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Research Article

Strong Convergence Algorithms of the Split Common Fixed Point Problem for Total Quasi-Asymptotically Pseudocontractive Operators

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We present a new algorithm for solving the two-set split common fixed point problem with total quasi-asymptotically pseudocontractive operators and consider the case of quasi-pseudocontractive operators. Under some appropriate conditions, we prove that the proposed algorithms have strong convergence. The results presented in this paper improve and extend the previous algorithms and results of Censor and Segal (2009), Moudafi (2011 and 2010), Mohammed (2013), Yang et al. (2011), Chang et al. (2012), and others.

1. Introduction

Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. To allow for constraints both in the domain and range of A, Censor and Elfving [1] originally formulated the split feasibility problem (SFP), which is to find a member of set

$$\Omega = \{x \in C : Ax \in Q\} \neq \emptyset. \tag{1}$$

A recent generalization, due to Censor and Segal in [2], is called the split common fixed point problem (SCFPP), which is to find a point x^* satisfying

$$x^* \in C := \bigcap_{i=1}^{t} \operatorname{Fix}(U_i), \qquad Ax^* \in Q := \bigcap_{j=1}^{r} \operatorname{Fix}(T_j), \tag{2}$$

where $U_i : H_1 \to H_1$ $(i = 1, 2, \ldots t)$ and $T_j : H_2 \to H_2$ $(j = 1, 2, \ldots, r)$ are some nonlinear operators and $A : H_1 \to H_2$ is also a bounded linear operator. Denote the solution set of SCFPP by

$$\Gamma = \{x^* \in C \mid Ax^* \in Q\}. \tag{3}$$

In particular, if $t = r = 1$, problem (2) is reduced to the two-set SCFPP, where $C := \operatorname{Fix}(U)$ and $Q := \operatorname{Fix}(T)$, and the SFP can be retrieved by picking as operators U and T orthogonal projections.

Censor and Segal [2] invented the following CQ-algorithm with directed operators to solve the two-set SCFPP:

$$\forall x_0 \in H_1, \quad x_{n+1} = U\left(x_n - \gamma A^*(I - T)Ax_n\right), \quad n \geq 0, \tag{4}$$

where $x_0 \in H$ and $\gamma \in (0, 2/L)$; L is the largest eigenvalue of the matrix A^*A.

Inspired by the work of Censor and Segal, for $\alpha_n \in (0, 1)$, Moudafi presented the following iteration with the demicontractive mappings and quasi-nonexpansive operators in papers [3] and [4], respectively:

$$\begin{aligned} u_n &= x_n - \gamma A^*(I - T)Ax_n, \\ x_{n+1} &= (1 - \alpha_n)u_n + \alpha_n U(u_n), \quad \forall x_0 \in H_1,\ n \geq 0. \end{aligned} \tag{5}$$

Moudafi's results are weak convergence. In [5, 6], Mohammed utilized the strongly quasi-nonexpansive operators and quasi-nonexpansive operators to solve recursion (5) and obtain weak and strong convergence, respectively. Strong

convergence of (5) with pseudo-demicontractive and firmly pseudo-demicontractive mappings can be found in [7, 8]. Furthermore, for several different strong convergence recursions with nonexpansive operators for solving the SCFPP see [9, 10]. For the purpose of generalization, papers [11–13] discussed the total asymptotically strictly pseudocontractive mappings and asymptotically strict pseudocontractive mappings for solving (2) and multiple-set fixed point problem (MSSFP) by the following iteration:

$$\begin{aligned} u_n &= x_n - \gamma A^* \left(I - T^n\right) A x_n, \\ x_{n+1} &= \left(1 - \alpha_n\right) u_n + \alpha_n U^n \left(u_n\right), \quad \forall x_1 \in H_1,\ n \geq 1, \end{aligned} \tag{6}$$

which is of weak convergence; when U is semicompact, strong convergence of (6) can be deduced. Obviously, (5) is the particular case of (6). On the other hand, papers [14, 15] presented cyclic algorithms of the SCFPP for directed operators and demicontractive mappings, and the results converge weakly.

