\theta) + \tau \lambda_{\max}(\tilde{P}_3) \phi^T(\theta) R \phi(\theta) \]

\[ \leq \lambda_{\max}(\tilde{P}_1) c_1 + h \lambda_{\max}(\tilde{P}_2) c_1 + h \lambda_{\max}(\tilde{P}_3) c_1 \]

\[ + \tau \lambda_{\max}(\tilde{P}_4) c_1 + \tau \lambda_{\max}(\tilde{P}_5) c_1. \]  

Then, formula (27) can be written as

\[ x^T(t) Rx(t) \]

\[ \leq c_1 \left[ \lambda_2 + h \lambda_3 + h \lambda_4 + \tau \lambda_5 + \tau \lambda_6 \right] + \delta \left(1 - e^{-\alpha T}\right), \]

which can be guaranteed by condition (16). This completes the proof. \( \square \)

According to Theorem 7, we will obtain the resilient robust finite-time \( L_\infty \) controller for a class of uncertain neutral system with mixed time-varying delays.

**Theorem 8.** Given positive scalars \( c_1, T, \delta, \) and \( \alpha, \) positive definite symmetric matrix \( R, \) and time-delay parameters \( h > 0, \) \( h_d > 0, \) \( \tau > 0, \) and \( \tau_d > 0, \) the closed-loop neutral system \( \Sigma \) is FTF with respect to \( (c_1, c_2, \delta, R, T) \) and satisfies the cost function (11) for all admissible disturbance \( w(t), \) if there exist positive scalars \( \beta_1 \) and symmetric positive definite matrices \( \tilde{P}_1, i = 1, 2, \ldots, 6, \) \( Q_i, i = 1, 2, \ldots, 4, \) \( W_i, i = 1, 2, \ldots, 6, \) such that conditions (15) and (16) and the following LMI hold:

\[ \Psi = \begin{bmatrix} -P_1 & \tilde{F}^T \\ -\beta I \end{bmatrix} < 0. \]  

**Proof.** Similar to the proof of Theorem 7, (29) can be rewritten as

\[ e^{-\alpha t} V(t) < \alpha \int_0^t e^{-\alpha \tau} w^T(\tau) w(\tau) d\tau. \]  

Recalling formula (24) and Lemmas 4 and 5 and using Schur complement, we can get

\[ \dot{V}(t) - aV(t) - \alpha w^T(t) w(t) \leq \xi^T(t) \Pi \xi(t) < 0; \]

that is,

\[ \dot{V}(t) < aV(t) + \alpha w^T(t) w(t). \]  

Pre- and postmultiplying (27) by \( e^{-\alpha t}, \) we have

\[ \frac{d}{dt} \left(e^{-\alpha t} V(t)\right) < \alpha e^{-\alpha t} w^T(t) w(t). \]  

Then integrating the aforementioned inequality from 0 to \( t, \) where \( t \in [0, T], \) it yields

\[ e^{-\alpha t} V(t) - V(0) < \alpha \int_0^t e^{-\alpha \tau} w^T(\tau) w(\tau) d\tau. \]
Then, we have

\[ x^T(t) P_1 x(t) \leq V(t) \leq \alpha e^{\alpha T} \int_0^t w^T(\tau) w(\tau) d\tau. \]  

(35)

From (33), we can obviously get

\[ \bar{F}^T \bar{F} < \beta P_1. \]  

(36)

Considering system \( \Sigma \), we have

\[ y^T(t) y(t) = \left[ \bar{F} x(t) \right]^T \left[ \bar{F} x(t) \right] = x^T(t) \bar{F}^T \bar{F} x(t). \]  

(37)

Combining (35)–(37), we can obtain

\[ x^T(t) \bar{F}^T \bar{F} x(t) \leq \beta x^T(t) P_1 x(t) \leq \beta \alpha e^{\alpha T} \int_0^t w^T(\tau) w(\tau) d\tau, \]

that is,

\[ y^T(t) y(t) \leq \beta \alpha e^{\alpha T} \int_0^t w^T(\tau) w(\tau) d\tau. \]  

(39)

Letting \( y^2 = \beta \alpha e^{\alpha T} \), we have \( \|y(t)\|_{\infty}^2 < y^2 \|w(t)\|_{\infty}^2 \). This completes the proof.

