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Research Article

Ground States for the Schrödinger Systems with Harmonic Potential and Combined Power-Type Nonlinearities

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We consider a class of coupled nonlinear Schrödinger systems with potential terms and combined power-type nonlinearities. We establish the existence of ground states, by using a variational method. As an application, some symmetry results for ground states of Schrödinger systems with harmonic potential terms are obtained.

1. Introduction

In this paper, we are interested in the steady state of the coupled nonlinear Schrödinger system

$$\begin{aligned} -i\varphi_{1t} &= \Delta\varphi_1 - Q_1(x)\varphi_1 + \mu_1|\varphi_1|^p\varphi_1 \\ &\quad + \sum_{k=1}^{m}\alpha_k|\varphi_1|^{p_k-2}|\varphi_2|^{q_k}\varphi_1, \\ -i\varphi_{2t} &= \Delta\varphi_2 - Q_2(x)\varphi_2 + \mu_2|\varphi_2|^p\varphi_2 \\ &\quad + \sum_{k=1}^{m}\beta_k|\varphi_1|^{p_k}|\varphi_2|^{q_k-2}\varphi_1, \end{aligned} \tag{1}$$

in $\mathbb{R}^N$, where $\mu_k, \alpha_k, \beta_k, p_k$, and q_k are real constants, $p_k+q_k = p+2$, $\alpha_k q_k = \beta_k p_k (k = 1, 2, \ldots, m)$, and $1 < p < 4/(N-2)^+$, ($(N-2)^+ = N-2$ when $N \geq 3$ and $4/(N-2)^+ = \infty$ when $N = 1, 2$). System (1) has applications in many physical problems, especially in the Hartree-Fock theory for a double Bose-Einstein condensate with interparticle interactions under the magnetic trap. In physic, $Q_i(x)$ is the trapping potential for the ith species, whose role is to confine the movement of particles. We remark that the harmonic potential $|x|^2$ is a widely used trapping potential in current experiments [1].

We call solutions of forms $\varphi_1(x,t) = e^{i\omega_1 t}u(x)$, $\varphi_2(x,t) = e^{i\omega_2 t}v(x)(\omega_i > 0)$ standing wave solutions to (1), where (u, v) solves the following elliptic system:

$$\begin{aligned} -\Delta u + V_1(x)u &= \mu_1|u|^p u + \sum_{k=1}^{m}\alpha_k|u|^{p_k-2}|v|^{q_k}u, \\ -\Delta v + V_2(x)v &= \mu_2|v|^p v + \sum_{k=1}^{m}\beta_k|u|^{p_k}|v|^{q_k-2}v. \end{aligned} \tag{2}$$

Here $V_i(x) = Q_i(x) + \omega_i (i = 1, 2)$.

The energy functional of (2) is

$$\begin{aligned} I(u,v) := \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2 + V_1(x)u^2 + |\nabla v|^2 + V_2(x)v^2\right) \\ - \frac{1}{p+2}\int_{\mathbb{R}^N}\left(\mu_1|u|^{p+2} + \mu_2|v|^{p+2}\right. \\ \left. + \sum_{k=1}^{m}(\alpha_k+\beta_k)|u|^{p_k}|v|^{q_k}\right) \end{aligned} \tag{3}$$

and the work space $E := \{(u,v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V_1(x)u^2dx + \int_{\mathbb{R}^N} V_2(x)v^2dx < \infty\}$. Since $E \hookrightarrow L^{p+2}(\mathbb{R}^N) \times L^{p+2}(\mathbb{R}^N)$ compactly (for $0 \leq p < 4/(N-2)$) and E is a Hilbert space with norm $\|(u,v)\|_E^2 = \int_{\mathbb{R}^N}(|\nabla u|^2 +$

$V_1(x)u^2)dx + \int_{\mathbb{R}^N}(|\nabla v|^2 + V_2(x)v^2)dx$ (see [2, 3]), the energy functional $I(u,v)$ makes sense in the work space.

We say that $(u,v) \in E$ is a nontrivial ($u \not\equiv 0$ or $v \not\equiv 0$) bound state of (2) if (u,v) is a nontrivial critical point of I. The ground state is usually defined as the positive minimizer of the following minimization problem:

$$\min_{(w,z)\in E} \{I(w,z) \mid dI(w,z) = 0\}. \tag{4}$$

The existence and structure of ground states for Schrödinger equation or systems have been investigated by many authors (see [4–19] and the references therein).

In the case of a single nonlinear Schrödinger equation $-\Delta u = f(u)$, $x \in \mathbb{R}^N$, under some appropriate conditions on f, the ground state exists and is radially symmetric [5, 8, 19]. For the single equation with potential terms

$$-\Delta u + V(x)u = f(x,u), \quad u > 0, \text{ in } \mathbb{R}^N, \tag{5}$$

Rabinowitz [14] used variational methods based on variants of mountain pass theorem to prove that (5) has a positive ground state, for those $V(x)$ satisfying $\lim_{|x|\to\infty} V(x) = +\infty$ and $V(x) \geq c > 0$. Similar results were also obtained in [6].

