Research Article

Hankel Operators on the Weighted L^p -Bergman Spaces with Exponential Type Weights

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We characterize the boundedness and compactness of the Hankel operator with conjugate analytic symbols on the weighted L^p -Bergman spaces with exponential type weights.

1. Introduction

Let $\mathbb{D}$ be the unit disc in the complex plane $\mathbb{C}$ and $dA(z)$ the area measure on $\mathbb{D}$, and denote by $H(\mathbb{D})$ the space of all analytic functions in $\mathbb{D}$. Let $\varphi \in C^2(\mathbb{D})$ with $\Delta\varphi > 0$. For $0 < p \leq \infty$, the weighted Bergman space A^p_φ is the space of functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{p,\varphi}^p = \int_{\mathbb{D}} |f(z) e^{-\varphi(z)}|^p dA(z) < \infty \quad \text{for } 0 < p < \infty,$$

$$\|f\|_{\infty,\varphi} = \sup_{z \in \mathbb{D}} |f(z)| e^{-\varphi(z)}.$$
(1)

Note that A^p_φ is the closed subspace of $L^p_\varphi := L^p(\mathbb{D}, e^{-p\varphi} dA)$ consisting of analytic functions. Since the space A^2_φ is a reproducing kernel Hilbert space, for each $z \in \mathbb{D}$, there are functions $K_z \in A^2_\varphi$ with $f(z) = \langle f, K_z \rangle$, where $\langle \cdot, \cdot \rangle$ is the usual inner product in L^2_φ . The orthogonal projection from L^2_φ to A^2_φ is given by

$$Pf(z) = \int_{\mathbb{D}} f(w) K(w, z) e^{2\varphi} dA(w), \quad (2)$$

where $K(w, z) = \overline{K_z(w)}$.

Given $\sigma \in C^1(\mathbb{D})$ so that there exists a dense subset $\mathcal{D}$ of A^2_φ with $\sigma f \in L^2_\varphi$ for $f \in \mathcal{D}$, the big Hankel operator H_σ with symbol σ is densely defined by

$$H_\sigma f = \sigma f - P(\sigma f), \quad f \in \mathcal{D}, \quad (3)$$

where P is the orthogonal projection of L^2_φ onto A^2_φ .

We write $\bar{\partial} = \partial/\partial\bar{z}$. Then the $\bar{\partial}$ -equation can be written by

$$\bar{\partial}u = f. \quad (4)$$

For $f \in A^2_\varphi$, we look for a solution $v \in L^2_\varphi$ of minimal L^2_φ -norm. Notice that the solution of minimal norm is the one that is orthogonal to the kernel of $\bar{\partial}$ on L^2_φ ; that is, $v \perp A^2_\varphi$.

Then, if $u \in L^2_\varphi$ solves (4), we get

$$v = (I - P)u. \quad (5)$$

The linear operator $N : A^2_\varphi \rightarrow L^2_\varphi$ given by

$$N(f) = v \quad (6)$$

is called the canonical solution operator to $\bar{\partial}$ on A^2_φ .

For any $f \in A^2_\varphi$, obviously $\bar{\partial}(\bar{z}f) = f$ and

$$N(f) = (I - P)(\bar{z}f) = H_{\bar{z}}f. \quad (7)$$

That is, the canonical solution operator coincides with the big Hankel operator acting on A_φ^2 with symbol $\bar{z}$. Motivated by this fact, we now consider Hankel operators with conjugate analytic symbols on A_φ^2 . For $f, g \in A_\varphi^2$, we do not necessarily have $\bar{g}f \in L_\varphi^2$. Let $\mathcal{D} := \text{Span}\{K_z : z \in \mathbb{D}\}$. Then $\mathcal{D}$ is dense in A_φ^2 . For symbol $g \in A_\varphi^2$ such that

$$\bar{g}K_z \in L_\varphi^2 \quad \forall z \in \mathbb{D}, \quad (8)$$

we consider the densely defined big Hankel operator on A_φ^2 given by

$$H_{\bar{g}}f = (I - P)(\bar{g}f). \quad (9)$$

A positive function τ on $\mathbb{D}$ is said to belong to the class $\mathcal{E}$ if it satisfies the following three properties.