However, we found that the strong convergence of (6) needs the condition of U to be semicompact. In order to obtain strong algorithm for the two-set SCFPP without more constraints on U or T and continue to generalize the operators, in this paper, we propose a different iteration, which can ensure the strong convergence with more general case when the operators are total quasi-asymptotically pseudocontractive, demiclosed at the origin. We can choose an initial data $x_1 \in H_1$ arbitrarily and define the sequence $\{x_n\}$ by the recursion:

$$\begin{aligned} u_n &= x_n - \gamma A \left(I - T^n\right) A x_n, \\ y_n &= \left(1 - \beta\right) u_n + \beta U^n \left(u_n\right), \\ x_{n+1} &= \alpha_n \psi \left(y_n\right) + \left(1 - \alpha_n\right) y_n, \quad n \geq 1, \end{aligned} \tag{7}$$

where $\psi : H_1 \to H_1$ is a δ-contraction with $\delta \in (0, 1)$, T and U are total quasi-asymptotically pseudocontractive mappings, and $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three real sequences satisfying appropriate conditions. Under some mild conditions, we prove that the sequence $\{x_n\}$ generated by (7) converges strongly to the solution of the two-set SCFPP.

2. Preliminaries

In order to reach the main results, we first recall the following facts.

Let C be a nonempty closed and convex subset of a real Hilbert space H with the inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Denote by $\mathrm{Fix}(T)$ the set of fixed points of a mapping T; that is, $\mathrm{Fix}(T) = \{x \in C : Tx = x\}$.

Definition 1 (see [2, 3, 16, 17]). (i) Recalled that $T : C \to C$ is said to be a directed or firmly quasi-nonexpansive operator; if $p \in \mathrm{Fix}(T)$, then

$$\left\|Tx - p\right\|^2 \leq \left\|x - p\right\|^2 - \left\|x - Tx\right\|^2, \quad \forall x \in C. \tag{8}$$

(ii) Let D be a closed convex nonempty set of C; $T : C \to C$ is nonexpansive; we say that T is attracting with respect to D, if, for every $x \in C \setminus D$, $p \in D$,

$$\left\|Tx - p\right\| < \left\|x - p\right\|. \tag{9}$$

(iii) A mapping $T : C \to C$ is said to be paracontracting or quasi-nonexpansive; if $p \in \mathrm{Fix}(T)$, then

$$\left\|Tx - p\right\| \leq \left\|x - p\right\|. \tag{10}$$

(iv) A mapping $T : C \to C$ is said to be demicontractive or strictly quasi-pseudocontractive; for $p \in \mathrm{Fix}(T)$, there exists a constant $\beta \in [0, 1)$ such that

$$\left\|Tx - p\right\|^2 \leq \left\|x - p\right\|^2 + \beta \|x - Tx\|^2, \quad \forall x \in C. \tag{11}$$

Definition 2 (see [11, 18]). (i) Let $T : C \to C$ be a total quasi-asymptotically strictly pseudocontractive if $F(T) \neq \emptyset$, and there exist a constant $\beta \in [0, 1]$, sequences $\{\mu_n\} \subset [0, \infty)$, and $\{\xi_n\} \subset [0, \infty)$ with $\mu_n \to 0$ and $\xi_n \to 0$ as $n \to \infty$ such that

$$\begin{aligned} \left\|T^n x - p\right\|^2 \leq \left\|x - p\right\|^2 + \beta \left\|x - T^n x\right\|^2 + \mu_n \phi \left(\left\|x - p\right\|\right) + \xi_n, \\ \forall n \geq 1,\ x \in C,\ p \in \mathrm{Fix}\left(T\right), \end{aligned} \tag{12}$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and strictly increasing function with $\phi(0) = 0$.

(ii) A mapping $T : C \to C$ is said to be total quasi-asymptotically pseudocontractive if $F(T) \neq \emptyset$, and there exist sequences $\{\mu_n\} \subset [0, \infty)$ and $\{\xi_n\} \subset [0, \infty)$ with $\mu_n \to 0$ and $\xi_n \to 0$ as $n \to \infty$ such that

$$\begin{aligned} \left\|T^n x - p\right\|^2 \leq \left\|x - p\right\|^2 + \left\|x - T^n x\right\|^2 + \mu_n \phi \left(\left\|x - p\right\|\right) + \xi_n, \\ \forall n \geq 1,\ x \in C,\ p \in \mathrm{Fix}\left(T\right), \end{aligned} \tag{13}$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and strictly increasing function with $\phi(0) = 0$.

(iii) A mapping $T : C \to C$ is said to be quasi-pseudocontractive if $\mathrm{Fix}(T) \neq \emptyset$, such that

$$\left\|Tx - p\right\|^2 \leq \left\|x - p\right\|^2 + \|x - Tx\|^2,\ x \in C,\ p \in \mathrm{Fix}\left(T\right). \tag{14}$$

(iv) A mapping $T : C \to C$ is said to be uniformly k-Lipschitzian if there is a constant $k > 0$, such that

$$\left\|T^n x - T^n y\right\| \leq k \left\|x - y\right\|, \quad \forall n \geq 1,\ \forall x, y \in C. \tag{15}$$

Remark 3. Note that the classes of directed operators and attracting operators belong to the class of paracontracting operators. The class of paracontracting operators belongs to the class of demicontractive operators, while the class of quasi-pseudocontractive operators includes the class of demicontractive operators. Further, the class of total quasi-asymptotically pseudocontractive operators, with quasi-pseudocontractive operators as a special case, includes the class of total quasi-asymptotically strictly pseudocontractive operators.