\( \square \)

**Theorem 9.** Given positive scalars \( c_i, T, \delta, \), and \( \alpha \), positive definite symmetric matrix \( R \), and time-delay parameters \( h > 0, h_d > 0, \tau > 0 \), and \( \tau_d > 0 \), the closed-loop neutral system \( \Sigma \) is FTF with respect to \( (c_1, c_2, \delta, R, T) \), satisfies the cost function (11) for all admissible disturbance \( w(t) \), and exists as a state feedback controller in the form of (7) with \( K = UP_i^{-1} \), if there exist positive scalars \( \beta, \epsilon_i, i = 1, 2, \ldots, 4 \), and \( \mu_i, i = 1, 2, \ldots, 5 \), and symmetric positive definite matrices \( L_i, i = 1, 2, \ldots, 5 \), \( P_i, i = 2, 3, \ldots, 5 \), \( \bar{P}_6 \), and \( U \), such that the following LMIs are feasible:

\[
\begin{bmatrix}
\bar{P}_1 & \bar{P}_2 & \bar{P}_3 & \bar{P}_4 & \bar{P}_5 & \bar{P}_6 & \bar{P}_7 \\
* & \bar{P}_1 & \bar{P}_3 & \bar{P}_5 & \bar{P}_7 & \bar{P}_9 & \bar{P}_{10} \\
* & * & \bar{P}_3 & \bar{P}_5 & \bar{P}_7 & \bar{P}_9 & \bar{P}_{11} \\
* & * & * & \bar{P}_5 & \bar{P}_7 & \bar{P}_9 & \bar{P}_{12} \\
* & * & * & * & \bar{P}_7 & \bar{P}_9 & \bar{P}_{13} \\
\end{bmatrix} < 0,
\]

(40)

\[
\begin{bmatrix}
-L_1 & L_1 F^T + U^T G^T L_1 H_1^T L_1 S^T & \cdots & \cdots & \cdots \\
* & \Psi & \cdots & \cdots & \cdots \\
* & * & \cdots & \cdots & \cdots \\
* & * & * & \cdots & \cdots \\
* & * & * & * & \cdots \\
\end{bmatrix} < 0,
\]

(41)

\[
\mu_1 R^{-1} < L_1 < R^{-1},
\]

(42)

\[
0 < P_2 < \mu_2 R,
\]

(43)

\[
0 < P_3 < \mu_3 R,
\]

(44)

\[
0 < P_4 < \mu_4 R,
\]

(45)

\[
0 < P_5 < \mu_5 R,
\]

(46)

\[
\left[ \frac{c_1 \left[ h (\mu_2 + \mu_3) + \tau (\mu_4 + \mu_5) \right] + \delta (1 - e^{-\alpha T}) - c_2 e^{-\alpha T}}{-\mu_1} \right] < 0,
\]

(47)

where
\[
\Pi_2 = \begin{bmatrix}
hT_1 & hT_2 & hT_3 & \tauT_4 & \tauT_5 & \tauT_6 \\
hT_1 & hT_2 & hT_3 & \tauT_4 & \tauT_5 & \tauT_6 \\
hT_1 & hT_2 & hT_3 & \tauT_4 & \tauT_5 & \tauT_6 \\
hT_1 & hT_2 & hT_3 & \tauT_4 & \tauT_5 & \tauT_6 \\
hT_1 & hT_2 & hT_3 & \tauT_4 & \tauT_5 & \tauT_6 \\
0 & 0 & 0 & 0 & 0 & 0 
\end{bmatrix},
\]
\[
\Pi_3 = \begin{bmatrix}
L_1A^T + U^TB^T & hL_1A^T + hU^TB^T & hL_4A^T + hU^TB^T & \tauL_1A^T + \tauU^TB^T & \tauL_4A^T + \tauU^TB^T \\
L_1A^T & hL_1A^T & hL_4^T & \tauL_1^T & \tauL_4^T \\
L_1C^T & hC^TQ_1 & hC^TQ_2 & \tauC^TQ_3 & \tauC^TQ_4 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
D^T & hD^T & hD^T & \tauD^T & \tauD^T 
\end{bmatrix},
\]
\[
\Pi_4 = \text{diag} \{-hL_7 - hL_7 - hL_7 - hL_9 - hL_10\}, \quad \Pi_5 = [0]_{6 \times 5}.
\]
\[
\Pi_6 = \text{diag} \{-\Delta_6 - h\Delta_7 - \Delta_8 - \Delta_9 - \Delta_{10}\}, \quad \Pi_7 = \begin{bmatrix}
L_1H_1^* & L_1S^T & \epsilon_1M_1 & \epsilon_2BN \\
L_1H_2^T & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
L_1H_3^T & 0 & 0 & 0 
\end{bmatrix},
\]
\[
\Pi_8 = [0]_{6 \times 4}, \quad \Pi_{10} = \text{diag} \{-\epsilon_1I - \epsilon_1I - \epsilon_2I - \epsilon_2I\}, \quad \Pi_{22} = -\beta I + \epsilon_3M^T_M + \epsilon_4GN^TNG^T.
\]