(a) There exists a constant $C_1 > 0$ such that

$$\tau(z) \leq C_1(1 - |z|), \quad \text{for } z \in \mathbb{D}. \quad (10)$$

(b) There exists a constant $C_2 > 0$ such that

$$|\tau(z) - \tau(w)| \leq C_2|z - w|, \quad \text{for } z, w \in \mathbb{D}. \quad (11)$$

(c) For each $m \geq 1$, there are constants $b_m > 0$ and $0 < t_m < 1/m$ such that

$$\tau(z) \leq \tau(w) + t_m|z - w|, \quad \text{for } |z - w| > b_m\tau(w). \quad (12)$$

In this paper, we characterize the boundedness and compactness of the Hankel operator with conjugate analytic symbols on the weighted L^p -Bergman spaces with exponential type weights as follows.

Theorem 1. Let $1 \leq p < \infty$ and $g \in A_\varphi^p$. Let $\varphi \in C^2(\mathbb{D})$ with $\Delta\varphi > 0$, and the function $\tau(z) = (\Delta\varphi(z))^{-1/2}$ is in the class $\mathcal{E}$. Then $H_{\bar{g}}$ extends to a bounded linear operator on A_φ^p if and only if

$$\sup_{z \in \mathbb{D}} \tau(z) |g'(z)| < \infty. \quad (13)$$

Theorem 2. Let $1 \leq p < \infty$ and $g \in A_\varphi^p$. Let $\varphi \in C^2(\mathbb{D})$ with $\Delta\varphi > 0$, and the function $\tau(z) = (\Delta\varphi(z))^{-1/2}$ is in the class $\mathcal{E}$. Then $H_{\bar{g}}$ extends to a compact linear operator on A_φ^p if and only if

$$\lim_{|z| \rightarrow 1^-} \tau(z) |g'(z)| = 0. \quad (14)$$

In [1], Luecking firstly proved the same results in the context of the ordinary L^2 -Bergman spaces. For L^2 -Bergman spaces with exponential type weights, the same results were proved in [2–4]. Moreover, Schatten-class Hankel operators are also indicated in their papers.

The expression $f \lesssim g$ means that there is a constant C independent of the relevant variables such that $f \leq Cg$, and $f \approx g$ means that $f \lesssim g$ and $g \lesssim f$.

2. Preliminaries

From now on we assume that $\varphi \in C^2(\mathbb{D})$, $\Delta\varphi > 0$, and the function $\tau(z) = (\Delta\varphi(z))^{-1/2}$ is in the class $\mathcal{E}$. The following notations will be frequently used:

$$m_\tau = \frac{\min(1, C_1^{-1}, C_2^{-1})}{4}, \quad (15)$$

where C_1 and C_2 are the constants in the conditions (a) and (b) in Section 1 and

$$d_\tau(z, w) = \frac{|z - w|}{\min[\tau(z), \tau(w)]}. \quad (16)$$

Lemma 3 (see [5]). For each $M \geq 1$, there exists a constant $C > 0$ (depending on M) such that for $z, w \in \mathbb{D}$, one has

$$|K(w, z)| \leq C \frac{e^{\varphi(z) + \varphi(w)}}{\tau(z)\tau(w)} \min\left[1, \frac{1}{d_\tau(z, w)}\right]^M. \quad (17)$$

By using the upper estimates for $K(w, z)$ in Lemma 3, Arroussi and Pau [5] proved that the orthogonal projection P projects L_φ^p boundedly onto A_φ^p for $1 \leq p \leq \infty$.

Lemma 4 (see [6]). Let $0 < \rho \leq m_\tau$ and $w \in \mathbb{D}$. Then,

$$\frac{3}{4}\tau(w) \leq \tau(z) \leq \frac{5}{4}\tau(w), \quad z \in D^\rho(w). \quad (18)$$

By using the third Green formula, we get the following two approximation results.

Lemma 5 (see [7]). For small $K > 0$, there exists $A = A(K) > 0$ such that

$$\sup_{w \in D^K(z)} |\varphi(w) - \varphi(z) - h_z(w)| \leq A, \quad (19)$$

where h_z is a harmonic function in $D^K(z)$ with $h_z(z) = 0$.

Lemma 6 (see [7]). For small $K > 0$, one has the estimate

$$\left| \frac{\partial\varphi}{\partial w}(w) - \frac{\partial h_z}{\partial w}(w) \right| \leq \frac{1}{\tau(w)} \quad \text{for } w \in D^K(z), \quad (20)$$

where h_z is defined in Lemma 5.