Remark 4. Let $T : C \to C$ be a total quasi-asymptotically pseudocontractive, if $F(T) \neq \emptyset$, for each $x \in C$ and $q \in \mathrm{Fix}(T)$; from (13) we can easily obtain the following equivalent inequalities:

$$\langle x - T^n x, x - p\rangle \geq -\frac{\mu_n}{2}\phi(\|x-p\|) - \frac{\xi_n}{2}; \tag{16}$$

$$\langle x - T^n x, p - T^n x\rangle \leq \|x - T^n x\|^2 + \frac{\mu_n}{2}\phi(\|x-p\|) + \frac{\xi_n}{2}; \tag{17}$$

$$\langle x - p, T^n x - p\rangle \leq \|x-p\|^2 + \frac{\mu_n}{2}\phi(\|x-p\|) + \frac{\xi_n}{2}. \tag{18}$$

Lemma 5 (see [19]). *Consider*

(i) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2$, *for all* $x, y \in H$;

(ii) $\|(1-t)x + ty\|^2 = (1-t)\|x\|^2 + t\|y\|^2 - t(1-t)\|x-y\|^2$, *for all* $x, y \in H$ *and* $t \in \mathbb{R}$.

Lemma 6 (see [18]). *Let C be a bounded and closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a uniformly L-Lipschitz and total quasi-asymptotically pseudocontractive mapping with* $\mathrm{Fix}(T) \neq \emptyset$. *Suppose there exist positive constants M and M^*, for the function ϕ in* (13), $\phi(\zeta) \leq M^*\zeta^2$ *for all* $\zeta \geq M$ *such that*

$$\phi(\zeta) \leq \phi(M) + M^*\zeta^2. \tag{19}$$

Then $\mathrm{Fix}(T)$ *is a closed convex subset of C.*

Lemma 7 (see [20]). *A mapping $I - T : C \to C$ is said to be demiclosed at zero, if for any sequence $\{x_n\} \in C$, such that $x_n \rightharpoonup x^* \in C$ and $(I-T)x_n \to 0$ as $n \to \infty$; then $(I-T)x^* = 0$.*

Lemma 8 (see [21]). *Let $\{r_n\}$, $\{s_n\}$, and $\{t_n\}$ be sequences of nonnegative real numbers satisfying*

$$r_{n+1} \leq (1 + t_n) r_n + s_n, \quad n \geq 1. \tag{20}$$

If $\sum_{n=1}^{\infty} t_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$, then the limit $\lim_{n\to\infty} r_n$ exists.

Lemma 9 (see [22]). *Let a sequence $\{t_n\} \in [0, 1)$ satisfy $\lim_{n\to\infty} t_n = 0$ and $\sum_{n=1}^{\infty} t_n = \infty$. Let $\{a_n\}$ be a sequence of nonnegative real numbers that satisfies any of the following conditions.*

(i) *For all $\varepsilon > 0$, there exists an integer $N \geq 1$ such that, for all $n \geq N$,*

$$a_{n+1} \leq (1 - t_n) a_n + t_n \varepsilon; \tag{21}$$

(ii) $a_{n+1} \leq (1 - t_n)a_n + o_n$, $n \geq 0$, *where $o_n \geq 0$ satisfies* $\lim_{n\to\infty} o_n/t_n = 0$;

(iii) $a_{n+1} \leq (1 - t_n)a_n + t_n c_n$, *where* $\overline{\lim}_{n\to\infty} c_n \leq 0$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

In this section, we will prove the strong convergence of (7) to solve the two-set SCFPP.