In the case of nonlinear Schrödinger systems, Lin and Wei [11] considered two-component systems of nonlinear Schrödinger equations with trap potentials:

$$\begin{gathered} -\epsilon^2\Delta u_j + V_j(x)u_j = \sum_{j=1}^{2}\beta_{ij}u_i^2u_j, \quad x \in \mathbb{R}^N, \\ u_j > 0, \quad u_j \in H^1(\mathbb{R}^N), \quad j = 1,2, \end{gathered} \tag{6}$$

where $N = 2,3$, $\epsilon > 0$. Among other things, they showed that there is some $\beta_0 > 0$ such that if $\beta_{11} > 0$, $\beta_{22} > 0$, $\beta_{12} = \beta_{21} \in (-\infty, \beta_0)$, $\lim_{|x|\to\infty} V_j(x) = +\infty$, and $\inf_{x\in\mathbb{R}^N} V_j > 0$, $j = 1,2$, then the ground state solution to (6) always exists.

Sirakov [17] studied a system of two equations,

$$\begin{gathered} -\Delta u + u = \mu_1 u^3 + \beta u v^2, \quad x \in \mathbb{R}^N, \\ -\Delta v + \lambda v = \mu_2 v^3 + \beta u^2 v, \quad x \in \mathbb{R}^N, \end{gathered} \tag{7}$$

and found that there are always ranges of positive parameters in (7), for which it has a ground state, and ranges of positive parameters for which it does not have ground state.

Maia et al. [13] considered the weakly couple nonlinear elliptic system

$$\begin{gathered} -\Delta u + u = |u|^{2q-2}u + b|v|^q|u|^{q-2}u, \quad x \in \mathbb{R}^N, \\ -\Delta v + \omega^2 v = |v|^{2q-2}v + b|u|^q|v|^{q-2}v, \quad x \in \mathbb{R}^N, \end{gathered} \tag{8}$$

where $1 < q < N/(N-2)$ for $N \geq 3$ and $q > 1$ for $N = 1,2$. They showed that if b is sufficiently large, then there exists a nontrivial positive ground state of (8).

Ma and Zhao [12] considered

$$\begin{gathered} -\Delta u + u = \mu_1|u|^{2p}u + \beta_1|v|^{p+1}|u|^{p-1}u, \quad x \in \mathbb{R}^N, \\ -\Delta v + v = \mu_2|v|^{2p}v + \beta_2|u|^{p+1}|v|^{p-1}v, \quad x \in \mathbb{R}^N, \end{gathered} \tag{9}$$

in which $0 < p < 2/(n-2)^+$. Under assumptions $\mu_1, \mu_2 \leq 0$, $\beta_1, \beta_2 > 0$, $\mu_1\beta_2^p = \mu_2\beta_2^p$, and $\mu_1 + \beta_2^{(p+1)/2}/\beta_1^{(p-1)/2} > 0$, or $\mu_2 + \beta_1^{(p+1)/2}/\beta_2^{(p-1)/2} > 0$, then the ground state of (9) exists and is unique up to translations.

Song [18] obtained the existence of ground states for a system of Schrödinger equations with combined power-type nonlinearities and with no trap potentials. It is natural to consider similar results for Schrödinger system with potential terms and the combined power-type nonlinearities.

Motived by the above work, in this paper we focus on the existence and symmetric properties for the ground states of Schrödinger system (2). For simplicity, we only prove the existence result for ground state of

$$\begin{gathered} -\Delta u + V(x)u = \mu|u|^p u + \sum_{k=1}^{m}\alpha_k|u|^{p_k-2}|v|^{q_k}u, \quad \text{in } \mathbb{R}^N, \\ -\Delta v + V(x)v = \mu|v|^p v + \sum_{k=1}^{m}\beta_k|u|^{p_k}|v|^{q_k-2}v, \quad \text{in } \mathbb{R}^N, \end{gathered} \tag{10}$$

where $1 < p < 4/(N-2)^+$, $N \leq 5$, and μ_k, α_k, β_k, p_k, and q_k are real constants satisfying

$$1 < p_k < p+1, \quad 1 < q_k < p+1, \tag{11}$$

$$p_k + q_k = p+2, \quad \alpha_k q_k = \beta_k p_k, \quad (k = 1,2,\ldots,m). \tag{12}$$

To be precise, our first result reads as follows.

Theorem 1. *Consider* (11) *and* (12), *where*

$$\mu < \frac{\sum_{k=1}^{m}(\alpha_k + \beta_k)}{2(2^{p/2} - 1)}, \tag{13}$$

$$\mu > -\sum_{k=1}^{m}\min\{0, \alpha_k\}, \qquad \mu > -\sum_{k=1}^{m}\min\{0, \beta_k\}. \tag{14}$$

Suppose the potential term $V(x)$ satisfies

$$\begin{gathered} V(x) \in C^1(\mathbb{R}^N, \mathbb{R}), \qquad V(x) \geq c_0 > 0, \\ \lim_{|x|\to\infty} V(x) = +\infty. \end{gathered} \tag{15}$$

Then the ground state of system (10) *exists.*

Remark 2. Condition (13) is the same as (1.10) and (1.11) in [18], in which Schrödinger systems (2) with $V(x) \equiv 0$ were considered. The result in [18] is suitable for the case with all coefficients being positive, while our result can be applied to system (2) with some negative α_k and β_k.