Proof. Replacing \(\hat{A}, \hat{A}_d, \hat{C}, \) and \(\hat{D}\) in (15) with \(\hat{A} = A + \Delta A(t), \hat{A}_d = A_d + \Delta A_d(t), \hat{C} = C + \Delta C(t),\) and \(\hat{D} = D + \Delta D(t)\), respectively, we have

\[
\Pi = \bar{\Pi} + \Delta \bar{\Pi} < 0,
\]

where

\[
\bar{\Pi} = \begin{bmatrix}
\Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} & \Pi_{15} & \Pi_{16} & \Pi_{17} \\
* & \Pi_{22} & \Pi_{23} & \Pi_{24} & \Pi_{25} & \Pi_{26} & \Pi_{27} \\
* & * & \Pi_{33} & \Pi_{34} & \Pi_{35} & \Pi_{36} & \Pi_{37} \\
* & * & * & \Pi_{44} & \Pi_{45} & \Pi_{46} & \Pi_{47} \\
* & * & * & * & \Pi_{55} & \Pi_{56} & \Pi_{57} \\
* & * & * & * & * & \Pi_{66} & \Pi_{67} \\
* & * & * & * & * & * & \Pi_{77} 
\end{bmatrix},
\]

\[
\Pi_{11} = \bar{A}_{d}^T P_1 + P_1 \bar{A} + P_2 + P_3 + P_4 + P_5 + W_1 + W_3 + W_4 + W_6 + W_4^T + W_3^T + W_6^T + W_4^T + W_3^T + W_6^T - \alpha P_1 - \frac{\alpha}{\tau} P_2 - 2\alpha Q_1 - 2\alpha Q_2 - 2\alpha Q_3 - 2\alpha Q_4,
\]

\[
\Pi_{12} = P_1 A_d - W_1 + W_2 + W_3^T + W_4^T + W_6^T,
\]

\[
\Pi_{13} = P_1 C + W_1 + W_3^T + W_4^T + W_6^T.
\]
Consider the following matrix equations:

\[
\Pi_3 = \begin{bmatrix}
A^T P_6 & hA^T Q_1 & hA^T Q_2 & \tau A^T Q_3 & \tau A^T Q_4 \\
A^T P_6 & hA^T Q_1 & hA^T Q_2 & \tau A^T Q_3 & \tau A^T Q_4 \\
C^T P_6 & hC^T Q_1 & hC^T Q_2 & \tau C^T Q_3 & \tau C^T Q_4 \\
D^T P_6 & hD^T Q_1 & hD^T Q_2 & \tau D^T Q_3 & \tau D^T Q_4
\end{bmatrix},
\]

\[
\Delta \Pi = \begin{bmatrix}
\Delta \Pi_1 & 0 & \Delta \Pi_2 & 0 & \Delta \Pi_3 & 0 & \Delta \Pi_4 & 0 & \Delta \Pi_5 & 0 & \Delta \Pi_6 & 0 \\
\end{bmatrix} < 0,
\]

bring formulas (4) and (8) into \(\Delta \Pi\), and according to Lemma 6, we have

\[
\Gamma_{d_1} \sigma(t) \Gamma_{e_1} + (\Gamma_{d_1} \sigma(t) \Gamma_{e_1})^T \leq \varepsilon_1 \Gamma_{d_1} \Gamma_{e_1} + \varepsilon_1^{-1} \Gamma_{e_1} \Gamma_{d_1},
\]

where

\[
\Gamma_{d_1} = \begin{bmatrix}
M^T P_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
M^T P_6 & hM^T Q_1 & hM^T Q_2 & \tau M^T Q_3 & \tau M^T Q_4
\end{bmatrix}^T,
\]

\[
\Gamma_{e_1} = \begin{bmatrix}
H_1 & H_2 & H_3 & 0 & 0 & 0 & H_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
N^T B^T P_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
N^T B^T P_6 & hN^T B^T Q_1 & hN^T B^T Q_2 & \tau N^T B^T Q_3 & \tau N^T B^T Q_4
\end{bmatrix}^T,
\]

\[
\Gamma_{d_2} = \begin{bmatrix}
S & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
N^T B^TP_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
N^T B^TP_6 & hN^T B^T Q_1 & hN^T B^T Q_2 & \tau N^T B^T Q_3 & \tau N^T B^T Q_4
\end{bmatrix}^T,
\]