The following is a certain submean value property of $|f(z)|e^{-\varphi(z)}$. We follow the proof of ([8], Lemma 19).

Lemma 7. Let $0 < p < \infty$. For any small $r > 0$, there exists $C = C(r) > 0$ such that for any $f \in A_\varphi^p$ and $z \in \mathbb{D}$

$$\begin{aligned} (a) & |f(z)|e^{-\varphi(z)} \leq C((1/\tau(z))^2 \int_{D^r(z)} |f|^p e^{-p\varphi} dA)^{1/p}; \\ (b) & |\nabla(|f|e^{-\varphi})(z)| \leq C(1/\tau(z))((1/\tau(z))^2 \int_{D^r(z)} |f|^p e^{-p\varphi} dA)^{1/p}, \text{ provided } f(z) \neq 0. \end{aligned}$$

Proof. (a) By Lemma 5, there exists some constant $A = A(K) > 0$ such that

$$\sup_{w \in D^K(z)} |\varphi(w) - \varphi(z) - h_z(w)| \leq A, \quad (21)$$

for $z \in \mathbb{D}$. Since h_z is harmonic, there is an analytic function Φ_z on $D^K(z)$ such that $\Phi_z(z) = 0$ and $\operatorname{Re} \Phi_z = h_z$ on $D^K(z)$. Thus we have $|e^{\Phi_z}| = e^{h_z}$. Hence by the submean value property together with Lemma 5, we get

$$\begin{aligned} |f(z)|^p &= \left| f(z) e^{-(1/p)\Phi_z(z)} \right|^p \\ &\leq \frac{1}{(\pi K \tau(z))^2} \int_{D^K(z)} \left| f e^{-(1/p)\Phi_z} \right|^p dA \\ &= \frac{1}{(\pi K \tau(z))^2} \int_{D^K(z)} \left| f e^{-(1/p)h_z} \right|^p dA \\ &\leq \frac{e^{\varphi(z)}}{\tau(z)^2} \int_{D^K(z)} \left| f e^{-(1/p)\varphi} \right|^p dA. \end{aligned} \quad (22)$$

(b) We begin as follows:

$$\begin{aligned} |\nabla(|f|e^{-\varphi})| &= \left| \frac{1}{2} \frac{1}{|f|} f' \bar{f} e^{-\varphi} - |f| e^{-\varphi} \frac{\partial \varphi}{\partial w} \right| \\ &\leq \left| f' e^{-\varphi} - 2f e^{-\varphi} \frac{\partial \varphi}{\partial w} \right| \\ &\leq \left| f' e^{-\varphi} - 2f e^{-\varphi} \frac{\partial h_z}{\partial w} \right| + \left| 2f e^{-\varphi} \frac{\partial h_z}{\partial w} - 2f e^{-\varphi} \frac{\partial \varphi}{\partial w} \right| \\ &\leq \left| f' - 2f \frac{\partial h_z}{\partial w} \right| e^{-\varphi} + |f| e^{-\varphi} \left| \frac{\partial h_z}{\partial w} - \frac{\partial \varphi}{\partial w} \right|. \end{aligned} \quad (23)$$

Since h_z is harmonic, there is an analytic function $\Phi_z \in H(D^K(z))$ such that

$$\Phi_z(z) = 0, \quad \Phi'_z = 2 \frac{\partial h_z}{\partial w}, \quad |e^{\Phi_z}| = e^{h_z}. \quad (24)$$

Note that

$$\begin{aligned} |\nabla(f e^{-\Phi_z})(z)| &= |f'(z) - f(z) \Phi'_z(z)| \\ &= \left| f'(z) - 2f(z) \frac{\partial h_z}{\partial w}(z) \right|. \end{aligned} \quad (25)$$

On the other hand,

$$\left| \nabla(f e^{-\Phi_z})(z) \right| \leq \left| \int_{|z-\zeta|=\tau(z)} \frac{f(\zeta) e^{-\Phi_z(\zeta)}}{(z-\zeta)^2} d\zeta \right| \quad (26)$$

$$\leq \frac{1}{\tau(z)^2} \int_{|z-\zeta|=\tau(z)} |f(\zeta)| e^{-h_z(\zeta)} |d\zeta|.$$