As an application, we give a symmetric result for system with harmonic potential in $\mathbb{R}^3$:

$$\begin{aligned} -\Delta u + \left(|x|^2+1\right)u &= \mu|u|^p u + \sum_{k=1}^{m}\alpha_k|u|^{p_k-2}|v|^{q_k}u, \quad \text{in } \mathbb{R}^3, \\ -\Delta v + \left(|x|^2+1\right)v &= \mu|v|^p v + \sum_{k=1}^{m}\beta_k|u|^{p_k}|v|^{q_k-2}v, \quad \text{in } \mathbb{R}^3, \end{aligned} \tag{16}$$

where $0 < p < 4$ and μ_k, α_k, β_k, p_k, and q_k are real constants which satisfy (11), (12), (13), and (14). In this case the work space

$$\Sigma := \left\{(u,v) \in H^1\left(\mathbb{R}^3\right) \times H^1\left(\mathbb{R}^3\right) \mid \int_{\mathbb{R}^3}\left(|x|^2|u|^2+|x|^2|v|^2\right)dx < +\infty\right\}. \tag{17}$$

We first show that each solution of (16) is classic and decays at infinity.

Theorem 3. *Suppose $(u,v) \in \Sigma$ solves (16) with $2 < p < 4$, μ_k, α_k, β_k, p_k, and q_k being real constants satisfying $2 < p_k < p+1$, $2 < q_k < p+1$, and $p_k + q_k = p+2$. Then*

(a) *$u, v \in W^{2,l}(\mathbb{R}^3)$, for every $2 \le l < \infty$;*

(b) *$\lim_{|x|\to\infty} u(x) = 0$ and $\lim_{|x|\to\infty} v(x) = 0$;*

(c) *$u, v \in C^2(\mathbb{R}^3)$.*

Using the above regularity and decay result, one can easily show that the ground state is radial symmetric.

Corollary 4. *Assume $\alpha_k > 0$, $\beta_k > 0$, $2 < p_k < p+1$, $2 < q_k < p+1$, $p_k + q_k = p+2$, (12), (13), and (14). Then there exists a radial symmetric ground state of system (16).*

An outline of this paper is as follows. We devote Section 2 to some preparations and the proof of Theorem 1. The proofs of Theorem 3 and Corollary 4 will be given in Section 3.

2. Existence of Ground State

In this section, we always assume (11), (12), (13), (14), and (15). Also, we denote by $\|\cdot\|_p$ the $L^p(\mathbb{R}^N)$ norm for $1 \le p \le \infty$. Recall that the work space $E := \{(u,v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V(x)(u^2+v^2)dx < \infty\}$ is a Hilbert space with norm $\|(u,v)\|_E = \left(\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2)dx + \int_{\mathbb{R}^N}(|\nabla v|^2 + V(x)v^2)dx\right)^{1/2}$.

We now introduce some lemmas, which will be needed in the proof of Theorem 1.

Lemma 5. *All critical points of energy function $I(u,v)$ in E are weak solutions of system (10).*

Proof. Suppose $(u,v) \in E$ is a critical point of I; that is,

$$dI(u,v)[\varphi_1,\varphi_2] = 0, \quad \forall \varphi_1,\varphi_2 \in E. \tag{18}$$

Direct computation shows that, for all $\varphi_1, \varphi_2 \in E$, we have

$$\begin{aligned} 0 &= dI(u,v)[\varphi_1,\varphi_2] \\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(I(u+\epsilon\varphi_1, v+\epsilon\varphi_2) - I(u,v)\right) \\ &= \int_{\mathbb{R}^N}\left(\nabla u\cdot\nabla\varphi_1 + \nabla v\cdot\nabla\varphi_2 + V(x)u\cdot\varphi_1 + V(x)v\cdot\varphi_2\right)dx \\ &\quad - \int_{\mathbb{R}^N}\left(\mu|u|^p u\varphi_1 + \mu|v|^p v\varphi_2\right)dx \\ &\quad - \int_{\mathbb{R}^N}\frac{\sum_k(\alpha_k+\beta_k)p_k}{p+2}|u|^{p_k-2}u|v|^{q_k}\varphi_1 dx \\ &\quad - \int_{\mathbb{R}^N}\frac{\sum_k(\alpha_k+\beta_k)q_k}{p+2}|u|^{p_k}|v|^{q_k-2}v\varphi_2 dx. \end{aligned} \tag{19}$$

Using (12), we have

$$\frac{\sum_k(\alpha_k+\beta_k)p_k}{p+2} = \sum_k\alpha_k, \qquad \frac{\sum_k(\alpha_k+\beta_k)q_k}{p+2} = \sum_k\beta_k. \tag{20}$$

Hence, for all $\varphi_1, \varphi_2 \in E$

$$\begin{aligned} 0 &= dI(u,v)[\varphi_1,\varphi_2] \\ &= \int_{\mathbb{R}^N}\left(\nabla u\cdot\nabla\varphi_1 + V(x)u\cdot\varphi_1 - \mu|u|^p u\cdot\varphi_1 - \sum_k\alpha_k|u|^{p_k-2}u|v|^{q_k}\cdot\varphi_1\right)dx \\ &\quad + \int_{\mathbb{R}^N}\left(\nabla v\cdot\nabla\varphi_2 + V(x)v\cdot\varphi_2 - \mu|v|^p v\cdot\varphi_2 - \sum_k\beta_k|u|^{p_k}|v|^{q_k-2}v\cdot\varphi_2\right)dx. \end{aligned} \tag{21}$$