Theorem 10. *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $U : H_1 \to H_1$ be a uniformly k_1-Lipschitz and $(\{\mu_n^{(1)}\}, \{\xi_n^{(1)}\}, \phi_1)$-total quasi-asymptotically pseudocontractive mapping, $T : H_2 \to H_2$ a uniformly k_2-Lipschitz, and $(\{\mu_n^{(2)}\}, \{\xi_n^{(2)}\}, \phi_2)$-total quasi-asymptotically pseudocontractive mappings satisfying the following conditions:*

(C_1) $C := \mathrm{Fix}(U) \neq \emptyset$, $Q := \mathrm{Fix}(T) = \emptyset$;

(C_2) $\mu_n = \max\{\mu_n^{(1)}, \mu_n^{(2)}\}$, $\xi_n = \max\{\xi_n^{(1)}, \xi_n^{(2)}\}$, $n \geq 1$, *and* $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \xi_n < \infty$;

(C_3) $\phi = \max\{\phi_1, \phi_2\}$ *and* $\exists M, M^* > 0$.

Let $\psi : H_1 \to H_1$ be a δ-contraction with $\delta \in (0, 1)$. Let $A : H_1 \to H_2$ be a bounded linear operator. For $\forall x_1 \in H_1$, sequence $\{x_n\}$ can be generated by the iteration (7), *where the sequence $\{\alpha_n\} \subset (0, 1)$ satisfies (i) $\lim_{n\to\infty}\alpha_n = 0$ and (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\{\beta\} \subset [a, b]$ with $a, b \in (0, 1/(1 + k_1))$, and $\{\gamma\} \subset (0, 2/L)$ with L being the largest eigenvalue of the matrix $A^T A$. Assume that $I - U$ and $I - T$ are demiclosed at zero. If $\Gamma \neq \emptyset$, then $\{x_n\}$ generated by* (7) *converges strongly to a solution of the two-set SCFPP.*

Proof. (1) First of all, we show that, for $\forall p \in \Gamma$, $\{x_n\}$ generated by (7) is bounded.

From (7), (16), and Lemma 6, we have

$$\begin{aligned}\|u_n - p\|^2 &= \|x_n - p + \gamma A^*(T^n - I)Ax_n\|^2 \\ &= \|x_n - p\|^2 + \gamma^2\|A^*(I - T^n)Ax_n\|^2 \\ &\quad + 2\gamma\langle Ax_n - Ap, T^n Ax_n - Ax_n\rangle \end{aligned} \tag{22}$$

$$\begin{aligned}&\leq \|x_n - p\|^2 + \gamma^2 L^2\|(I - T^n)Ax_n\|^2 \\ &\quad + \gamma\mu_n\phi(\|Ax_n - Ap\|) + \gamma\xi_n \end{aligned} \tag{23}$$

$$\begin{aligned}&\leq (1 + \gamma\mu_n M^* L)\|x_n - p\|^2 \\ &\quad + \gamma^2 L^2\|(I - T^n)Ax_n\|^2 + \gamma\mu_n\phi(M) + \gamma\xi_n. \end{aligned} \tag{24}$$

Since

$$\begin{aligned}\|(I - T^n)Ax_n\| &\leq \|Ax_n - Ap\| + \|T^n Ax_n - Ap\| \\ &\leq (\|A\| + k_2\|A\|)\|x_n - p\|, \end{aligned} \tag{25}$$

substituting (25) into (24), we have

$$\begin{aligned}\|u_n - p\|^2 &\leq \left[1 + \gamma\mu_n M^* L + \gamma^2 L^2(\|A\| + k_2\|A\|)^2\right] \\ &\quad \times \|x_n - p\|^2 + \gamma\mu_n\phi(M) + \gamma\xi_n \\ &= (1 + a_n)\|x_n - p\|^2 + \gamma\mu_n\phi(M) + \gamma\xi_n, \end{aligned} \tag{26}$$

where $a_n = \gamma\mu_n M^* L + \gamma^2 L^2 (\|A\| + k_2\|A\|)^2$; by condition (C_2), we know

$$\sum_{n=1}^{\infty} a_n < \infty. \tag{27}$$

Next, from (7), (13), and Lemma 5, we can get

$$\begin{aligned}
\|y_n - p\|^2 &= \|(1-\beta)(u_n - p) + \beta(U^n(u_n) - p)\|^2 \\
&= (1-\beta)\|u_n - p\|^2 + \beta\|U^n(u_n) - p\|^2 \\
&\quad - \beta(1-\beta)\|u_n - U^n(u_n)\|^2 \\
&\le (1-\beta)\|u_n - p\|^2 + \beta\left[\|u_n - p\|^2 + \|u_n - U^n(u_n)\|^2\right. \\
&\qquad \left. + \mu_n\phi(\|u_n - p\|) + \xi_n{}^2\right] \\
&\quad - \beta(1-\beta)\|u_n - U^n(u_n)\|^2 \\
&= \|u_n - p\|^2 + \beta^2\|u_n - U^n(u_n)\|^2 \\
&\quad + \beta\mu_n\phi(\|u_n - p\|) + \beta\xi_n;
\end{aligned} \tag{28}$$