For $|z-\zeta| = \tau(z)$, we have

$$\begin{aligned} |f(\zeta)| e^{-h_z(\zeta)} &\leq |f(\zeta)| e^{-\varphi(\zeta)+\varphi(z)} \\ &\leq \left(\frac{1}{\tau(z)^2} \int_{D^K(z)} |f|^p e^{-p\varphi} dA \right)^{1/p} e^{\varphi(z)}. \end{aligned} \quad (27)$$

Hence we have

$$\left| \nabla(f e^{-\Phi_z})(z) \right| \leq \frac{1}{\tau(z)} \left(\frac{1}{\tau(z)^2} \int_{D^K(z)} |f|^p e^{-p\varphi} dA \right)^{1/p} e^{\varphi(z)}. \quad (28)$$

Thus

$$\left| \nabla(|f|e^{-\varphi})(z) \right| \leq \frac{1}{\tau(z)} \left(\frac{1}{\tau(z)^2} \int_{D^K(z)} |f|^p e^{-p\varphi} dA \right)^{1/p}. \quad (29)$$

Despite that the next result was proved in [4], we give the proof of different method by using Lemma 7.

Proposition 8. *There is an $r > 0$ independent of z such that*

$$|K(w, z)| \geq \frac{e^{\varphi(z)+\varphi(w)}}{\tau(z)\tau(w)}, \quad w \in D^r(z). \quad (30)$$

Proof. Let $r > 0$. By (b) of Lemma 7, we have

$$\begin{aligned} & \left| |K(w, z)| e^{-\varphi(w)} - |K(z, z)| e^{-\varphi(z)} \right| \\ & \leq \frac{|w-z|}{\tau(z)} \left(\frac{1}{\tau(z)^2} \int_{D^r(z)} |K(\zeta, z)|^2 e^{-2\varphi(\zeta)} dA(\zeta) \right)^{1/2} \\ & \leq \frac{r}{\tau(z)} K(z, z)^{1/2}, \quad w \in D^r(z). \end{aligned} \quad (31)$$

Hence it follows that

$$\begin{aligned} |K(w, z)| e^{-\varphi(w)} &\geq K(z, z) e^{-\varphi(z)} - \frac{r}{\tau(z)} K(z, z)^{1/2}, \\ & \quad w \in D^r(z). \end{aligned} \quad (32)$$

Note that by Lemma 3 and (a) of Lemma 7, we have

$$K(z, z) \approx \tau(z)^{-2} e^{2\varphi(z)}, \quad z \in \mathbb{D}. \quad (33)$$

Thus we have

$$\begin{aligned} |K(w, z)| e^{-\varphi(w)} &\geq \tau(z)^{-2} e^{\varphi(z)} - r \tau(z)^{-2} e^{\varphi(z)} \\ &\geq (1-r) \tau(z)^{-2} e^{\varphi(z)}, \\ & \quad w \in D^r(z), \end{aligned} \quad (34)$$

if we choose small $r > 0$. $\square$

3. Hankel Operators on A_φ^p

For the proof of boundedness of Hankel operator on L^2 -Bergman spaces with exponential type weights in [2–4], they used Hörmander's L^2 -estimates for $\bar{\partial}$. However, for L^p -Bergman spaces, we need the following L^p -estimates for $\bar{\partial}$.

Theorem 9 (see [6]). *Let $\varphi \in C^2(\mathbb{D})$ with $\Delta\varphi > 0$. Suppose that the function $\tau(z) = (\Delta\varphi(z))^{-1/2}$ satisfies conditions (a) and (b) in Section 1. Let $1 \leq p \leq \infty$. Then there is a solution u to the equation $\bar{\partial}u = f$ such that*

$$\int_{\mathbb{D}} |ue^{-\varphi}|^p dA(z) \leq \int_{\mathbb{D}} \left| \frac{fe^{-\varphi}}{\sqrt{\Delta\varphi}} \right|^p dA(z), \quad (35)$$

provided the right hand side integral is finite.