Therefore, (u,v) is a weak solution of system (10). □

Define $T_1 := \{(u,v) \in E \mid u \not\equiv 0 \text{ or } v \not\equiv 0\}$ and the Nehari manifold

$$\begin{aligned} N_1 := \Bigg\{(u,v) &\in T_1 \mid \int_{\mathbb{R}^N}\left(|\nabla u|^2 + V(x)u^2 + |\nabla v|^2 + V(x)v^2\right)dx \\ &= \int_{\mathbb{R}^N}\left(\mu|u|^{p+2} + \mu|v|^{p+2} + \sum_{k=1}^{m}(\alpha_k+\beta_k)|u|^{p_k}|v|^{q_k}\right)dx\Bigg\}. \end{aligned} \tag{22}$$

Note that $N_1 \neq \emptyset$ when (11), (12), (14), and (15) are satisfied. Actually, by using (12) and (14), one can prove that, for all $(u,v) \in T_1$,

$$\int_{\mathbb{R}^N} \left(\mu|u|^{p+2} + \mu|v|^{p+2} + \sum_{k=1}^{m} (\alpha_k + \beta_k) |u|^{p_k} |v|^{q_k} \right) dx > 0. \tag{23}$$

Therefore, we can choose

$$\begin{aligned} t = \Bigg(& \left(\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x) u^2 + |\nabla v|^2 + V(x) v^2 \right) dx \right) \\ & \times \left(\int_{\mathbb{R}^N} \left(\mu|u|^{p+2} + \mu|v|^{p+2} \right.\right. \\ & \left.\left. + \sum_{k=1}^{m} (\alpha_k + \beta_k) |u|^{p_k} |v|^{q_k} \right) dx \right)^{-1} \Bigg)^{1/p} > 0 \end{aligned} \tag{24}$$

such that $(tu, tv) \in N_1$.

The following lemma shows that distance between the Nehari manifold N_1 and $(0,0)$ is positive in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ or in $L^{p+2}(\mathbb{R}^N) \times L^{p+2}(\mathbb{R}^N)$.

Lemma 6. *Assume* $1 < p < 4/(N-2)^+$, $N \leq 5$, $\mu > 0$, *and* (15). *Then there are positive constants* $\epsilon_1, \epsilon_2 > 0$, *such that*

$$\begin{aligned} \|u\|_{H^1(\mathbb{R}^N)} + \|v\|_{H^1(\mathbb{R}^N)} \geq \epsilon_1 > 0, \quad \forall (u,v) \in N_1, \\ \|u\|_{p+2}^{p+2} + \|v\|_{p+2}^{p+2} \geq \epsilon_2 > 0, \quad \forall (u,v) \in N_1. \end{aligned} \tag{25}$$

Proof. Choose $0 < e < \min\{c_0, 1\}$. For each $(u,v) \in N_1$, by using (15) and Young's inequality, we have

$$\begin{aligned} & e\left(\|u\|_{H^1(\mathbb{R}^N)}^2 + \|v\|_{H^1(\mathbb{R}^N)}^2 \right) \\ & \leq \int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x) u^2 + |\nabla v|^2 + V(x) v^2 \right) \\ & = \int_{\mathbb{R}^N} \left(\mu|u|^{p+2} + \mu|v|^{p+2} + \sum_{k=1}^{m} (\alpha_k + \beta_k) |u|^{p_k} |v|^{q_k} \right) dx \\ & \leq \mu \left(\|u\|_{p+2}^{p+2} + \|v\|_{p+2}^{p+2} \right) \\ & \quad + \sum_{k=1}^{m} |\alpha_k + \beta_k| \int_{\mathbb{R}^N} \left(\frac{p_k}{p+2} |u|^{p+2} + \frac{q_k}{p+2} |v|^{p+2} \right) dx \\ & \leq C \left(\|u\|_{p+2}^{p+2} + \|v\|_{p+2}^{p+2} \right). \end{aligned} \tag{26}$$

Since $H^1(\mathbb{R}^N)$ is embedded into $L^{p+2}(\mathbb{R}^N)(0 < p < 4/(N-2)^+)$, we know that

$$\begin{aligned} \|u\|_{p+2}^{p+2} + \|v\|_{p+2}^{p+2} & \leq C' \left(\|u\|_{H^1(\mathbb{R}^N)}^{p+2} + \|v\|_{H^1(\mathbb{R}^N)}^{p+2} \right) \\ & \leq C' \left(\|u\|_{H^1(\mathbb{R}^N)}^2 + \|v\|_{H^1(\mathbb{R}^N)}^2 \right)^p, \end{aligned} \tag{27}$$

where we have used the fact that $p > 1$.