we also can see that

$$\begin{aligned}
\|u_n - U^n(u_n)\| &\le \|u_n - p\| + \|U^n(u_n) - p\| \\
&\le (1 + k_1)\|u_n - p\|;
\end{aligned} \tag{29}$$

then substituting (29) into (28) and from (26), we have

$$\begin{aligned}
\|y_n - p\|^2 &\le \left[1 + \beta^2(1+k_1)^2\right]\|u_n - p\|^2 \\
&\quad + \beta\mu_n\phi(\|u_n - p\|) + \beta\xi_n \\
&\le \left[1 + \beta^2(1+k_1)^2 + \beta\mu_n M^*\right]\|u_n - p\|^2 \\
&\quad + \beta\mu_n\phi(M) + \beta\xi_n \\
&\le (1+b_n)(1+a_n)\|x_n - p\|^2 + \mu_n\phi(M) \\
&\quad \times [(1+b_n)\gamma + \beta] + \xi_n[(1+b_n)\gamma + \beta],
\end{aligned} \tag{30}$$

where $b_n = \beta^2(1+k_1)^2 + \beta\mu_n M^*$, and we also know that

$$\sum_{n=1}^{\infty} b_n < \infty. \tag{31}$$

From (7) and Lemma 5, we also have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n[\psi(y_n) - p] + (1-\alpha_n)(y_n - p)\|^2 \\
&\le \alpha_n\|\psi(y_n) - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\
&\le (1 + 2\alpha_n\delta)\|y_n - p\|^2 \\
&\quad + 2\alpha_n\|\psi(p) - p\|^2 - \alpha_n\|y_n - p\|^2 \\
&\le (1 + 2\alpha_n\delta)\|y_n - p\|^2 + 2\alpha_n\|\psi(p) - p\|^2.
\end{aligned} \tag{32}$$

Substituting (30) into (32) and simplifying it we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le (1 + 2\alpha_n\delta)(1+b_n)(1+a_n) \\
&\quad \times \|x_n - p\|^2 + 2\alpha_n\|\psi(p) - p\|^2 \\
&\quad + (1 + 2\alpha_n\delta)\{\mu_n\phi(M)[(1+b_n)\gamma + \beta] \\
&\qquad + \xi_n[(1+b_n)\gamma + \beta]\} \\
&= [1 + a_n + (1+a_n)(b_n + 2a_n\delta + 2a_n b_n\delta)] \\
&\quad \times \|x_n - p\|^2 + 2\alpha_n\|\psi(p) - p\|^2 \\
&\quad + (1 + 2\alpha_n\delta)\{\mu_n\phi(M)[(1+b_n)\gamma + \beta] \\
&\qquad + \xi_n[(1+b_n)\gamma + \beta]\};
\end{aligned} \tag{33}$$

set

$$\begin{aligned}
t_n &= a_n + (1+a_n)(b_n + 2a_n\delta + 2a_n b_n\delta), \\
s_n &= 2\alpha_n\|\psi(p) - p\|^2 + (1 + 2\alpha_n\delta) \\
&\quad \times \{\mu_n\phi(M)[(1+b_n)\gamma + \beta] + \xi_n[(1+b_n)\gamma + \beta]\};
\end{aligned} \tag{34}$$

(33) can be rewritten as

$$\|x_{n+1} - p\|^2 \le (1 + t_n)\|x_n - p\|^2 + s_n; \tag{35}$$

by condition (C_2), (27), and (31), we know that $\sum_{n=1}^{\infty} t_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Thus it follows from Lemma 8 that the following limit exists:

$$\lim_{n\to\infty}\|x_n - p\|. \tag{36}$$

Therefore, we obtain that $\{x_n\}$ is bounded, so is $\{u_n\}$. Set $z_n = U^n(u_n)$. Then $\{z_n\}$ is also bounded.

(2) Next we prove $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, $\lim_{n\to\infty}\|u_{n+1} - u_n\| = 0$.