Let g be an analytic function in $\mathbb{D}$ satisfying the condition (8). Let

$$k_{\zeta}(z) = \frac{K(\zeta, z)}{\sqrt{K(\zeta, \zeta)}}, \quad \zeta, z \in \mathbb{D}. \quad (36)$$

By the reproducing formula in A_{φ}^2 we get

$$H_{\bar{g}}k_{\zeta}(z) = (\overline{g(z)} - \overline{g(\zeta)})k_{\zeta}(z), \quad \zeta, z \in \mathbb{D}. \quad (37)$$

Lemma 10. *Let $0 < p < \infty$. Then*

$$\|k_z\|_{p,\varphi}^p \approx \tau(z)^{2-p}. \quad (38)$$

Proof. Consider

$$\|k_z\|_{p,\varphi}^p = \int_{|z-w| \leq \delta_0 \tau(z)} |k_z(w)|^p e^{p\varphi(w)} dA(w) + \int_{|z-w| > \delta_0 \tau(z)} |k_z(w)|^p e^{p\varphi(w)} dA(w). \quad (39)$$

First,

$$\begin{aligned} & \int_{|z-w| \leq \delta_0 \tau(z)} |k_z(w)|^p e^{p\varphi(w)} dA(w) \\ & \approx \int_{|z-w| \leq \delta_0 \tau(z)} \frac{dA(w)}{\tau(w)^p} \approx \tau(z)^{2-p}. \end{aligned} \quad (40)$$

Now,

$$\begin{aligned} & \int_{|z-w| > \delta_0 \tau(z)} |k_z(w)|^p e^{p\varphi(w)} dA(w) \\ & \leq \int_{|z-w| > \delta_0 \tau(z)} \frac{1}{\tau(w)^p} \left(\frac{\min[\tau(z), \tau(w)]}{|z-w|} \right)^{pM} dA(w). \end{aligned} \quad (41)$$

We take a large constant $M > 1$ so that $pM > 2$. Then

$$\begin{aligned} & \int_{|z-w| > \delta_0 \tau(z)} \frac{1}{\tau(w)^p} \left(\frac{\min[\tau(z), \tau(w)]}{|z-w|} \right)^{pM} dA(w) \\ & \leq \tau(z)^{pM-p} \int_{\delta_0 \tau(z) < |z-w| < 2} \frac{1}{|z-w|^{pM}} dA(w) \\ & \leq \tau(z)^{pM-p} \int_{\delta_0 \tau(z)}^2 \frac{1}{r^{pM-1}} dr \\ & \leq \tau(z)^{2-p}. \end{aligned} \quad (42)$$

Thus we get the result. $\square$

Theorem 11. *Let $1 \leq p < \infty$. Let $g \in A_{\varphi}^p$. Then $H_{\bar{g}}$ extends to a bounded linear operator on A_{φ}^p if and only if*

$$\sup_{z \in \mathbb{D}} \tau(z) |g'(z)| < \infty. \quad (43)$$

Proof. Assume first that

$$\sup_{z \in \mathbb{D}} \tau(z) |g'(z)| < \infty. \quad (44)$$

By Theorem 9, there is a solution u of the equation $\bar{\partial}u = f\bar{\partial}\bar{g}$ such that

$$\|u\|_{p,\varphi} \leq C \|\tau f \bar{\partial}\bar{g}\|_{p,\varphi}. \quad (45)$$

Since $H_{\bar{g}}f$ is the minimal L_{φ}^2 -norm solution of the $\bar{\partial}$ -equation, we have $H_{\bar{g}}f = (I - P)u$. In [5], Arroussi and Pau proved that the orthogonal projection P projects L_{φ}^p boundedly onto A_{φ}^p for $1 \leq p \leq \infty$. Thus we have

$$\|H_{\bar{g}}f\|_{p,\varphi} = \|(I - P)u\|_{p,\varphi} \leq \|u\|_{p,\varphi}. \quad (46)$$

By (45) and (46), we have

$$\|H_{\bar{g}}f\|_{p,\varphi} \leq \|\tau f \bar{\partial}\bar{g}\|_{p,\varphi} \leq \sup_{z \in \mathbb{D}} \tau(z) |g'(z)| \|f\|_{p,\varphi}, \quad (47)$$

which shows that $H_{\bar{g}}$ can be extended to a bounded linear operator on A_{φ}^p .