Therefore, combining the above inequality with (26), we obtain

$$\|u\|_{H^1(\mathbb{R}^N)}^2 + \|v\|_{H^1(\mathbb{R}^N)}^2 \geq \epsilon_1 := \left(\frac{e}{CC'} \right)^{1/(p-1)} > 0. \tag{28}$$

Putting (28) into (26), we obtain

$$\|u\|_{p+2}^{p+2} + \|v\|_{p+2}^{p+2} \geq \epsilon_2 := \left(\frac{e}{C} \right) \epsilon_1 > 0. \tag{29}$$

□

For each $(u,v) \in T_1$, define

$$f_{(u,v)}(t) := I(tu, tv), \quad t \in [0, +\infty). \tag{30}$$

Since $f_{(u,v)}(0) = 0$, $f_{(u,v)}(+\infty) = -\infty$, $f'_{(u,v)}(0) = 0$, and $f''_{(u,v)}(0) > 0$, there holds $f_{(u,v)}(t) > 0$ for $t > 0$ small and $f_{(u,v)}(t) < 0$ for t large. Therefore,

$$\max_{t \geq 0} f_{(u,v)}(t) > 0 \tag{31}$$

and it is achieved at some $t = \psi(u,v) > 0$. Let

$$\varphi(u,v) := \max_{t \geq 0} f_{(u,v)}(t). \tag{32}$$

Solving $f'_{(u,v)}(\psi(u,v)) = 0$, we obtain

$$\begin{aligned} & \psi(u,v) \\ & = \Bigg(\left(\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x) u^2 + |\nabla v|^2 + V(x) v^2 \right) \right) \\ & \quad \times \left(\int_{\mathbb{R}^N} \left(\mu|u|^{p+2} + \mu|v|^{p+2} \right.\right. \\ & \quad \left.\left. + \sum_{k=1}^{m} (\alpha_k + \beta_k) |u|^{p_k} |v|^{q_k} \right) \right)^{-1} \Bigg)^{1/p}. \end{aligned} \tag{33}$$

Recalling (24), we can see $(\psi(u,v)u, \psi(u,v)v) \in N_1$.

To conclude, for each $(u,v) \in T_1$, there hold

$$\begin{aligned} I(\psi(u,v)u, \psi(u,v)v) = \max_{t \geq 0} I(tu, tv) = \varphi(u,v) > 0, \\ (\psi(u,v)u, \psi(u,v)v) \in N_1. \end{aligned} \tag{34}$$

Define

$$d := \inf_{(u,v) \in T_1} \varphi(u,v) = \inf_{(u,v) \in T_1} \max_{t \geq 0} I(tu, tv), \tag{35}$$

$$c := \inf_{(u,v) \in N_1} I(u,v). \tag{36}$$

Proposition 7. *Consider $d = c \geq 0$.*

Proof. Firstly, for all $(u, v) \in N_1$,

$$I(u,v) = \frac{p}{2(p+2)} \int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2 + |\nabla v|^2 + V(x)v^2\right) > 0, \tag{37}$$

which implies $c \geq 0$.

Next, we show that $d = c$. On one hand, for all $(u, v) \in N_1 \subset T_1$, we know that $\psi(u, v) = 1$ and $f'_{(u,v)}(1) = 0$, which imply

$$I(u,v) = \max_{t\geq 0} I(tu, tv) = \varphi(u,v) \geq \inf_{(u,v)\in T_1} \varphi(u,v) = d. \tag{38}$$

That is, $c \geq d$. On the other hand, for all $(u, v) \in T_1$,

$$\max_{t\geq 0} I(tu, tv) = I(\psi(u,v)u, \psi(u,v)v) \geq c, \tag{39}$$

where we have used the fact that $(\psi(u, v)u, \psi(u, v)v) \in N_1$. That is, $d \geq c$. Therefore, $d = c \geq 0$. □

Lemma 8. *$\varphi(u, v)$ is sequentially weakly lower semicontinuous on N_1 with respect to E.*

Proof. Let $(u_n, v_n) \rightharpoonup (u_0, v_0)$ weakly in E. By the uniform boundedness theorem, $\{(u_n, v_n)\}$ is bounded in E. Moreover, since the embedding $E \hookrightarrow L^{p+2}(\mathbb{R}^N) \times L^{p+2}(\mathbb{R}^N)$ is compact (see [2, 3]), there is a subsequence $(u_n, v_n) \to (u_0, v_0)$ strongly in $L^{p+2}(\mathbb{R}^N) \times L^{p+2}(\mathbb{R}^N)$. By using Fatou's lemma, we obtain

$$\begin{aligned}
\varphi(u_0, v_0) &= \max_{t\geq 0} I(tu_0, tv_0) \\
&= I(\psi(u_0,v_0)u_0, \psi(u_0,v_0)v_0) \\
&\leq \liminf_{n\to\infty} I(\psi(u_0,v_0)u_n, \psi(u_0,v_0)v_n) \\
&\leq \liminf_{n\to\infty} \max_{t\geq 0} I(tu_n, tv_n) \\
&= \liminf_{n\to\infty} \varphi(u_n, v_n),
\end{aligned} \tag{40}$$

which ends the proof. □

Lemma 9. *c is achieved on N_1.*

Proof. Let $\{(u_n, v_n)\} \subset N_1$ be a minimizing sequence for c; that is, $\lim_{n\to\infty} I(u_n, v_n) = c$. We have $I(u_n, v_n) = (p/2(p+1)) \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)u_n^2 + |\nabla v_n|^2 + V(x)v_n^2) = (p/2(p+1))(\|u_n\|_E^2 + \|v_n\|_E^2)$, which implies $\{(u_n, v_n)\}$ is bounded in E. Since $E \hookrightarrow L^{p+2}(\mathbb{R}^N) \times L^{p+2}(\mathbb{R}^N)$ compactly, up to a subsequence, there is $(u_0, v_0) \in E$ satisfying

$$\begin{aligned}
&(u_n, v_n) \rightharpoonup (u_0, v_0) \quad \text{weakly in } E, \\
&(u_n, v_n) \longrightarrow (u_0, v_0), \\
&\text{strongly in } L^{p+2}(\mathbb{R}^N) \times L^{p+2}(\mathbb{R}^N).
\end{aligned} \tag{41}$$

By using Lemma 6, $\|u_n\|_{p+2}^{p+2} + \|v_n\|_{p+2}^{p+2} \geq \epsilon_2$, we have $(u_0, v_0) \neq (0, 0)$. Hence, $(u_0, v_0) \in T_1$.