For each $n \ge 1$, $\forall u_n \in H_1$, assume there exists $v_i^{(n)} \in C$ $(i = 1, 2)$ such that $u_n = wv_1^{(n)} + (1-w)v_2^{(n)}$ for $w \in (0, 1)$. Then for all $q \in C$, and by virtue of (16), we have

$$\begin{aligned}
\|u_n - U^n(u_n)\|^2 &= \langle u_n - U^n(u_n), u_n - U^n(u_n)\rangle \\
&= \frac{1}{\beta}\langle u_n - y_n, u_n - U^n(u_n)\rangle \\
&= \frac{1}{\beta}\langle u_n - U^n(u_n) - (y_n - U^n(y_n)), u_n - y_n\rangle \\
&\quad + \frac{1}{\beta}\langle y_n - U^n(y_n), u_n - y_n\rangle \\
&\le \frac{1}{\beta}(\|u_n - y_n\| + \|U^n(u_n) - U^n(y_n)\|) \\
&\quad \times \|u_n - y_n\| + \frac{1}{\beta}\langle u_n - q, y_n - U^n(y_n)\rangle
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\beta}\langle q-y_n, y_n-U^n(y_n)\rangle \\
&\le \frac{1+k_1}{\beta}\|u_n-y_n\|^2+\frac{1}{\beta}\langle u_n-q, y_n-U^n(y_n)\rangle \\
&+\frac{1}{\beta}\left[\frac{\mu_n}{2}\phi(\|y_n-q\|)+\frac{\xi_n}{2}\right] \\
&\le (1+k_1)\beta\|u_n-U^n(u_n)\|^2 \\
&+\frac{1}{\beta}\langle u_n-q, y_n-U^n(y_n)\rangle \\
&+\frac{1}{\beta}\left[\frac{\mu_n}{2}\left[M^*\|y_n-q\|^2+\phi(M)\right]+\frac{\xi_n}{2}\right],
\end{aligned}
\tag{37}
$$

which implies that

$$
\begin{aligned}
&\beta[1-(1+k_1)\beta]\|u_n-U^n(u_n)\|^2 \\
&\quad\le \langle u_n-z, y_n-U^n(y_n)\rangle \\
&\quad+\left[\frac{\mu_n}{2}\left[M^*\|y_n-z\|^2+\phi(M)\right]+\frac{\xi_n}{2}\right].
\end{aligned}
\tag{38}
$$

Now we take $q=v_i^{(n)}$ $(i=1,2)$ in (38); multiplying w and $(1-w)$ on the two side of (38), respectively, and then adding up, we can obtain

$$
\begin{aligned}
&\beta[1-(1+k_1)\beta]\|u_n-U^n(u_n)\|^2 \\
&\quad\le \left[\frac{\mu_n}{2}\left[M^*\|y_n-z\|^2+\phi(M)\right]+\frac{\xi_n}{2}\right].
\end{aligned}
\tag{39}
$$

Letting $n\to\infty$ in (39), we have

$$
\lim_{n\to\infty}\|u_n-U^n(u_n)\|=0. \tag{40}
$$

From (7), we know that

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &= \|y_n-p+\alpha_n(\psi(y_n)-y_n)\|^2 \\
&= \|y_n-p\|^2+2\alpha_n\langle y_n-p, \psi(y_n)-y_n\rangle \\
&\quad+\alpha_n^2\|\psi(y_n)-y_n\|^2.
\end{aligned}
\tag{41}
$$

Letting $n\to\infty$ in (41) and by condition (i) in Theorem 10, we know

$$
\lim_{n\to\infty}\|x_n-p\|=\lim_{n\to\infty}\|y_n-p\|. \tag{42}
$$

Similarly,

$$
\begin{aligned}
\|y_n-p\|^2 &= \|u_n-p+\beta(u_n-U^n(u_n))\|^2 \\
&= \|u_n-p\|^2+2\beta\langle u_n-p, u_n-U^n(u_n)\rangle \\
&\quad+\beta^2\|u_n-U^n(u_n)\|^2;
\end{aligned}
\tag{43}
$$

from (40) the limit of $\|y_n-p\|$ exists and

$$
\lim_{n\to\infty}\|x_n-p\|=\lim_{n\to\infty}\|y_n-p\|=\lim_{n\to\infty}\|u_n-p\|. \tag{44}
$$

Therefore, when we take limit on both sides of (22), we can deduce that

$$
\lim_{n\to\infty}\|Ax_n-T^nAx_n\|=0. \tag{45}
$$

Then,

$$
\begin{aligned}
\|x_{n+1}-x_n\| &\le \|y_n-x_n\|+\alpha_n\|\psi(y_n)-y_n\| \\
&\le \|u_n-x_n\|+\beta\|u_n-U^n(u_n)\| \\
&\quad+\alpha_n\|\psi(y_n)-y_n\| \\
&\le \gamma\|A\|\|Ax_n-T^nAx_n\|+\beta\|u_n-U^n(u_n)\| \\
&\quad+\alpha_n\|\psi(y_n)-y_n\|.
\end{aligned}
\tag{46}
$$