Conversely, assume that $H_{\bar{g}}$ is bounded on A_{φ}^p . Then we have

$$\left\| H_{\bar{g}} \left(\frac{k_z}{\|k_z\|_{p,\varphi}} \right) \right\|_{p,\varphi} < M \quad \text{for } z \in \mathbb{D}. \quad (48)$$

Using Proposition 8 and Lemma 3, there exists $r > 0$ such that

$$|k_{\zeta}(z)| = \frac{|K(\zeta, z)|}{\sqrt{K(\zeta, \zeta)}} \geq \frac{e^{\varphi(z)}}{\tau(\zeta)} \quad \text{on } z \in D^r(\zeta). \quad (49)$$

Hence we have

$$\begin{aligned} M^p & > \frac{1}{\|k_z\|_{p,\varphi}^p} \|H_{\bar{g}}k_{\zeta}\|_{p,\varphi}^p \\ & \approx \tau(z)^{p-2} \int_{\mathbb{D}} |g(z) - g(\zeta)|^p |k_{\zeta}(z)|^p e^{-p\varphi(z)} dA(z) \\ & \geq \tau(z)^{p-2} \int_{D^r(\zeta)} |g(z) - g(\zeta)|^p |k_{\zeta}(z)|^p e^{-p\varphi(z)} dA(z) \\ & \geq \frac{1}{\tau(z)^2} \int_{D^r(\zeta)} |g(z) - g(\zeta)|^p dA(z). \end{aligned} \quad (50)$$

Since g is an analytic function in $\mathbb{D}$, by the Cauchy estimates applied to $g_{\zeta}(z) := g(z) - g(\zeta)$, we can now conclude

$$\begin{aligned} & \tau(\zeta)^p |g'(\zeta)|^p \\ & \leq \frac{1}{\tau(\zeta)^2} \int_{D^r(\zeta)} |g(z) - g(\zeta)|^p dA(z) \leq M^p, \end{aligned} \quad (51)$$

$\zeta \in \mathbb{D}$.

Thus we get the result. $\square$

Lemma 12. *Let $0 < p < \infty$. Then*

$$\frac{k_z}{\|k_z\|_{p,\varphi}} \longrightarrow 0 \quad \text{uniformly on compact subsets, } |z| \longrightarrow 1^-. \quad (52)$$

Proof. Let K be a compact subset of $\mathbb{D}$. We choose a large constant M so that $M - (2/p) - 1 > 0$. Then we have for $w \in K$

$$\begin{aligned} \frac{|k_z(w)|}{\|k_z\|_{p,\varphi}} &\leq \tau(z)^{(2/p)-1} \frac{e^{\varphi(w)}}{\tau(w)} \frac{\min[\tau(z), \tau(w)]^M}{|z-w|} \\ &\leq \tau(z)^{M-(2/p)-2} \frac{1}{\text{dist}(K, z)} \sup_{w \in K} e^{\varphi(w)} \\ &\leq (1-|z|)^{M-(2/p)-2} \frac{1}{\text{dist}(K, z)} \sup_{w \in K} e^{\varphi(w)} \longrightarrow 0, \\ &|z| \longrightarrow 1^-, \end{aligned} \quad (53)$$

where $\text{dist}(K, z) = \min\{|z-w| : w \in K\}$. $\square$

Theorem 13. Let $1 \leq p < \infty$. Let $g \in A_\varphi^p$. Then $H_{\bar{g}}$ extends to a compact linear operator on A_φ^p if and only if

$$\lim_{|z| \rightarrow 1^-} \tau(z) |g'(z)| = 0. \quad (54)$$

Proof. Suppose now that $H_{\bar{g}}$ is compact on A_φ^p . Then by Riesz-Tamarkin compactness theorem, we have

$$\lim_{r \rightarrow 1^-} \frac{1}{\|k_\zeta\|_{p,\varphi}^p} \int_{r < |z| < 1} |H_{\bar{g}} k_\zeta(z)|^p e^{-p\varphi(z)} dA(z) = 0, \quad (55)$$

uniformly in $\zeta \in \mathbb{D}$. Now, by Lemma 12,

$$\begin{aligned} &\frac{1}{\|k_\zeta\|_{p,\varphi}^p} \int_{|z| \leq r} |H_{\bar{g}} k_\zeta(z)|^p e^{-p\varphi(z)} dA(z) \\ &\leq \sup_{|z| \leq r} \left(\frac{|k_\zeta(z)|}{\|k_\zeta\|_{p,\varphi}} \right)^p \\ &\times \int_{|z| \leq r} |g(z) - g(\zeta)|^p e^{-p\varphi(z)} dA(z) \longrightarrow 0, \end{aligned} \quad (56)$$

as $|\zeta| \rightarrow 1^-$. Thus we have

$$\begin{aligned} &\left\| H_{\bar{g}} \left(\frac{k_z}{\|k_z\|_{p,\varphi}} \right) \right\|_{p,\varphi} \\ &= \frac{1}{\|k_\zeta\|_{p,\varphi}} \|H_{\bar{g}} k_\zeta\|_{p,\varphi} \longrightarrow 0, \quad |\zeta| \longrightarrow 1^-. \end{aligned} \quad (57)$$