Using Lemma 8, Proposition 7, and the fact that $\varphi(u_n, v_n) = I(u_n, v_n)$, we obtain

$$\begin{gathered}
c = d \leq \varphi(u_0, v_0) \leq \liminf_{n\to\infty} \varphi(u_n, v_n) = \liminf_{n\to\infty} I(u_n, v_n) = c, \\
\varphi(u_0, v_0) = c.
\end{gathered} \tag{42}$$

By using (36)

$$c \leq \varphi(\psi(u_0,v_0)u_0, \psi(u_0,v_0)v_0) = \varphi(u_0, v_0) = c. \tag{43}$$

Let

$$(u_c, v_c) := (\psi(u_0,v_0)u_0, \psi(u_0,v_0)v_0). \tag{44}$$

We obtain $(u_c, v_c) \in N_1$ and it is a minimizer for c. □

Lemma 10. *(u_c, v_c) defined as in (44) is a critical point of I.*

Proof. Define

$$\begin{aligned}
G(u,v) &:= \int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2 + |\nabla v|^2 + V(x)v^2\right) dx \\
&\quad - \int_{\mathbb{R}^N} \left(\mu|u|^{p+2} + \mu|v|^{p+2} + \sum_k (\alpha_k + \beta_k)|u|^{p_k}|v|^{q_k}\right) dx.
\end{aligned} \tag{45}$$

From the proof of Lemma 9, we know that (u_c, v_c) is a minimizer for $\min_{(u,v)\in T_1} I(u, v)$ under the constrained condition $G(u, v) = 0$. Hence, there is a Lagrange multiplier λ such that

$$(dI(u_c, v_c) + \lambda dG(u_c, v_c))[\varphi_1, \varphi_2] = 0, \quad \forall (\varphi_1, \varphi_2) \in E. \tag{46}$$

Choosing $(\varphi_1, \varphi_2) = (u_c, 0)$, we have

$$\begin{aligned}
0 &= \int_{\mathbb{R}^N} \left(|\nabla u_c|^2 + V(x)u_c^2 - \mu|u_c|^{p+2} - \sum_{k=1}^m \alpha_k |u_c|^{p_k}|v_c|^{q_k}\right) dx \\
&\quad + \lambda \int_{\mathbb{R}^N} \left(2|\nabla u_c|^2 + 2V(x)u_c^2 - (p+2)\mu|u_c|^{p+2} - \sum_{k=1}^m (\alpha_k + \beta_k) p_k |u_c|^{p_k}|v_c|^{q_k}\right) dx.
\end{aligned} \tag{47}$$

Choosing $(\varphi_1, \varphi_2) = (0, v_c)$, we have

$$\begin{aligned} 0 = \int_{\mathbb{R}^N} \Big(& |\nabla v_c|^2 + V(x) v_c^2 - \mu |v_c|^{p+2} \\ & - \sum_{k=1}^{m} (\alpha_k + \beta_k) |u_c|^{p_k} |v_c|^{q_k} \Big) dx \\ + \lambda \int_{\mathbb{R}^N} \Big(& 2|\nabla v_c|^2 + 2V(x) v_c^2 - (p+2) \mu |v_c|^{p+2} \\ & - \sum_{k=1}^{m} (\alpha_k + \beta_k) q_k |u_c|^{p_k} |v_c|^{q_k} \Big) dx. \end{aligned} \tag{48}$$

Combining (47) with (48) and using $p_k + q_k = p+2$, we obtain

$$\begin{aligned} 0 = & G(u_c, v_c) + 2\lambda G(u_c, v_c) \\ & - \lambda p \int_{\mathbb{R}^N} \Big(\mu |u_c|^{p+2} + \mu |v_c|^{p+2} \\ & + \sum_{k=1}^{m} (\alpha_k + \beta_k) |u_c|^{p_k} |v_c|^{q_k} \Big) dx. \end{aligned} \tag{49}$$

By the fact that $(u_c, v_c) \in N_1$, namely, $G(u_c, v_c) = 0$, that is, $\int_{\mathbb{R}^N} (\mu |u_c|^{p+2} + \mu |v_c|^{p+2} + \sum_{k=1}^{m} (\alpha_k + \beta_k) |u_c|^{p_k} |v_c|^{q_k}) = \|u_c\|_E^2 + \|v_c\|_E^2$, we obtain

$$0 = \lambda p \left(\|u_c\|_E^2 + \|v_c\|_E^2 \right). \tag{50}$$

Since $(u_c, v_c) \in T_1$, $(u_c, v_c) \neq (0,0)$, which leads to $\lambda = 0$. Therefore, $dI(u_c, v_c) = 0$. □

Let us now give a proof of Theorem 1.