In view of (40) and (45) we have that

$$
\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{47}
$$

Similarly, it follows from (7), (45), and (47) that

$$
\begin{aligned}
&\|u_{n+1}-u_n\| \\
&\quad=\left\|x_{n+1}-x_n-\gamma A^*\left(I-T^{n+1}\right)Ax_{n+1}-\gamma A^*\left(I-T^n\right)Ax_n\right\| \\
&\quad\le \|x_{n+1}-x_n\|+\gamma\|A^*\|\left\|Ax_{n+1}T^{n+1}-Ax_{n+1}\right\| \\
&\qquad+r\|A^*\|\|Ax_nT^n-Ax_n\|\longrightarrow 0\ \ (n\longrightarrow\infty).
\end{aligned}
\tag{48}
$$

(3) Next we prove that $\|x_n-U(u_n)\|\to 0$, as $n\to\infty$. From (40) and (48), we have

$$
\begin{aligned}
&\|u_n-U(u_n)\| \\
&\le \|u_n-U^n(u_n)\|+\|U^n(u_n)-U(u_n)\| \\
&\le \|u_n-U^n(u_n)\|+k_1\left\|U^{n-1}(u_n)-u_n\right\| \\
&\le \|u_n-U^n(u_n)\|+k_1\left[\left\|U^{n-1}(u_n)-U^{n-1}(u_{n-1})\right\|\right. \\
&\qquad\qquad\left.+\left\|U^{n-1}(u_{n-1})-u_n\right\|\right] \\
&\le \|u_n-U^n(u_n)\|+k_1^2\|u_n-u_{n-1}\| \\
&\quad+k_1\left[\left\|U^{n-1}(u_{n-1})-u_{n-1}\right\|+\|u_{n-1}-u_n\|\right] \\
&\longrightarrow\infty\ \ (n\to\infty).
\end{aligned}
\tag{49}
$$

By the same way, from (45) and (47) we can also prove that

$$
\|Ax_n-TAx_n\|\longrightarrow 0,\quad n\longrightarrow\infty. \tag{50}
$$

Therefore, from (44) and (49), we know

$$\begin{aligned}\|x_n - U(u_n)\| &\le \|x_n - u_n\| + \|u_n - U(u_n)\| \\ &\longrightarrow 0 \quad (n \longrightarrow \infty).\end{aligned} \tag{51}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to a point x^*. Without loss of generality, we may assume that $\{x_n\}$ converges weakly to x^*. Therefore, from (49)–(51) and Lemma 7, we have $x^* \in \mathrm{Fix}(U)$.

(4) Finally, we prove that $x_n \to x^*$ in norm. To do this, we calculate

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &\quad = \langle \alpha_n \psi(y_n) + (1-\alpha_n) y_n - x^*, x_{n+1} - x^* \rangle \\ &\quad = \langle \alpha_n \psi(y_n) + (1-\alpha_n) y_n - x^*, x_{n+1} - x^* \rangle \\ &\quad = \alpha_n \langle \psi(y_n) - x^*, x_{n+1} - x^* \rangle \\ &\qquad + (1-\alpha_n) \langle y_n - x^*, x_{n+1} - x^* \rangle \\ &\quad \le \alpha_n \langle \psi(y_n) - \psi(x^*), x_{n+1} - x^* \rangle \\ &\qquad + \alpha_n \langle \psi(x^*) - x^*, x_{n+1} - x^* \rangle \\ &\qquad + \frac{1-\alpha_n}{2}\|y_n - x^*\|^2 + \frac{1-\alpha_n}{2}\|x_{n+1} - x^*\|^2 \\ &\quad \le \frac{\alpha_n \delta}{2}\|y_n - x^*\|^2 + \frac{\alpha_n}{2}\|x_{n+1} - x^*\|^2 \\ &\qquad + \alpha_n \langle \psi(x^*) - x^*, x_{n+1} - x^* \rangle \\ &\qquad + \frac{1-\alpha_n}{2}\|y_n - x^*\|^2 + \frac{1-\alpha_n}{2}\|x_{n+1} - x^*\|^2 \\ &\quad = \frac{1-(1-\delta)\alpha_n}{2}\|y_n - x^*\|^2 + \frac{1}{2}\|x_{n+1} - x^*\|^2 \\ &\qquad + \alpha_n \langle \psi(x^*) - x^*, x_{n+1} - x^* \rangle.\end{aligned} \tag{52}$$

Therefore, we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le (1-(1-\delta)\alpha_n)\|y_n - x^*\|^2 \\ &\quad + 2\alpha_n \langle \psi(x^*) - x^*, x_{n+1} - x^* \rangle.\end{aligned} \tag{53}$$