Using Schur complement, equality (54) can be rewritten as

\[
\Pi + \varepsilon_1 \Gamma_{d_1} \Gamma_{d_1}^T + \varepsilon_1^{-1} \Gamma_{e_1} \Gamma_{e_1} + \varepsilon_2 \Gamma_{d_2} \Gamma_{d_2}^T + \varepsilon_2^{-1} \Gamma_{e_2} \Gamma_{e_2} < 0.
\]
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\[ \hat{\Pi}_7 = \begin{bmatrix} S_T^T & e_1 P_M & e_2 P_BN \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \]

\[ \hat{\Pi}_8 = \begin{bmatrix} 0 & e_1 P_M & e_2 P_BN \\ 0 & 0 & e_2 Q_BN \\ 0 & 0 & e_2 Q_BN \\ 0 & 0 & e_2 Q_BN \end{bmatrix}. \]

Using the Schur complement, equality (61) can be written and

\[ \Gamma = \begin{bmatrix} 0 \Gamma & \Gamma_3 \\ \Gamma_3^T & \Gamma_4 \end{bmatrix}. \]

Multiplying inequality (62) by block-diagonal matrix \( \Psi \) and combining with formulas (5) and (8), we have

\[ \Delta \Psi = \Gamma_{d3} \sigma(t) \Gamma_{e3} + \Gamma_{d3} \eta(t) \Gamma_{e3}^T + \Gamma_{d4} \eta(t) \Gamma_{e4}^T \]

\[ \leq e_3 \Gamma_{d3} \Gamma_{d3}^T + e_3^{-1} \Gamma_{e3} \Gamma_{e3} + e_4 \Gamma_{d4} \Gamma_{d4}^T + e_4^{-1} \Gamma_{e4} \Gamma_{e4}^T, \]

where \( \Gamma_{d3} = \begin{bmatrix} 0 & M_2^T & 0 \end{bmatrix}^T, \Gamma_{e3} = \begin{bmatrix} H_1 & 0 \end{bmatrix}, \Gamma_{d4} = \begin{bmatrix} 0 & N^T G^T \end{bmatrix}^T, \) and \( \Gamma_{e4} = [S \ 0] \).

Then, we can get the following inequality which ensures (58):

\[ \vec{\Psi} + \varepsilon_3 \Gamma_{d3} \Gamma_{d3}^T + \varepsilon_3^{-1} \Gamma_{e3} \Gamma_{e3} + \varepsilon_4 \Gamma_{d4} \Gamma_{d4}^T + \varepsilon_4^{-1} \Gamma_{e4} \Gamma_{e4}^T < 0, \]

Using the Schur complement, equality (61) can be rewritten as

\[ \Psi = \begin{bmatrix} -A & F \Gamma^T & H_1^T & S^T \\ -A & 0 & 0 & 0 \\ 0 & * & -e_2 I & 0 \\ 0 & * & * & -e_2 I \end{bmatrix} < 0. \]

Then, we can obtain condition (40) by pre- and post-multiplying inequality (62) by block-diagonal matrix \( \text{diag} \{P_1, I, I, I \} \).

Denoting \( \bar{L}_4 = R_1^{-1/2} L_1 R_2^{-1/2}, \bar{P}_2 = R_1^{-1/2} P_2 R_1^{-1/2}, \bar{P}_3 = R_1^{-1/2} P_3 R_2^{-1/2}, \bar{P}_4 = R_1^{-1/2} P_4 R_1^{-1/2}, \) and \( \bar{P}_5 = R_1^{-1/2} P_5 R_2^{-1/2}, \) we know that condition (16) is equivalent to (47) according to conditions (42)–(46). This completes the proof. \( \square \)