Proof of Theorem 1. Lemma 10 shows that (u_c, v_c) is the critical point of energy functional I. By Lemma 5, (u_c, v_c) is a solution of system (10). We now claim that $u_c \not\equiv 0$ and $v_c \not\equiv 0$. Hence $(|u_c|, |v_c|)$ is a ground state of system (10).

Assume for contradiction that $u_c \not\equiv 0$, $v_c \equiv 0$. On one hand, one can see that u_c is a solution of scalar equation

$$-\Delta u + V(x) u = \mu |u|^p u. \tag{51}$$

Using Lemma 9 and Proposition 7, we have $d = c = I(u_c, 0) = J(u_c)$, where

$$J(u) := \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x) u^2 \right) dx - \frac{\mu}{p+2} \int_{\mathbb{R}^N} |u|^{p+2} dx. \tag{52}$$

Let $M = \{v \in H^1(\mathbb{R}^N) | \int_{\mathbb{R}^N} V(x) v^2 dx < +\infty, \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x) v^2) dx = \mu \int_{\mathbb{R}^N} v^{p+2} dx\}$. Let w be the ground state of (51) (see [14] for the existence). There hold

$$\int_{\mathbb{R}^N} \left(|\nabla w|^2 + V(x) w^2 \right) dx = \mu \int_{\mathbb{R}^N} w^{p+2} dx \tag{53}$$

and the least energy

$$\begin{aligned} J_0 := & \min_{v \in M} J(v) = J(w) \\ = & \frac{p}{2(p+2)} \int_{\mathbb{R}^N} \left(|\nabla w|^2 + V(x) w^2 \right) dx > 0. \end{aligned} \tag{54}$$

By the fact that $u_c \in M$ and (54), we know that

$$J_0 \le J(u_c) = c. \tag{55}$$

Notice that $(\psi(w,w)w, \psi(w,w)w) \in N_1$. Using (24), (53), (13), and (14), we get $2\mu + \sum_{k=1}^{m} (\alpha_k + \beta_k) > 0$ and

$$\psi(w,w) = \left(\frac{2\mu}{2\mu + \sum_{k=1}^{m} (\alpha_k + \beta_k)} \right)^{1/p} < \frac{1}{2} < 1. \tag{56}$$

By using the definition of c, (53), (54), and (56), we compute

$$\begin{aligned} c & \le I(\psi(w,w) w, \psi(w,w) w) \\ & = \frac{\psi(w,w)^2}{2} \int_{\mathbb{R}^N} 2 \left(|\nabla w|^2 + V(x) w^2 \right) dx \\ & \quad - \frac{\psi(w,w)^{p+2}}{p+2} \int_{\mathbb{R}^N} \left(2\mu |w|^{p+2} + \sum_k (\alpha_k + \beta_k) |w|^{p+2} \right) dx \\ & = \psi(w,w)^2 J_0 \\ & \quad \times \left(\frac{2(p+2)}{p} - \frac{\psi(w,w)^p \cdot 2 \left(2\mu + \sum_k (\alpha_k + \beta_k) \right)}{\mu p} \right) \\ & = 2\psi(w,w)^2 J_0 \\ & < J_0, \end{aligned} \tag{57}$$

which contradicts (55).

Hence, we obtain $u_c \not\equiv 0$ and $v_c \not\equiv 0$. Therefore $(|u_c|, |v_c|)$ is a ground state of (10). □

3. Symmetry of Ground State

In this section we first use a bootstrap argument similar to in [20] (see also Theorem 8.1.1 of [21]) to obtain a regularity result (Theorem 3). Then we use this regularity result to prove that the ground state of (16) is symmetric about origin.

Lemma 11. *Let $u, v \in L^q(\mathbb{R}^3)$ (for some $p+1 < q < \infty$) be a solution of* (16). *Then $u, v \in W^{2,q/(p+1)}(\mathbb{R}^3)$.*

Proof. Since $u, v \in L^q(\mathbb{R}^3)$, we have

$$|u|^p u, |v|^p v, |u|^{p_k - 2} |v|^{q_k} u, |u|^{p_k} |v|^{q_k - 2} v \in L^{q/(p+1)}(\mathbb{R}^3), \tag{58}$$

by using Hölder's inequality. By the fact that $(-\Delta + |x|^2)$ is a maximal accretive operator in $L^{q/(p+1)}$ (see Theorem 2.5 in [22]), we have

$$\begin{aligned} & \left(-\Delta + |x|^2 + 1 \right)^{-1} L^{q/(p+1)}(\mathbb{R}^3) \\ & \quad = \left\{ f \in W^{2,q/(p+1)}(\mathbb{R}^3) \mid |x|^2 f \in L^{q/(p+1)}(\mathbb{R}^3) \right\}, \end{aligned} \tag{59}$$

which implies $u, v \in W^{2,q/(p+1)}(\mathbb{R}^3)$. □

Proof of Theorem 3. Consider the sequence q_j defined by

$$\frac{1}{q_j} = (p+1)^j \left(\frac{1}{p+2} - \frac{2}{3p} + \frac{2}{3p(1+p)^j} \right). \tag{60}$$