Substituting (23) into (28), we have

$$\begin{aligned}\|y_n - x^*\|^2 &\le \|x_n - p\|^2 + \gamma^2 L^2 \|(I - T^n) A x_n\|^2 \\ &\quad + \beta^2 \|u_n - U^n(u_n)\|^2 \\ &\quad + \mu_n [\gamma\phi(\|Ax_n - Ap\|) + \beta\phi(\|u_n - p\|)] \\ &\quad + \xi_n(\gamma + \beta).\end{aligned} \tag{54}$$

Since $(1-\delta)\alpha_n \in (0,1)$ and substituting (53) into (51), we get

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &\quad \le (1-(1-\delta)\alpha_n)\|x_n - p\|^2 \\ &\qquad + (1-(1-\delta)\alpha_n) \\ &\quad \times \{\gamma^2 L^2 \|(I - T^n) A x_n\|^2 + \beta^2 \|u_n - U^n(u_n)\|^2 \\ &\qquad + \mu_n [\gamma\phi(\|Ax_n - Ap\|) + \beta\phi(\|u_n - p\|)] + \xi_n(\gamma + \beta)\}.\end{aligned} \tag{55}$$

Let

$$\begin{aligned}o_n = {} & (1-(1-\delta)\alpha_n) \\ & \times \{\gamma^2 L^2 \|(I - T^n) A x_n\|^2 + \beta^2 \|u_n - U^n(u_n)\|^2 \\ & + \mu_n [\gamma\phi(\|Ax_n - Ap\|) + \beta\phi(\|u_n - p\|)] + \xi_n(\gamma + \beta)\}.\end{aligned} \tag{56}$$

Equation (55) can be rewritten as

$$\|x_{n+1} - x^*\|^2 \le (1-(1-\delta)\alpha_n)\|x_n - p\|^2 + o_n. \tag{57}$$

Evidently, from (40), (45), and Lemma 9 (ii), we can conclude that $x_{n+1} - x^* \to 0$ $(n \to \infty)$.

This completes the proof. □

The following theorem can be concluded from Theorem 10 immediately.

Theorem 11. *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $U : H_1 \to H_1$ be a uniformly k_1-Lipschitz and quasi-pseudocontractive mapping with $C := \mathrm{Fix}(U) \neq \emptyset$. Let $T : H_2 \to H_2$ be a uniformly k_2-Lipschitz and quasi-pseudocontractive mapping with $Q := \mathrm{Fix}(T) = \emptyset$. Let $\psi : H_1 \to H_1$ be a δ-contraction with $\delta \in (0,1)$. Let $A : H_1 \to H_2$ be a bounded linear operator. For $\forall x_1 \in H_1$, sequence $\{x_n\}$ can be generated by the iteration:*

$$\begin{aligned}u_n &= x_n - \gamma A(I - T) A x_n, \\ y_n &= (1-\beta) u_n + \beta U(u_n), \\ x_{n+1} &= \alpha_n \psi(y_n) + (1-\alpha_n) y_n, \quad n \ge 1,\end{aligned} \tag{58}$$

where the sequence $\{\alpha_n\} \subset (0,1)$ satisfies (i) $\lim_{n\to\infty}\alpha_n = 0$ and (ii) $\sum_{n=0}^{\infty}\alpha_n = \infty$, $\{\beta\} \subset [a,b]$ with $a, b \in (0, 1/(1+k_1))$, and $\{\gamma\} \subset (0, 2/L)$ with L being the largest eigenvalue of the matrix $A^T A$. Assume that $I - U$ and $I - T$ are demiclosed at zero. If $\Gamma \neq \emptyset$, then $\{x_n\}$ generated by (58) converges strongly to a solution of the two-set SCFPP.

Proof. For each $p \in \Gamma$, if we take $T = T^n$, $U = U^n$, $\mu_n \to 0$ and $\xi_n \to 0$, and follow the proof of Theorem 10, we can also prove that $\{x_n\}$ converges strongly to $x^* \in \Gamma$ by the same way. □

Remark 12. Algorithm (7) and Theorems 10 and 11 improve and extend the corresponding results of Censor and Segal [2], Moudafi [3, 4], Mohammed [5, 6], Chang et al. [11, 13], Yang et al. [12], and others.

4. Concluding Remarks

In this work, we develop the split common fixed point problem with more general classes of total quasi-asymptotically pseudocontractive and quasi-pseudocontractive operators; corresponding algorithms are improved based on the viscosity iteration; thus we can obtain strong convergence without more constraints on operators.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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