We choose $\rho > 0$ so that

$$|k_\zeta(z)| = \frac{|K(\zeta, z)|}{\sqrt{K(\zeta, \bar{\zeta})}} \geq \frac{e^{\varphi(z)}}{\tau(z)} \quad \text{on } z \in D^r(\zeta). \quad (58)$$

Then

$$\begin{aligned} &\frac{1}{\|k_\zeta\|_{p,\varphi}^p} \int_{\mathbb{D}} |H_{\bar{g}} k_\zeta(z)|^p e^{-p\varphi(z)} dA(z) \\ &\geq \frac{1}{\tau(z)^2} \int_{D^r(\zeta)} |g(z) - g(\zeta)|^p dA(z) \\ &\geq \tau(\zeta)^p |g'(\zeta)|^p. \end{aligned} \quad (59)$$

This implies that

$$\lim_{|z| \rightarrow 1^-} \tau(z) |g'(z)| = 0. \quad (60)$$

For $|\zeta| > r + \rho$, the inclusion $D^\rho(\zeta) \subset \{r < |z| < 1\}$ holds, and

$$\begin{aligned} &\frac{1}{\|k_\zeta\|_{p,\varphi}^p} \int_{r < |z| < 1} |H_{\bar{g}} k_\zeta(z)|^p e^{-p\varphi(z)} dA(z) \\ &= \frac{1}{\|k_\zeta\|_{p,\varphi}^p} \int_{r < |z| < 1} |g(z) - g(\zeta)|^p |k_\zeta(z)|^p e^{-p\varphi(z)} dA(z) \\ &\geq \frac{1}{\tau(z)^2} \int_{D^r(\zeta)} |g(z) - g(\zeta)|^p dA(z) \\ &\geq (\tau(\zeta) |g'(\zeta)|)^p. \end{aligned} \quad (61)$$

This implies that

$$\lim_{|z| \rightarrow 1^-} \tau(z) |g'(z)| = 0. \quad (62)$$

Assume now that

$$\lim_{|z| \rightarrow 1^-} \tau(z) |g'(z)| = 0. \quad (63)$$

It is enough to show that for any sequence $\{f_n\}$ that is bounded in norm and converges uniformly to zero on compact subsets, we have $\|H_{\bar{g}} f_n\|_{p,\varphi} \rightarrow 0$ as $n \rightarrow \infty$. As in relation (46), we have

$$\|H_{\bar{g}} f_n\|_{p,\varphi}^p \leq \int_{\mathbb{D}} |g'|^p \tau^p |f_n|^p e^{-p\varphi} dA. \quad (64)$$

Now

$$\int_{\mathbb{D}} |g'|^p \tau^p |f_n|^p e^{-p\varphi} dA = \int_{|z| \leq r} + \int_{r < |z| < 1} |g'|^p \tau^p |f_n|^p e^{-p\varphi} dA. \quad (65)$$

Since $\{f_n\}$ converges uniformly to zero on compact subsets,

$$\begin{aligned} &\int_{|z| \leq r} |g'|^p \tau^p |f_n|^p e^{-p\varphi} dA \\ &\leq \sup_{|z| \leq r} |f_n|^p \int_{|z| \leq r} |g'|^p \tau^p e^{-p\varphi} dA \longrightarrow 0, \\ &n \longrightarrow \infty. \end{aligned} \quad (66)$$

Now

$$\begin{aligned} &\int_{r < |z| < 1} |g'|^p \tau^p |f_n|^p e^{-p\varphi} dA \\ &\leq \sup_{r < |z| < 1} |g'|^p \tau^p \int_{\mathbb{D}} |f_n|^p e^{-p\varphi} dA \longrightarrow 0, \\ &r \longrightarrow 1^-. \end{aligned} \quad (67)$$

Hence $H_{\bar{g}}$ is compact. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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