Set $\delta = 2/3p - 1/(p+2) = (4-p)/3p(p+2)$. Since $p < 4$, $\delta > 0$. Direct computation shows

$$\frac{1}{q_{j+1}} - \frac{1}{q_j} = -(p+1)^j p\delta \le -\delta, \tag{61}$$

that is to say, $1/q_j$ is decreasing and $1/q_j \to -\infty (j \to \infty)$. Since $q_0 = p+2 > 0$, it follows that there exists $j_0 \ge 0$ such that

$$\frac{1}{q_l} > 0, \quad \forall 0 \le l \le j_0; \ \frac{1}{q_{j_0+1}} \le 0. \tag{62}$$

We claim that $u, v \in L^{q_{j_0}}(\mathbb{R}^3)$. $u, v \in H^1(\mathbb{R}^3)$, by Sobolev embedding $u, v \in L^{q_0}(\mathbb{R}^3)$. If $u, v \in L^{q_l}(\mathbb{R}^3)$, for some $l \le j_0 - 1$, we have by Lemma 11

$$u, v \in W^{2, q_l/(p+1)}\left(\mathbb{R}^3\right). \tag{63}$$

Using Sobolev embedding again $u, v \in L^q(\mathbb{R}^3)$, for all $q \ge q_l/(p+1)$ such that $1/q \ge (p+1)/q_l - 2/3 = 1/q_{l+1}$. In particular, $u, v \in L^{q_{l+1}}(\mathbb{R}^3)$. By induction, $u, v \in L^{q_{j_0}}(\mathbb{R}^3)$.

Using Lemma 11 and Sobolev embedding again, we have

$$\begin{gathered} u \in L^q\left(\mathbb{R}^3\right), \quad \forall q \ge \frac{q_{j_0}}{p+1} \\ \text{such that } \frac{1}{q} \ge \frac{p+1}{q_{j_0}} - \frac{2}{3} = \frac{1}{q_{j_0+1}} < 0. \end{gathered} \tag{64}$$

Therefore

$$u, v \in \bigcap_{2 \le q < \infty} L^q\left(\mathbb{R}^3\right). \tag{65}$$

Part (a) then follows form Lemma 11.

By Sobolev's embedding, $u, v \in W^{1,\infty}(\mathbb{R}^3)$, which implies u, v are uniformly Lipschitz continuous. Since $u, v \in L^2(\mathbb{R}^3)$, there holds that u, v decay to zero at infinity; that is, (b) is valid.

To prove (c), we first take derivative to obtain for every $j \in \{1, 2, 3\}$

$$\begin{aligned} &\left(-\Delta + |x|^2 + 1\right) u_j \\ &= -2x_j u + \mu(p+1)|u|^p u_j \\ &\quad + \sum_{k=1}^{m} \alpha_k \left((p_k - 1)|u|^{p_k-2}|v|^{q_k} u_j + q_k |u|^{p_k-2} u |v|^{q_k-1} v_j \right), \\ &\left(-\Delta + |x|^2 + 1\right) v_j \\ &= -2x_j v + \mu(p+1)|v|^p v_j \\ &\quad + \sum_{k=1}^{m} \beta_k \left((q_k - 1)|v|^{q_k-2}|u|^{p_k} v_j + p_k |v|^{q_k-2} v |u|^{p_k-1} u_j \right). \end{aligned} \tag{66}$$

By (a) and Sobolev's embedding, $u, v \in W^{1,4}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$. Hence, $|u|^p u_j$, $|u|^{p_k-2}|v|^{q_k} u_j$, and $|u|^{p_k-2} u |v|^{q_k-1} v_j \in L^4(\mathbb{R}^3)$. Using (a), we know that $u, v \in L^{4(p+1)}(\mathbb{R}^3)$. By the fact that

$$\begin{aligned} &\left(-\Delta + |x|^2 + 1\right)^{-1} L^4\left(\mathbb{R}^3\right) \\ &\quad = \left\{ f \in W^{2,4}\left(\mathbb{R}^3\right) \mid |x|^2 f \in L^4\left(\mathbb{R}^3\right) \right\} \end{aligned} \tag{67}$$

(see Theorem 2.5 in [22]), we have

$$\int \left|x_j u\right|^4 dx \le \int \left(|x|^2 |u|\right)^4 dx < +\infty; \tag{68}$$

that is, $x_j u \in L^4(\mathbb{R}^3)$. Therefore, the right hand side of the first equation in (66) belongs to $L^4(\mathbb{R}^3)$. Using Lemma 11, $u \in W^{3,4}(\mathbb{R}^3) \hookrightarrow C^2(\mathbb{R}^3)$. Similarly, $v \in C^2(\mathbb{R}^3)$. □

Proof of Corollary 4. This result is a corollary of Theorem 1 and Corollary 3 in [23]. By using Theorem 1, there exists a ground state (u, v) of (16). From Theorem 3, the ground state is of class C^2 and satisfies $\lim_{|x| \to \infty} u = 0$, $\lim_{|x| \to \infty} v = 0$. The maximum principle applied to each single equation in (16) suggests that $u > 0, v > 0$. Then, by using Theorem 1 in [24] (see also Corollary 3 in [23]), all positive and decay solutions of (16) are radially symmetric about the origin, which proves this corollary. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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