4. Simulation Example

In this part, we consider a class of neutral time-varying delayed systems with parameters described as

\[ A = \begin{bmatrix} 1.5 & 0.2 \\ 2.1 & 0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1.1 & -0.2 \\ -0.1 & -1.1 \end{bmatrix}, \quad B = \begin{bmatrix} 1.0 \\ 0.8 \end{bmatrix}, \]

\[ C = \begin{bmatrix} -0.2 & 0 \\ 0.2 & -0.1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0.1 & 0.2 \\ -0.2 & 0.1 \end{bmatrix}, \quad F = \begin{bmatrix} 1.5 & 1.7 \\ 0.2 & 0.9 \end{bmatrix}, \]

\[ G = \begin{bmatrix} 2 \\ -1.5 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 1.1 \\ -0.7 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0.8 \\ -0.4 \end{bmatrix}, \]

\[ H_1 = \begin{bmatrix} 1.4 & 0.8 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0.4 & 1.1 \end{bmatrix}, \quad H_3 = \begin{bmatrix} 0.7 & 0.2 \end{bmatrix}, \quad H_4 = \begin{bmatrix} 0.5 & 1.3 \end{bmatrix}, \]

\[ N = \begin{bmatrix} 0.2 \end{bmatrix}, \quad S = \begin{bmatrix} 0.2 & 0.6 \end{bmatrix}. \]

In this note, we choose the initial values for \( c_1 = 1, T = 5, \alpha = 0.3, \) and \( \delta = 1.0 \) and the upper bounds on the delays are \( \tau = 0.8, h = 0.5, h_1 = 0.9, \) and \( r_1 = 0.9. \) By using the LMI toolbox in MATLAB to solve LMIs (40)–(47), we can get the finite-time \( L_2-L_{\infty} \) controller gain as follows:

\[ L_4 = \begin{bmatrix} 0.6515 & -0.1789 \\ -0.1789 & 0.3827 \end{bmatrix}, \quad U = \begin{bmatrix} -0.3115 \\ -0.0343 \end{bmatrix}, \]

\[ K = U L_1^{-1} = \begin{bmatrix} -0.5768 \\ -0.3593 \end{bmatrix}. \]

with constraint conditions \( \beta = 14.7085, \gamma = 0.9923, \) and \( c_2 = 124.6975. \)

Selecting \( h(t) = 0.9/(1 + t^2), \tau(t) = 0.11/(3 + t^2), \sigma(t) = (0.9/(1 + t^2)) I, \eta(t) = (1.5/(1 + t^2)) I, \) and \( h(t) = 0.9/(1 + t^2), \)

\[ \hat{\Pi}_8 = [0]_{6 \times 4}, \]

\[ \hat{\Pi}_{10} = \text{diag} \{-e_1 I, -e_2 I, -e_1 I, -e_2 I\}. \]
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2 4 6 8 10 12 14 16 18 20

Figure 1: The trajectories of open-loop controlled system state $x(t)$.

$-3.5 -3 -2.5 -2 -1.5 -1 -0.5 0 0.5 1$

$t_0$ $2$ $4$ $6$ $8$ $10$ $12$ $14$ $16$ $18$ $20$

$-6 -5 -4 -3 -2 -1 0 1$

Figure 2: The trajectories of closed-loop controlled system state $x(t)$.

$-6 -5 -4 -3 -2 -1 0 1$

$0 5 10 15 20 25 30 35$

$-3.5 -3 -2.5 -2 -1.5 -1 -0.5 0 0.5 1$

$t_0$ $2$ $4$ $6$ $8$ $10$ $12$ $14$ $16$ $18$ $20$

Figure 3: The trajectories of closed-loop controlled system output $y(t)$.

$0 1$

$-12 -10 -8 -6 -4 -2 0 2$

Figure 4: The graph of $x^T(t)Rx(t)$ ($t \in [0, T]$) of closed-loop controlled system.

$t \in [0, 20]$, and setting the initial states $x_0 = [-0.5 0.8]^T$ and $w_0 = [0.04 0.08]^T$, we have the open-loop controlled system state simulation graph and the trajectories of closed-loop controlled system state and output as shown in Figures 1, 2, and 3, respectively. Figure 4 shows the evolution of function $x^T(t)Rx(t)$ ($t \in [0, 20]$) of the uncertain neutral time-delayed system $\Sigma_0$. Based on comparison between result in Figure 1 and result in Figure 2, we noted that the design finite-time $L_2$-$L_{\infty}$ controller can make the closed-loop controlled system achieve FTB.

5. Conclusions

This paper studied the delay-dependent resilient robust finite-time $L_2$-$L_{\infty}$ control problem for a class of uncertain neutral time-delayed system with mixed time-varying delays. A state feedback controller is designed by using LMI technique and free weighting matrices, such that the closed-loop controlled system is FTB and satisfies the input-output $L_2$-$L_{\infty}$ performance matrices. The simulation results verify the effectiveness of the design method. We will consider the finite-time observer for neutral time-delayed system in the future.

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

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